NOTES ON SPECIAL CUBE COMPLEXES
YGGT 2024

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Contents

Very rough lecture notes; please direct corrections to `markfhagen@posteo.net` if you have them. There are already many good expository treatments of this topic, so it's unlikely to get tidied up too much more, but the hope is that it is useful in the following two ways: there is a lightly-annotated list of references in the last section, and the penultimate section contains exercises. There are also a few exercises in the notes themselves (some of which duplicate the ones at the end).

1. RAAGs

1.1. Reminders about residual finiteness.

Definition 1.1 (Separable, residually finite). A subset S of a group G is *separable* if for all $g \in G - S$, there exists $G' \leqslant G$ with $[G : G'] < \infty$ and $S \subset G'$ and $g \notin G'$. An important special case: if $\{1\}$ is separable, then G is *residually finite* □

Exercise 1.2. If G is finitely presented and residually finite, it has solvable word problem.

Definition 1.3. G is *linear* over the field F if there is an embedding $G \to GL_n(F)$ for some n. We say G is $\mathbb{Z}$*–linear* if there is an embedding $G \to SL_n(\mathbb{Z})$ for some n. □

Date: April 8, 2024.

Exercise 1.4. If G is virtually linear (resp. $\mathbb{Z}$–linear) then it is linear (resp. $\mathbb{Z}$–linear).

Theorem 1.5 (Mal'cev, 1940)**.** *Finitely generated linear groups are residually finite.*

Exercise 1.6. Prove directly from the definition: $SL_n(\mathbb{Z})$ is residually finite.

Problem 1.7 (Gromov 1987)**.** Find a hyperbolic group G that is not residually finite.

Non-linear hyperbolic groups are known [Kap05], some of which are definitely residually finite [TT22], but there seem to be relatively few examples.

Consider the presentations

$$\Gamma = \langle x, y \mid [[x, y], y]x \rangle$$

and

$$\langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = ba \rangle.$$

Exercise 1.8. Show that these define the same group.

The first presentation makes Γ one-relator group. The second shows that Γ is the mapping torus of the (non-surjective) injective endomorphism $\phi : F(a, b) \to F(a, b)$ given by $\phi(a) = ab, \phi(b) = ba$. There are various ways to see (e.g. [Kap00]):

Lemma 1.9. *Γ is hyperbolic.*

Druţu-Sapir [DS04] asked:

Question 1.10. Is Γ linear?

Sapir observed[1] that if Γ is linear (over, say, $\mathbb{C}$), then we have invertible matrices x, y satisfying $[[x, y], y] = x^{-1}$. There was some evidence that this should force x^n to be a unipotent matrix for some $n > 0$. This in turns yields a non-cyclic nilpotent subgroup, contradicting Lemma 1.9. But it turns out that Γ is linear. In fact, the only known answer to "Sapir's linear algebra question" (no, x need not have a unipotent power)[2] is via virtually embedding Γ in a RAAG.

1.2. RAAGs and RACGs.

Definition 1.11 (Graph product, RAAG, RACG)**.** Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simplicial (no loops, no bigons) undirected graph. For each $v \in V(\Gamma)$, let G_v be a (finitely generated) group. The *graph product* $G(\Gamma)$ is

$$G(\Gamma) = \ast_{v \in V(\Gamma)} G_v / \langle\langle \{[g_v, g_w] : \{v, w\} \in E(\Gamma),\ g_v \in G_v,\ g_w \in G_w\} \rangle\rangle.$$

When each $G_v \cong \mathbb{Z}/2$, denote $G(\Gamma)$ by $W(\Gamma)$; this is called a *right-angled Coxeter group.* When each $G_v \cong \mathbb{Z}$, denote $G(\Gamma)$ by $A(\Gamma)$; this is a *right-angled Artin group.* □

Theorem 1.12 (Linearity of RAAGs)**.** *Let Γ be a finite simplicial graph. Then*

- *$W(\Gamma)$ is linear (in fact $\mathbb{Z}$–linear).*
- *There exists a finite simplicial graph Λ with the property that $A(\Gamma)$ embeds in $W(\Lambda)$ (*[DJ00, HW99, Gre90]*).*

Hence $A(\Gamma)$ is ($\mathbb{Z}$–)linear and thus residually finite.

Sketch. Let $V(\Gamma) = \{v_1, \dots, v_n\}$. Define a bilinear form on $\mathbb{R}^n$ by

$$\langle e_i, e_j \rangle = \begin{cases} 1 \text{ if } \{v_i, v_j\} \notin E(\Gamma), \\ 0 \text{ otherwise,} \end{cases}$$

where $e_1, \dots, e_n$ are the standard basis vectors. For each v_i, let $f(v_i) \in End(\mathbb{R}^n)$ be $f(v_i)(x) = x - 2\langle x, e_i \rangle \cdot e_i$.

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[1] `https://mathoverflow.net/questions/44737/invertible-matrices-satisfying-x-y-y-x`

[2] As far as I know...

Exercise 1.13. Show that the $f(v_i)$ defined as above determine a unique homomorphism $f : W(\Gamma) \to End(\mathbb{R}^n)$ with image in $SL_n(\mathbb{Z})$. Show (or read about it in, for instance, [Cas]) that f is injective.

Now for the embedding of $A(\Gamma)$ in a Coxeter group. Define a graph Λ as follows. For $1 \leqslant i \leqslant n$, let v_i^+, v_i^- be vertices, and form edges by joining $v_i^\pm$ to $v_j^\pm$ and $v_j^\mp$ if and only if $\{v_i, v_j\} \in V(\Gamma)$. Define $i : A(\Gamma) \to W(\Lambda)$ by $i(v_i) = v_i^- v_i^+$. This induces a unique homomorphism. Injectivity is explained in [HW99] but it is also not too hard to see using cube complex stuff discussed shortly. □

Definition 1.14 ((Virtually) special group, first definition)**.** G is (virtually) special if (there is a finite index subgroup of) G that embeds in $A(\Gamma)$ for some Γ. □

From now on, unless stated otherwise, we restrict to finitely generated groups — if G is finitely generated, then we can insist on a finite Γ in the previous definition without affecting anything. So, (finitely generated) virtually special groups are $\mathbb{Z}$–linear and residually finite.

Now return to our discussion of separability:

Exercise 1.15 (Separability facts)**.** Fix a group G, a finite-index subgroup G', some $H \leqslant G$, and let $H' = G' \cap H$. Then:

(1) If H' is separable in G', then H is separable in G.
(2) If G is residually finite and there is a retraction $r : G \to G$ onto H, then H is separable in G.

Hence, if $H \leqslant G$ and $H \cap G'$ is a retract of G' for some finite index G', then H is separable in G. (*It is useful to prove this sort of thing once in your life from the definitions, and then go read about the profinite topology in the enumerated exercises at the end of the notes.*)

Corollary 1.16. *Let $\Lambda \subset \Gamma$ be an induced subgraph, and let $i : A(\Lambda) \to A(\Gamma)$ be the homomorphism induced by the inclusion $\Lambda \to \Gamma$. Then i is injective, and its image is separable in $A(\Gamma)$.*

Proof. Exercise; consider the map $r : A(\Gamma) \to A(\Lambda)$ given by $r(v) = v$ for $v \in \Lambda$ and $r(v) = 1$ otherwise. □

There are non-separable subgroups of RAAGs, as first observed in [Mih66]:

Example 1.17. Let $F = \langle a, b \mid \varnothing \rangle$ and $F' = \langle c, d \mid \varnothing \rangle$ and let $A = F \times F'$, which is $A(\Gamma)$ where Γ is a 4–cycle. Let Q be a 2–generated group that is not residually finite. Many Baumslag-Solitar groups give nice examples [Mes72]. So we have quotient maps $f : F \to Q, f' : F' \to Q'$ where Q' is a copy of Q. Let $i : Q \to Q'$ be the isomorphism given by $f(a) \mapsto f'(c), f(b) \mapsto f'(d)$. Let $\Delta \subset Q \times Q'$ be $\Delta = \{(x, i(x)) : x \in Q\}$. Let $H = (f \times f')^{-1}(\Delta)$.

Fix $Q = \langle a, b \mid ba^2b^{-1}a^{-3} \rangle$ and Q' the same but with a replaced by c and b by d. Then H is generated by $(a, c), (b, d)$ and $(ba^2b^{-1}a^{-3}, 1)$ and $(1, dc^2d^{-1}c^{-3})$, so it is finitely generated. One the other hand, H is not separable; see exercises at the end. □

Much of the importance of separability comes from its geometric/topological interpretation, observed by Scott [Sco78]; here is a geometric version of the statement, which I prefer:

Theorem 1.18 (Separability, geometric viewpoint)**.** *Let $\widetilde{X}$ be a connected graph and let G act (metrically) properly on $\widetilde{X}$. Let $H \leqslant G$ stabilise a connected subgraph $\widetilde{Y} \subset \widetilde{X}$, with H acting coboundedly on $\widetilde{Y}$. Then the following are equivalent:*

(1) H is separable in G.

(2) *For all* $r < \infty$ *there exists* $G' \leqslant_{fi} G$ *such that* $H \leqslant G'$ *and* $\mathsf{d}_{\widetilde{X}}(\widetilde{Y}, g\widetilde{Y}) > r$ *for all* $g \in G' - H$.

This could be stated a bit more generally but conveys the point.

Proof. Fix $y_0 \in \widetilde{Y}$ and $C < \infty$ such that $\widetilde{Y} \subseteq \bigcup_{h \in H} N_C(h \cdot y_0)$.

Assume that H is separable and let $r < \infty$ be arbitrary. Let $\mathcal{S} = \{g \in G - H : \mathsf{d}_{\widetilde{X}}(y_0, gy_0) \leqslant 10(r + C)\}$. Then $|\mathcal{S}| < \infty$ since G acts on $\widetilde{X}$ properly. Use separability to choose $G'_r \leqslant_{fi} G$ such that $H \leqslant G'_r$ and $\mathcal{S} \cap G'_r = \varnothing$. Suppose that $g \in G'_r$ and $\mathsf{d}_{\widetilde{X}}(g\widetilde{Y}, \widetilde{Y}) \leqslant r$. Then for some $a, b \in H$, we have $\mathsf{d}_{\widetilde{X}}(ga \cdot y_0, b \cdot y_0) \leqslant r + 2C$, so $b^{-1}ga \in \mathcal{S}$ or $b^{-1}ga \in H$. Now, $b^{-1}ga \in G'_r$ since $a, b \in H$ and $g \in G'_r$; this is a contradiction unless $b^{-1}ga \in H$, and hence $g \in H$.

Conversely, assume that for each r we have a G' with the second enumerated property. Let $g \in G - H$. Let $r = \mathsf{d}_{\widetilde{X}}(y_0, gy_0) + 1$ and let $\mathcal{S}$ be defined as above for the given r. Let $G' \leqslant_{fi} G$ contain H and have the property that every $a \in G' - H$ moves $\widetilde{Y}$ a distance more than r. Then $\mathsf{d}_{\widetilde{X}}(ay_0, y_0) > r$, so $g \notin G'$, as required to prove separability. □

Here is a useful special case. Let X be a connected CW complex based at a vertex v and let $G = \pi_1(X, v)$. Let (Y, y) be a based CW complex with $Y^{(1)}$ finite, and let $f : (Y, y) \to (X, v)$ be a continuous based map such that $f_* : \pi_1(Y, y) \to G$ is injective. Let $H = \mathrm{im}(f_*)$.

Let $\widetilde{X} \to X$ be the universal cover and let $\widetilde{Y} \to Y$ be the universal cover, and choose basepoints $\tilde{v} \in \widetilde{X}, \tilde{y} \in \widetilde{Y}$ mapping to v, y respectively under the covering maps.

Let $\tilde{f} : (\widetilde{Y}, \tilde{y}) \to (\widetilde{X}, \tilde{v})$ lift f; we abuse notation and denote its image by $\widetilde{Y}$, which we regard as a subcomplex of $\widetilde{X}$. Up to conjugation, $H \leqslant \mathrm{Stab}_G(\widetilde{Y})$, and $Y = H \backslash \widetilde{Y}$.

So, G acts freely by isometries on $\widetilde{X}^{(1)}$, and H stabilises, coboundedly, the subgraph $\widetilde{Y}^{(1)}$. As long as $X^{(1)}$ is finite, then G acts metrically properly (since $\widetilde{X}$ is locally compact in this case). Note that $\widetilde{Y}_C := N_C(\widetilde{Y})$ is H–cobounded for all $C < \infty$. Let $Y_C = H \backslash \widetilde{Y}_C$, so we have a π_1–injective map $Y_C \to X$ extending f and inducing the inclusion of H into G.

Now, by the theorem, the following are equivalent:

- H is separable in G, and
- for all C, there exists $G' \leqslant_{fi} G$ such that $H \leqslant G'$ and $Y_C \cap gY_C = \varnothing$ for $g \in G' - H$.

The second condition can be rephrased as follows. Let $X_C = G' \backslash \widetilde{X}$, so that the covering map $\widetilde{X} \to X = G \backslash \widetilde{X}$ factors as $\widetilde{X} \to X_C \to X$, where the second arrow is a finite-sheeted cover. Moreover, the inclusion $\widetilde{Y}_C \to \widetilde{X}$ descends to an *injective* map $Y_C \to X_C$, and there is a commutative diagram:

$$\begin{array}{ccc} & & X_C \\ & \nearrow & \downarrow \\ Y_C & \to & X \end{array}.$$

So, Y_C **embeds in a finite cover of** X; this is the point of separability (for our purposes).

Remark 1.19. Major motivation came from 3–manifolds: for instance, the virtual Haken problem for hyperbolic 3–manifolds asked, given a closed, oriented, hyperbolic 3–manifold M, for a finite cover $\hat{M} \to M$ containing an embedded π_1–injective closed surface. This is achieved (once one has surface subgroups, itself already a huge deal) by showing that the relevant subgroups of $\pi_1 M$ are separable, and this is in turn achieved by showing that $\pi_1 M$ is virtually special. More about this later. □

Next, we will discuss the combinatorial/topological viewpoint on virtual specialness, using cube complexes. We will show how it interacts well with the geometric characterisation of separability, and why it is equivalent to virtually embedding an a RAAG. Then we will discuss the recipe for proving virtual specialness, and how it relates to examples like Sapir's question.

2. Cube complexes, specialness, and finite covers

2.1. Cube complexes.

Definition 2.1 (Cube (as a metric CW complex))**.** For $n \geqslant 0$, an n–*cube* is defined inductively as follows. A 0–cube is a 0–cell, and a 1–cube is a 1–cell. For $n \geqslant 1$, an n–cube is the CW complex obtained by equipping the product of n 1–cubes with the product cell structure. Identifying each 1–cube c with $[-\frac{1}{2}, \frac{1}{2}]$ makes the n–cube c^n a metric space by equipping it with the ℓ_p metric, for $1 \leqslant p \leqslant \infty$. The most commonly-used values of p are $1, 2, \infty$. □

Definition 2.2 (Face, cube complex)**.** If $c = [-\frac{1}{2}, \frac{1}{2}]^n$ is an n–cube, a *face* f of c is a subspace obtained by restricting some coordinates to $+\frac{1}{2}$ or $-\frac{1}{2}$. The face f is naturally isometric (and isomorphic as a CW complex) to $[-\frac{1}{2}, \frac{1}{2}]^k$, where $k \leqslant n$ is the number of unrestricted coordinates. The inclusion $f \hookrightarrow c$ is a combinatorial map, and an isometric embedding in any of the above metrics $\ell_1, \ell_2, \ell_\infty$.

A *cube complex* is a CW complex X whose cells are cubes, with the property that for each n–cube c, the attaching map $\partial c \to X^{(n-1)}$ restricts on each proper face of c to a combinatorial isometry to some cube of $X^{(n-1)}$. □

Definition 2.3 (Link, induced map, nonpositively curved, CAT(0))**.** For each vertex $v \in X$, let $\mathrm{Lk}(v)$ be the simplex complex defined as follows.

Let $i \geqslant 0$ and let c be an $(i+1)$–cube with attaching map $\partial c \to X^{(i)}$. For each $p \in \partial c$ mapping to v, there is an i–simplex $\sigma(c, p)$ in $\mathrm{Lk}(v)$. We view $\sigma(c, p)$ as the $\frac{1}{3}$–ball in the ℓ_1 metric on c centred at the corner p.

The attaching maps define an equivalence relation on the disjoint union of all the $\sigma(c, p)$, and the quotient is a CW complex (made of simplices) called the *link* $\mathrm{Lk}(v)$ of v.

We say that X is *nonpositively curved* if $\mathrm{Lk}(v)$ is a simplicial complex (each simplex is uniquely determined by its 0–skeleton) and moreover a flag complex for all $v \in X^{(0)}$. (This means: if $x_0, \ldots, x_n$ are pairwise-adjacent vertices in $\mathrm{Lk}(v)$, then they span an n–simplex.)

If $f : X \to Y$ is a combinatorial map of cube complexes, then for each 0–cube $v \in X$, there is a natural combinatorial map $f_v : \mathrm{Lk}_X(v) \to \mathrm{Lk}_Y(f(v))$, since each $(i+1)$–cube incident to v in X is sent to an $(i+1)$–cube in Y incident to $f(v)$.

If X is simply connected and nonpositively-curved, it is a *CAT(0) cube complex.* □

Remark 2.4 (Metrics)**.** If X is a CAT(0) cube complex, then regarding each cube as a unit cube with the ℓ_p metric equips X with a length metric d_p in the usual way.

1. When X is finite-dimensional (i.e. $X = X^{(n)}$ for some n), d_2 is a CAT(0) metric inducing the CW topology [Bri91]; even if X is infinite-dimensional, d_2 is still a CAT(0) metric, although now the metric and CW topologies are only homotopy equivalent instead of homeomorphic [Lea13]. A useful consequence is that CAT(0) cube complexes are contractible. (One can also prove this using the other metrics, too.)
2. The metric d_1 is a median metric (see e.g. [Mie14]), and restricts on the 0–skeleton to the graph metric on $X^{(1)}$. The 1–skeleton is a *median graph* and uniquely determines X [Che00].
3. When X is finite-dimensional, d_∞ and d_1 and d_2 are all bilipschitz equivalent. Combinatorial automorphisms are isometries of all three metrics. d_∞ is an injective metric.

We aren't going to do much geometry on CAT(0) cube complexes. The median viewpoint is the most used to date for this sort of purpose, although the ℓ_∞ viewpoint is increasingly important. So the name "CAT(0) cube complex" is perhaps a bit misleading. However, it is the case that if the piecewise–ℓ_2 metric on a simply connected cube complex is CAT(0), then it satisfies the link condition. □

Definition 2.5. Let X be a cube complex. An *automorphism* is a continuous bijection $f : X \to X$ such that f sends 0–cubes to 0–cubes and open cubes homeomorphically to open cubes (necessarily of the same dimension). If X is CAT(0), automorphisms are isometries of all the above metrics, and group actions on X are always by automorphisms. □

We will mostly work with compact nonpositively curved cube complexes.

Example 2.6.

(1) Each graph Γ is a cube complex of dimension at most 1. CAT(0) cube complexes of dimension at most 1 are precisely trees.

(2) Let Γ be a finite simplicial graph, and let $A(\Gamma)$ be the associated RAAG. The *Salvetti complex* $X(\Gamma)$ is the cube complex that has one 0–cube, a 1–cube v for each $v \in V(\Gamma)$, and an n–cube spanned by the 1–cubes $v_1, \ldots, v_n$ if and only if $v_1, \ldots, v_n$ are pairwise-adjacent in Γ.

Exercise 2.7. Show that $X(\Gamma)$ is nonpositively curved and that $\pi_1 X(\Gamma) \cong A(\Gamma)$. □

2.2. **Hyperplanes.** These are the answer to "why cubes?".

Definition 2.8 (Hyperplane)**.** Let $c \cong [-\frac{1}{2}, \frac{1}{2}]^n$ for $n \geqslant 1$. A *(closed) midcube* of c is a subspace (isometric to $[-\frac{1}{2}, \frac{1}{2}]^{n-1}$) obtained by restricting exactly one coordinate to 0. Let X be a cube complex. For each cube c and each midcube m of c, consider the map $m \hookrightarrow c \to X$, where the second arrow is the characteristic map. If f is a face of c and n is a midcube of f such that $f \hookrightarrow c$ restricts to an inclusion $n \hookrightarrow m$, then write $n \prec m$. Form a cube complex *Hyp* whose cubes are all of the midcubes, where m, m' are attached by identifying $n \prec m$ with $n \prec m'$ via the identity $n \to n$ whenever such an n exists. The maps $m \to X$ induce a map $Hyp \to X$. A component h of *Hyp* is called an *immersed hyperplane* of X, and we have a map $h \to X$ that is an embedding on each open cube of h (open cubes get sent to open midcubes isometrically).

If $h \to X$ is an embedding, we call h a *hyperplane* (i.e. we drop "immersed"). □

Each 1–cube of X intersects exactly one immersed hyperplane, namely the immersed hyperplane h containing the midpoint of the 1–cube; h and the 1–cube are *dual*. Collected useful facts about hyperplanes (see e.g. [Sag95, Che00, Wis21, Hag22]):

Theorem 2.9. *Let X be a connected nonpositively curved cube complex, and let $\widetilde{X} \to X$ be the universal cover (so, $\widetilde{X}$ is a CAT(0) cube complex). Let $h \to X$ be an immersed hyperplane, and let $\widetilde{h} \to h$ be the universal cover. Then:*

(1) $\widetilde{h} \to h \to X$ lifts to an embedding $\widetilde{h} \to \widetilde{X}$, and $\widetilde{h}$ is a hyperplane of $\widetilde{X}$.

(2) $\widetilde{h}$ intersects each cube c of $\widetilde{X}$ either in $\varnothing$ or in a midcube of c.

(3) $\widetilde{h}$ is geodesically convex in $\widetilde{X}$ with both the d_1 and d_2 metrics.

(4) $\widetilde{X} - \widetilde{h}$ has exactly two components, the halfspaces *associated to $\widetilde{h}$. We denote these $\widetilde{h}_+, \widetilde{h}_-$. We say that $x, y \in \widetilde{X}$ are* separated *by $\widetilde{h}$ if $x \in \widetilde{h}_\pm$ and $y \in \widetilde{h}_\mp$.*

(5) If $v, w \in \widetilde{X}^{(0)}$, then $\mathsf{d}_1(v, w)$ is the cardinality of the set of hyperplanes separating v, w. And $\mathsf{d}_\infty(v, w)$ is the maximal possible cardinality of a set of pairwise-disjoint hyperplanes separating v, w.

(6) Let $N(\widetilde{h})$ be the union of all closed cubes of $\widetilde{X}$ that contribute midcubes to $\widetilde{h}$. Then there is a cubical isomorphism $\widetilde{h} \times [-\frac{1}{2}, \frac{1}{2}] \to N(\widetilde{h})$ sending $\widetilde{h} \times \{0\}$ to $\widetilde{h}$ via $(x, 0) \mapsto x$. The subcomplexes $\widetilde{h} \times \{\pm\frac{1}{2}\}$ are convex and called combinatorial hyperplanes.

(7) If $\widetilde{h}$ is a hyperplane, then $\widetilde{h}_\pm \cup N(\widetilde{h})$ is a convex (in all metrics) subcomplex of $\widetilde{X}$ called a combinatorial halfspace.

Let $h \to X$ be a hyperplane, and let $G = \pi_1 X$ act on $\widetilde{X}$ by deck transformations. Let $H = \mathrm{im}(\pi_1 h \to G)$ (imagine we chose basepoints), so that H stabilises an appropriately chosen lift $\widetilde{h} \subset \widetilde{X}$ of $\widetilde{h} \to h \to X$.

First, the theorem implies $\pi_1 h \to G$ is injective $\pi_1 h \cong H$. The covering map $\widetilde{h} \to h$ is therefore the quotient by the H–action, and the original map $h \to X$ is just the map $H\backslash\widetilde{h} \to G\backslash\widetilde{X}$ induced by $\widetilde{h} \hookrightarrow \widetilde{X}$. Let $N(h) = H\backslash N(\widetilde{h})$; the complex $N(h)$ may not have a product structure, although it is an interval bundle over h, since H might permute the two halfspaces associated to h. But we do have a cubical map $N(h) \to X$ which is again π_1–injective (and $N(h)$ is homotopy equivalent to h). We call $N(\widetilde{h})$ the *carrier* of $\widetilde{h}$ and $N(h)$ the *(immersed) carrier of h.*

2.3. Local isometries.

Definition 2.10. Let X, Y be cube complexes. The combinatorial map $f : Y \to X$ is *locally injective* if $f_v : \mathrm{Lk}(v) \to \mathrm{Lk}(f(v))$ is injective for all v. The locally injective map f is a *local isometry* if for all $v \in X$ and all $x_0, \ldots, x_n \in \mathrm{Lk}(v)^{(0)}$ such that the $f_v(x_i)$ span a simplex e in $\mathrm{Lk}(f(v))$, there is a simplex e' in $\mathrm{Lk}(v)$ spanned by the x_i and $f_v(e') = e$. □

Example 2.11. Covering maps of nonpositively curved cube complexes are local isometries. A big part of the point of specialness is that it will enable us to start with a local isometry $Y \to X$ and add stuff to Y to get a covering map, without having to add too much... □

Lemma 2.12. *Let $f : Y \to X$ be a local isometry of (connected, to avoid basepoints) cube complexes. If X is nonpositively-curved, so is Y. In this case, $\widetilde{Y} \to Y \xrightarrow{f} X$ lifts to an embedding $\widetilde{Y} \to \widetilde{X}$ with convex image (in d_1 and d_2 sense), and in particular f is π_1–injective.*

Exercise 2.13. Prove Lemma 2.12. Suggestions:

- Lifting criterion gives the lift $\tilde{f} : \widetilde{Y} \to \widetilde{X}$; check it's a local isometry.
- Injectivity of $\tilde{f}$: consider a combinatorial geodesic γ in $\widetilde{Y}$ that is embedded but has the property that $\tilde{f} \circ \gamma$ is a closed path. Use the part of Theorem 2.9 about separation to find two distinct edges e, e' of $\tilde{f} \circ \gamma$ that are dual to the same hyperplane of $\widetilde{X}$. Choose the e, e' to be a "smallest" counterexample in an appropriate sense, and derive a contradiction using that $\tilde{f}$ is an isometry.
- Convex image: verify that any $x \in \widetilde{X}^{(0)} - \tilde{f}(\widetilde{Y})$ is separated from $\tilde{f}(\widetilde{Y})$ by a hyperplane and use Theorem 2.9.

2.4. Specialness.

Definition 2.14 (Specialness, extrinsic definition)**.** The [compact] cube complex X is *[compact] special* if there exists a Salvetti complex $X(\Gamma)$ and a local isometry $X \to X(\Gamma)$. The [compact] cube complex X is *virtually [compact] special* if it has a finite-sheeted cover with this property.

The group G is *[compact] special* if $G = \pi_1 X$ for X a [compact] special cube complex. The group G is *virtually ([compact] special)* if there exists a finite-index $G' \leqslant G$ with G' [compact] special. □

Remark 2.15. The group G being virtually compact special is a weaker property than asking that $G = \pi_1 X$ where X is virtually compact special.

We insisted our RAAGs are finitely generated, but it's fine to allow $X(\Gamma)$ with Γ infinite, everything will work, and it's more natural from the next point of view. With this more expansive definition, one can say nice things like: CAT(0) cube complexes are all special, but not in general compact special, and not via local isometries to $X(\Gamma)$ with Γ finite. □

Proposition 2.16. *Suppose that X is a connected*[3] *special cube complex. Suppose that $h \to X$ and $w \to X$ are hyperplanes. Then:*

- *$h \to X$ is an embedding.*
- *$N(h) \cong h \times [-\frac{1}{2}, \frac{1}{2}]$.*
- *If $g \in \pi_1 X$ has the property that $gN(\widetilde{h}) \neq N(\widetilde{h})$ but $gN(\widetilde{h}) \cap N(\widetilde{h}) \neq \varnothing$, then the halfspaces associated to $\widetilde{h}$ can be labeled so that $g\widetilde{h}_+ \subset \widetilde{h}_+$.*
- *If h and w cross (i.e. there exist edges e, f of X that are dual to h and w respectively, and span a 2–cube in X), then for all edges e', f' dual to h, w and incident to a common vertex of X, there is a square in X spanned by e', f'.*

Proof. Salvetti complex exercise. □

Definition 2.17 (Specialness)**.** Let X be a nonpositively curved cube complex and suppose that the itemised conditions in Proposition 2.16 hold for all hyperplanes $h, w \to X$. Then X is *special.* □

Proposition 2.18. *If X is special in the sense of the preceding definition, then it admits a local isometry to $X(\Gamma)$ for some Γ.*[4]

Proof. Since each hyperplane is embedded, we regard the hyperplanes as subspaces of X. Let Γ be the graph whose vertices are hyperplanes of X, with an edge whenever two hyperplanes cross in X. We define a map $f : X \to X(\Gamma)$ as follows. First, f sends all vertices of X to the vertex $x_0 \in X(\Gamma)$. Next, the edges of $X(\Gamma)$ are labeled by the vertices of Γ, i.e. by the hyperplanes of X. For each hyperplane h of X, fix an identification of the immersed carrier $N(h) \to X$ with $h \times [-\frac{1}{2}, \frac{1}{2}]$, using the hypothesis about that, and note that this induces an orientation of the edges dual to h in X. Map each such edge e to the (oriented) edge e_h of $X(\Gamma)$ corresponding to h, by an orientation-preserving combinatorial map (after orienting edges in $X(\Gamma)$ arbitrarily). Now, if edges $e_1, \ldots, e_k$ of X span a cube, then their dual hyperplanes pairwise-cross and are in particular distinct, and $e_{h_1}, \ldots, e_{h_k}$ span a cube in $X(\Gamma)$, so we can extend to a combinatorial map $f : X \to X(\Gamma)$.

Fix $v \in X^{(0)}$. Suppose that $x, y \in \mathrm{Lk}(v)^{(0)}$. If x, y correspond to edges e, e' of X, then consider the hyperplanes h, w dual to e, e'. If h, w are distinct, then $e_h = f(e) \neq f(e') = e_w$, so $f_v(x) \neq f_v(y)$. If $h = w$, then x, y correspond respectively to the terminal end of e and the initial end of e' (or the same, but swapping initial and terminal), by the third bullet point, so $f_v(x) \neq f_v(y)$. Hence f is a local injection.

Since all the links involved are flag, we just have to check the fullness condition on 1–simplices. Let $x, y \in \mathrm{Lk}(v)^{(0)}$ and suppose that $f_v(x), f_v(y)$ are joined by an edge in $\mathrm{Lk}\, f(v)$. Let e, e' be as above. Then $f(e) = e_h$ and $f(e') = e_w$ span a square in $X(\Gamma)$, so by the definition of Γ, the hyperplanes h, w cross in X. So by the last bullet point, they cross at v, i.e. e, e' span a square, and we are done. □

Corollary 2.19. *Let G be a finitely generated group. Then the following are equivalent:*

- *G embeds in a (finitely generated) RAAG;*
- *$G = \pi_1 X$, where X is a nonpositively-curved cube complex admitting a local isometry to $X(\Gamma)$ for some finite Γ.*

[3] This assumption is just to avoid mentioning basepoints; the conclusions all hold in a disconnected complex if they hold in each component.

[4] If X has infinitely many hyperplanes, then we have to allow Γ to be infinite.

- *$G = \pi_1 X$, where X is a nonpositively-curved cube complex with the properties from the list in Proposition 2.16.*

Proof. Suppose $G \leqslant A(\Gamma)$ with Γ finite. Then $G \hookrightarrow A(\Gamma)$ is induced by a cover $\widehat{X}(\Gamma) \to X(\Gamma)$, and covers are local isometries. So the first statement implies the second. The second implies the third by Proposition 2.16. Now suppose that $G = \pi_1 X$, where X is a nonpositively-curved cube complex with the listed properties in Proposition 2.16. Consider the G–action on $\widetilde{X}$. Since G is finitely generated, there is a connected G–cocompact subgraph of $\widetilde{X}^{(1)}$, and so the hyperplanes crossing this subgraph fall into finitely many G–orbits. So by replacing $\widetilde{X}$ with the convex hull of the G–orbit, we can assume that X has finitely many hyperplanes. Apply Proposition 2.18 to get a local isometry $X \to X(\Gamma)$ (with Γ finite since its vertices correspond to the hyperplanes in X) and hence an embedding $G \to A(\Gamma)$. □

2.5. **Canonical completion and separability.** Showing that G is virtually special tells us that it is $\mathbb{Z}$–linear, residually finite, etc. But we can do much more, because we have an excellent tool for making finite covers that is not much more complicated than what one can classically (see [Sta83]) do for graphs.

2.5.1. *Canonical completion for wedges of circles.* Let B be a wedge of circles, with a vertex x_0 and oriented edges $e_1, \ldots, e_n$. Let Y be a finite graph, and let $f : Y \to X$ be a local isometry. Since the links are all discrete, this just means that f sends all vertices to x_0 and maps open edges homeomorphically, with no backtracking at vertices.

Pull back the labels and orientations of the edges to Y. A *e_i–path* in Y is a connected subgraph whose edges are all labeled e_i. Each e_i path is either a directed cycle, or an embedded directed interval, since f is a local isometry.

Let $\mathsf{C}\big(Y, B\big)$ be the graph obtained from Y as follows: for each i and each e_i–path that is not a cycle, add an edge joining the endpoints of the e_i–path. Label that edge e_i and orient it so that the resulting cycle is directed. This gives a local isometry $\mathsf{C}\big(Y, B\big) \to B$, and by construction, it is actually a covering map. By construction, Y embeds in $\mathsf{C}\big(Y, B\big)$ and the restriction to Y of the cover is f.

With this construction, one can prove easily that finitely generated subgroups of free groups are separable (compare to [Hal49]):

Corollary 2.20 (Effective Marshall Hall's theorem)**.** *Let $f : Y \to B$ be a local isometry with Y a finite connected graph, and let $g \in \pi_1 B - \mathrm{im}(f_*)$ be an element of length n. Then there is a subgroup F' of $\pi_1 B$ such that* $\mathrm{im}(f_*) \leqslant F'$*, and $g \notin F'$, and* $[\pi_1 B : F'] \leqslant |Y^{(0)}| + n$.

Sketch. Exercise: choose an appropriate enlargement of Y, depending on g, and use canonical completion. □

2.5.2. *Canonical completion for Salvetti complexes.* Next, we generalise canonical completion by allowing some higher-dimensional cubes. Let Γ be a finite connected graph and let Y be a compact connected nonpositively curved cube complex and $f : Y \to X(\Gamma)$ a local isometry.

Since f restricts to a graph immersion $f : Y^{(1)} \to X(\Gamma)^{(1)}$, and $X(\Gamma)^{(1)}$ is a wedge of circles, we have a finite cover $\mathsf{C}\big(Y^{(1)}, X(\Gamma)^{(1)}\big) \to X(\Gamma)^{(1)}$ extending f.

The key technical lemma (whose proof I don't find that illuminating of the canonical completion construction, so I am leaving it out):

Lemma 2.21. *For each 2–cube s of $X(\Gamma)$, let $\partial s \to X(\Gamma)^{(1)}$ be its boundary path. Then each lift of ∂s to $\mathsf{C}\big(Y^{(1)}, X(\Gamma)^{(1)}\big)$ is a 4–cycle.*

Proof. Exercise; case analysis. Or see [HW08] or [BRHP15] or various other places. □

By the lemma, we can add squares (and then higher cubes, which are easier) to get a cover $\mathsf{C}\big(Y, X(\Gamma)\big) \to X(\Gamma)$ where Y embeds and where the restriction to Y of the covering map is the original local isometry $Y \to X$. The construction is enough to show:

Corollary 2.22. *$\pi_1 Y$ is separable in $A(\Gamma)$.*

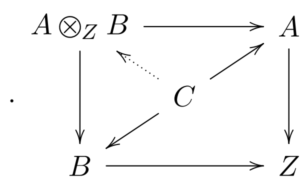
But we will take it a bit further.

2.5.3. *Arbitrary special codomains.* What if the codomain isn't a Salvetti complex?

Definition 2.23 (Fibre product)**.** Let A, B, Z be nonpositively-curved cube complexes and let $A, B \to Z$ be local isometries. Then there is a unique nonpositively-curved cube complex $A \otimes_Z B$, equipped with local isometries $A \otimes_Z B \to A, B$, such that

$$\begin{array}{ccc} A \otimes_Z B & \to & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & Z \end{array}$$

commutes, and such that the following holds: let C be a nonpositively-curved cube complex equipped with local isometries $C \to A, B$ so that the following diagram (with solid arrows) commutes, there is a unique local isometry $C \to A \otimes_Z B$ making the whole diagram commute:



We call $A \otimes_Z B$ the *fibre product* of $A, B \to Z$.[5] □

Exercise 2.24. Prove that fibre products exist. The 0–skeleton is the set of $(a, b) \in A^{(0)} \times B^{(0)}$ such that a, b have the same images under $A, B \to Z$. Then 0–cells $(a_0, b_0), \cdots, (a_n, b_n)$ form a cube if and only if $a_0, \ldots, a_n$ form a cube in A and $b_0, \ldots, b_n$ form a cube in B. Check that this works, and observe that $A \otimes_Z B$ is in general disconnected even if A, B, Z are connected (there's a more detailed exercise about this at the end).

Let Y be compact and connected and nonpositively curved, let X be special, let $X \to B$ be a local isometry with B a Salvetti complex, and let $f : Y \to X$ be a local isometry. The composition $Y \to X \to B$ is a local isometry, so we have the finite cover $\mathsf{C}\big(Y, B\big) \to B$ where Y embeds, constructed before. We thus have a (possibly disconnected) nonpositively-curved cube complex $X \otimes_B \mathsf{C}\big(Y, B\big)$ equipped with local isometries to X and to $\mathsf{C}\big(Y, B\big)$.

Lemma 2.25. *The map $X \otimes_B \mathsf{C}\big(Y, B\big) \to X$ is a finite cover, and the inclusion $Y \to \mathsf{C}\big(Y, B\big)$ lifts to an embedding $Y \to X \otimes_B \mathsf{C}\big(Y, B\big)$.*

Proof. We have given local isometries $f : Y \to X$ and $Y \hookrightarrow \mathsf{C}\big(Y, B\big)$, so the universal property provides a local isometry $Y \to X \otimes_B \mathsf{C}\big(Y, B\big)$ lifting both f and the inclusion. This has to be injective since the inclusion is. On the other hand, let $\widetilde{X} \to X \to B$ be the composition of the universal cover with the given map $X \to B$. Then since $\mathsf{C}\big(Y, B\big) \to B$ is a cover, the lifting criterion provides a lift $\widetilde{X} \to \mathsf{C}\big(Y, B\big)$ which is a local isometry. Hence the universal property gives a local isometry $\widetilde{X} \to X \otimes_B \mathsf{C}\big(Y, B\big)$ through which $\widetilde{X} \to X$ factors, so it has to be a cover. □

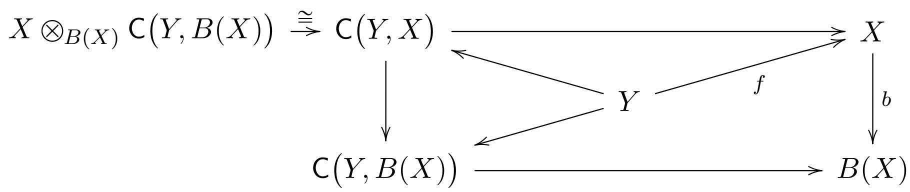
[5]It's a pullback in the category whose objects are nonpositively-curved cube complexes and whose morphisms are local isometries, if you're into that sort of thing, which I increasingly think is a good sort of thing to be into.

We can always choose $B = B(X)$ to be $X(\Gamma)$, where Γ is the crossing graph of X from the proof of Proposition 2.18. We let

$$\mathsf{C}\big(Y, X\big) := X \otimes_B \mathsf{C}\big(Y, B\big),$$

and call it the *canonical completion* of $Y \to X$.

Remark 2.26 (Canonical completion diagram)**.** Here is the commutative diagram summarising canonical completion of the local isometry $f : Y \to X$



where $b : X \to B(X)$ is the canonical map to a Salvetti complex given by the crossing graph, the maps $\mathsf{C}\big(Y, X\big) \to X, \mathsf{C}\big(Y, B(X)\big)$ are induced by the natural projections, the covering map $\mathsf{C}\big(Y, B(X)\big) \to B(X)$ is canonical completion, $Y \hookrightarrow \mathsf{C}\big(Y, B(X)\big)$ is the inclusion, and $Y \to \mathsf{C}\big(Y, X\big)$ is the injective local isometry given by the universal property. □

2.5.4. *Canonical retraction.* Consider $Y \xrightarrow{f} X \xrightarrow{b} B(X)$. First consider the canonical completion $\mathsf{C}\big(Y, B(X)\big) \to Y$ and the canonical inclusion $\iota : Y \to \mathsf{C}\big(Y, B(X)\big)$; abusing language we denote $\iota(Y)$ by Y. Define a map $r_Y : \mathsf{C}\big(Y, B(X)\big) \to Y$ as follows. Recall that ι is surjective on vertices and let r_Y be the identity on the vertices. If e is an edge of $\mathsf{C}\big(Y, B(X)\big)$, then one of the following holds:

- e is an edge of Y, and we let $r_Y|_e : e \to e$ be the identity.
- There is a (directed) cycle $\gamma \to \mathsf{C}\big(Y, B(X)\big)^{(1)}$ such that for some (directed) edge $\bar{e}$ of $B(X)$ (a loop), the covering map $\mathsf{C}\big(Y, B(X)\big) \to B(X)$ restricts to a cover $\gamma \to \bar{e}$, and $\gamma = \alpha e$, where α is a (possibly trivial) embedded combinatorial interval. In this case, r_Y maps e to α homeomorphically, fixing endpoints.

Thus far, we constructed a retraction $r_Y : \mathsf{C}\big(Y, (X)\big)^{(1)} \to Y^{(1)}$. Now suppose that $c = e_1 \times \cdots \times e_n$ is an (immersed) n–cube of $\mathsf{C}\big(Y, B(X)\big)$, viewed as a product of edges, mapping to $\bar{e}_1 \times \cdots \times \bar{e}_n$, which is a torus of $B(X)$ (an n–cube with opposite faces identified).

Exercise 2.27. Explain how to naturally extend r_Y from $c^{(1)}$ to all of c.

After the exercise, we have a retraction $r_Y : \mathsf{C}\big(Y, B(X)\big) \to Y$, and we define the *canonical retraction* $r : \mathsf{C}\big(Y, X\big) \to Y$ to be the composition $\mathsf{C}\big(Y, X\big) \to \mathsf{C}\big(Y, B(X)\big) \xrightarrow{r_Y} Y$.

The map r is not in general combinatorial; for many applications it is important to worry about conditions under which it can be made combinatorial, but we're not going into enough depth to worry about it.

2.5.5. *QCERF in the hyperbolic case.*

Definition 2.28. A hyperbolic group G is *QCERF* if every quasiconvex subgroup of G is separable. □

Theorem 2.29. *Let X be a special cube complex based at a vertex x, let Y be a compact cube complex based at a vertex y, and let $f : (Y, y) \to (X, x)$ be a based local isometry. Then $\pi_1(X, x)$ is residually finite and $\mathrm{im}(f_*)$ is separable in $\pi_1(X, x)$ by virtue of being a virtual retract.*

Proof. Let $\mathsf{C}(Y,X) \to X$ be the canonical completion of f, so that f lifts to an embedding $\hat{f} : (Y,y) \to (\mathsf{C}(Y,X), \hat{f}(y))$.[6] Let $G' \leqslant_{fi} G$ be the finite index subgroup corresponding to the based cover $(\mathsf{C}(Y,X), \hat{f}(y)) \to (X,x)$. Let $r : \mathsf{C}(Y,X) \to \hat{f}(Y)$ be the canonical retraction. Then $r_* : G' \to \mathrm{im}(f_*)$ is the desired retraction map. In view of Exercise 1.15, this will show that $\mathrm{im}(f_*)$ is separable once we know that G is residually finite.

To prove residual finiteness, we use a special case of the above: let $\widetilde{Y}_0 \subset \widetilde{X}$ be any finite subset of the universal cover $\widetilde{X}$ of X. Let $\widetilde{Y}$ be the cubical convex hull of $\widetilde{Y}_0$.

Exercise 2.30. Show that $\widetilde{Y}$ is a compact subcomplex of $\widetilde{X}$.

Since $\widetilde{Y}$ is convex, the restriction of the covering map $\widetilde{X} \to X$ to $\widetilde{Y}$ is a local isometry $\widetilde{Y} \to X$, so we get an embedding of $\widetilde{Y}$ in a finite cover $\mathsf{C}(\widetilde{Y}, X)$ of X. This verifies residual finiteness, so we are done. □

Generalising Exercise 2.30 in the hyperbolic setting:

Proposition 2.31. *Let X be a compact nonpositively curved cube complex with $\pi_1(X,x)$ a hyperbolic group. Let H be a quasiconvex subgroup of $\pi_1(X,x)$. Then there is a* ***compact*** *nonpositively curved cube complex Y and a based local isometry $f : (Y,y) \to (X,x)$ such that $\mathrm{im}(f_*) = H$.*

Proof. It's in the exercises. □

Corollary 2.32. *Let G be a hyperbolic group which is virtually compact special. Then G is QCERF.*

Remark 2.33. If X is compact, then each immersed hyperplane h is compact. In particular, if X is special, then by applying canonical completion/retraction to $N(h) \to X$, we get separability of $\pi_1 h$ in $\pi_1 X$. In [HW08], Haglund-Wise showed that if X is compact and nonpositively curved, then it is virtually special provided the following holds for all hyperplanes $h, v \to X$:

- $\pi_1 h$ is separable in $\pi_1 X$, and
- the double coset $\pi_1 h \cdot \pi_1 v$ is separable in $\pi_1 X$ (dealing with basepoints properly).

When G is hyperbolic then QCERFness of G implies that double cosets of quasiconvex subgroups are separable [Git99, Min06], so that in particular, if $G = \pi_1 X$ with X a compact nonpositively curved cube complex, then G is virtually special provided G is QCERF. As we will see, this is always the case for G a hyperbolic cubical group.

This has a very important consequence: if X, Y are compact nonpositively curved cube complexes and $G = \pi_1 X \cong \pi_1 Y$ is hyperbolic, then X has a finite special cover if and only if Y does. □

Corollary 2.34 (Largeness)**.** *Let G be a hyperbolic group which is virtually compact special. Then either G is virtually cyclic or G has a finite-index subgroup surjecting onto F_2.*

Proof. If G is not virtually $\mathbb{Z}$, a standard ping-pong argument gives a quasiconvex free subgroup $F \leqslant G$, which by Theorem 2.29 is a virtual retract. (There's a more general formulation in the exercises.) □

[6] As mentioned earlier, replacing Y with suitable locally convex thickenings and using Theorem 1.18, this is already enough to show that $\mathrm{im}(f_*)$ is separable.

2.5.6. *Wall-injectivity and the magic component.* Let $f : Y \to X$ be a local isometry of connected nonpositively-curved cube complexes with X special and Y compact. Recall the canonical local isometries $b_X : X \to B(X), b_Y : Y \to B(Y)$ to Salvetti complexes, coming from specialness of X and Y. Although specialness of Y follows from the existence of a local isometry $Y \to X \to B(X)$, the Salvetti complex $B(Y)$ is different from $B(X)$, since the crossing graph of Y is different from that of X.

Remark 2.35 (Magic component of $\mathsf{C}(Y, Y)$)**.** Consider (yes, really) the canonical completion of the identity map $I : Y \to Y$; recall that by definition, this

$$\mathsf{C}(Y, Y) := Y \otimes_{B(Y)} \mathsf{C}(Y, B(Y)).$$

Let $p_Y : \mathsf{C}(Y, Y) \to Y$ and $p : \mathsf{C}(Y, Y) \to \mathsf{C}(Y, B(Y))$ be the projection maps, and recall that p_Y is a covering map. We also have a canonical inclusion $\iota : Y \to \mathsf{C}(Y, Y)$ lifting the one to $\mathsf{C}(Y, B(Y))$, and a canonical retraction $s : \mathsf{C}(Y, Y) \to \iota(Y)$.

What does $\mathsf{C}(Y, Y)$ actually look like? Well, $\mathsf{C}(Y, B(Y)) = Y \cup P$, where P is a union of cubes whose 0–skeletons lie in Y. The maps $b_Y : Y \to B(Y)$ and $\mathsf{C}(Y, B(Y)) \to B(Y)$ both send all the 0–cubes to the unique 0–cube of $B(Y)$. Hence $\mathsf{C}(Y, Y)^{(0)} = Y^{(0)} \times Y^{(0)}$.

The 0–cubes in $\mathsf{C}(Y, Y)$ of the form (y, y), $y \in Y$ span a connected subcomplex of $\mathsf{C}(Y, Y)$ which is exactly $\iota(Y)$, because a collection of such 0–cubes (y_i, y_i) span a cube exactly when the y_i span a cube in both Y and $Y \cup P$, which is to say, when they span a cube in Y.

Since the covering map $\mathsf{C}(Y, Y) \to Y$ restricts to a degree–1 cover on $\iota(Y)$, we see that $\iota(Y)$ is the entire "diagonal" component.

So $\mathsf{C}(Y, Y) = \iota(Y) \sqcup M$, where $M \to Y$ is a degree–$(|Y^{(0)}| - 1)$ cover of Y. (Because $\mathsf{C}(Y, Y)$ has $|Y^{(0)}||\mathsf{C}(Y, B(Y))^{(0)}| = |Y^{(0)}|^2$ vertices, of which $|Y^{(0)}|$ are in $\iota(Y)$. So $|M^{(0)}| = |Y^{(0)}|(|Y^{(0)}|) - 1)$.) M may have various components. □

Definition 2.36 (Wall-injective)**.** Any local isometry $f : Y \to X$ of special cube complexes takes hyperplanes to hyperplanes, and we say f is *wall-injective* if the resulting map on sets of hyperplanes is injective. □

Exercise 2.37. The canonical inclusion $Y \to \mathsf{C}(Y, X)$ is wall-injective.

Solution. Consider $Y \hookrightarrow \mathsf{C}(Y, B(X))$. Let $h, h' \subset Y$ be hyperplanes (immersed hyperplanes in Y are embedded since Y is special, because X is). Let H, H' be the hyperplanes of $\mathsf{C}(Y, B(X))$ to which h, h' map. Note that if H, H' are distinct, then so are their elevations to $\mathsf{C}(Y, X)$, so it suffices to prove that H, H' are distinct.

Let $r : \mathsf{C}(Y, B(X)) \to Y$ be the canonical retraction. If $\hat{f}$ is an edge of $\mathsf{C}(Y, B(X))$, mapping to an edge f of $B(X)$ (viewed as a generator of the RAAG), then $r(\hat{f})$ is a path also mapping to f, by construction.

Suppose $H = H'$ and let e be the oriented edge of $B(X)$ dual to the hyperplane $\bar{H}$ of $B(X)$ to which H maps. Let $\hat{e}, \hat{e}'$ be edges of Y dual to h, h' respectively. Choose a path α in $N(H)$ so that $\hat{e}^{-1}\alpha\hat{e}'$ is a path in $\mathsf{C}(Y, B(X))$ crossing H twice. Then $\hat{e}^{-1}r(\alpha)\hat{e}'$ is a path in Y, and the edges of $r(\alpha)$ all map to edges of $B(X)$ corresponding to generators that commute with e, since that was true in α. Thus $r(\alpha)$ belongs to the carrier of the hyperplane H. We have just shown that $H \cap Y$ is connected, so $h = h'$. □

Note that when $f : Y \to X$ is wall-injective, then the induced map on crossing graphs gives an injective combinatorial map $B(Y) \to B(X)$. This need not be a local isometry, because disjoint hyperplanes of Y could map to crossing hyperplanes of X.

Suppose that f is *injective and* wall-injective, so we can just think of Y as a wall-injective locally convex subcomplex of X. Letting $p_X : \mathsf{C}(Y, X) \to X$ be the projection (i.e. the cover), consider $p^{-1}(Y)$. By restricting, we have a covering map $p^{-1}(Y) \to Y$ and we also have a local isometry $p^{-1}(Y) \hookrightarrow \mathsf{C}(Y, X) \to \mathsf{C}(Y, B(X))$, so the universal property gives a covering map $p^{-1}(Y) \to Y \otimes_{B(X)} \mathsf{C}(Y, B(X))$. On the other hand, the identity on $\mathsf{C}(Y, B(X))$ and the inclusion $Y \to X$ and the universal property give an inverse for this map, so $p^{-1}(Y) \to Y \otimes_{B(X)} \mathsf{C}(Y, B(X))$ is an isomorphism. But if c is a cube of Y and d is a cube of $\mathsf{C}(Y, B(X))$ such that c, d have the same image in $B(X)$, then that image $\bar{c}$ is a cube of $B(Y) \subset B(X)$ since c is a cube of Y. By the construction of $\mathsf{C}(Y, B(X))$, it follows that d is a cube of $\mathsf{C}(Y, B(Y))$, so the cube $\hat{c}$ of $Y \otimes_{B(X)} \mathsf{C}(Y, B(X))$ mapping to $c \times d$ is naturally a cube of $Y \otimes_{B(Y)} \mathsf{C}(Y, B(Y)) = \mathsf{C}(Y, Y)$. Conversely, if c is a cube of $\mathsf{C}(Y, Y)$, mapping to $\bar{c}$ of Y and $\bar{d}$ of $\mathsf{C}(Y, B(Y)) \subset \mathsf{C}(Y, B(X))$, then c gives a cube of $\mathsf{C}(Y, X)$ mapping to Y. Hence:

Proposition 2.38. *Let $Y \to X$ be an injective, wall-injective local isometry with Y compact and X special. Then the preimage of Y in $\mathsf{C}(Y, X)$ is $\mathsf{C}(Y, Y) = Y \sqcup M$, where Y is the image of the canonical inclusion $Y \to \mathsf{C}(Y, X)$ and M is the union of the magic components of $\mathsf{C}(Y, Y)$.*

One can get a good feel for this thing by looking at the proof in [Wis00] that free groups are *omnipotent*; in that context, Y and X are graphs, so "wall-injective plus injective" means the same as "injective".

Example 2.39. At this point, there were some examples, but one had to be there (for the time being). □

3. Some applications

Here are two closely related results which are the main sources of specialness for hyperbolic groups in practice.

The first is Agol's virtual specialness theorem [Ago13], which says that one can verify virtual specialness of a hyperbolic group just by cocompactly cubulating[7] it:

Theorem 3.1. *Let G be a hyperbolic group acting properly and cocompactly on a CAT(0) cube complex $\widetilde{X}$. Then there is a finite-index $G' \leqslant G$ such that $G' \backslash \widetilde{X}$ is special; in particular, G is virtually compact special.*

The second is Wise's quasiconvex hierarchy theorem [Wis21, Wis09].

Definition 3.2. Let QVH denote the class of groups defined as follows:

- $\{1\} \in$ QVH.
- If $G' \in$ QVH and $[G : G'] < \infty$, then $G' \in$ QVH.
- If $G = A *_C B$ and $A, B \in$ QVH and C is finitely generated and QI-embedded in G, then $G \in$ QVH.
- If $G = A *_C$ and $A \in$ QVH and C is f.g. and QIE in G, then $G \in$ QVH. □

Theorem 3.3. *If $G \in$ QVH is hyperbolic, then it is virtually compact special.*

Remark 3.4. The converse also holds. □

Corollary 3.5. *Let G be a hyperbolic group acting freely and cocompactly on a CAT(0) cube complex, or having a quasiconvex virtual hierarchy. Then*

- *G is $\mathbb{Z}$–linear.*

[7] Usage varies, but for me, to "cubulate" a group G is to construct a properly discontinuous (but not necessarily metrically proper) action on a CAT(0) cube complex; to "cocompactly cubulate" G is to construct a proper cocompact action.

- *Every quasiconvex subgroup of G is separable.*
- *G is large (i.e. either elementary or virtually surjects to F_2).*

Example 3.6 (One-relator groups)**.** Let $\mathcal{A}$ be a finite alphabet and w be a reduced word in $\mathcal{A}$. A one-relator group has the form $G = \langle \mathcal{A} \mid w \rangle$.

(1) Baumslag conjecture, 1968: one-relator groups with torsion are residually finite.
(2) If $n \geqslant 2$, $\langle \mathcal{A} \mid w^n \rangle$ is hyperbolic, and the only torsion is the "obvious" torsion.
(3) One relator groups have a hierarchy terminating in finite groups (Magnus-Moldavanskii).
(4) Wise [Wis21]: if $n \geqslant 2$, then the Magnus-Moldavanskii hierarchy is quasiconvex, so G is virtually compact special.
(5) Linton [Lin22a, Lin22b]: more general conditions ensuring G has a quasiconvex hierarchy and is therefore virtually special when G is a hyperbolic one-relator group (not necessarily with torsion). □

Example 3.7 (Mapping tori)**.** Let F be a hyperbolic group and $\phi : F \to F$ an injective endomorphism. Let $G = \langle F, t \mid tft^{-1} = \phi(f),\ f \in F \rangle$.

(1) If F is free, then G is hyperbolic if and only if it has no Baumslag-Solitar subgroup [Bri00, BF92, Kap00, Mut20].
(2) If F is a closed surface group, then by Thurston, G is hyperbolic if and only if ϕ is pseudo-Anosov, and in this case, Dufour shows G is virtually compact special [Duf12]. (G is the fundamental group of a fibred hyperbolic 3–manifold in this case.)
(3) If F is free, and G is hyperbolic, and ϕ is *irreducible*, then G is virtually compact special [HW16]. Proof inspired by Dufour's technique and subsequently improved so that irreducibility is unnecessary if ϕ is an automorphism [HW15].
(4) Many hyperbolic one-relator groups are in the preceding category, *using* virtual specialness [KL23].
(5) Dahmani-Krishna MS-Mutanguha, very recently: if F is any hyperbolic group and ϕ an automorphism such that G is hyperbolic, then G is virtually compact special [DMM23]. Uses a generalisation of [HW16] by Dahmani-Krishna MS [DM22]; in the case where F is free, they recover a much simpler proof of [HW15]. □

Remark 3.8. The group

$$\Gamma = \langle a, b, t \mid a^t = ab, b^t = ba \rangle = \langle x, t \mid [[x, y], y]x \rangle$$

is virtually compact special, as an irreducible hyperbolic mapping torus. This shows that Γ is linear (using Theorem 3.1), so the answer to Sapir's mathoverflow question is negative. **Thing to check:** does the one-relator presentation satisfy the hypotheses of Linton's theorems in [Lin22b], giving an alternate proof of virtual specialness? □

Example 3.9 (3–manifolds)**.** Let M be a closed oriented hyperbolic 3–manifold. Then $\pi_1 M$ is virtually compact special: Kahn-Markovic [KM12] and Bergeron-Wise [BW12] provide the action on a cube complex needed to use Agol's theorem [Ago13]. This was the "virtually Haken" conjecture for hyperbolic 3–manifolds, since a consequence is that M has a finite cover containing an embedded π_1–injective surface. Using another Agol result [Ago08], Wise [Wis21] had previously showed that this implies M virtually fibres.

If M is a finite-volume hyperbolic 3–manifold with cusps (so $\pi_1 M$ is only relatively hyperbolic), then Wise proved independently that $\pi_1 M$ is virtually compact special [Wis21]. This gets virtual fibring for M independently.

Non-hyperbolic (graph manifold) and mixed cases covered by various results, like [PW14, PW18, Liu13, HP15, Tid18]. The classic thing is the examples due to Rubinstein-Wang of 3–dimensional graph manifolds M with immersed surfaces $S \to M$ that do not embed in any finite cover, i.e. $\pi_1 S$ fails to be separable in $\pi_1 M$. This sort of thing happens in other 3–manifolds M even when $\pi_1 M$ is virtually special — there are even examples where $\pi_1 M$ is a RAAG. The point is that $\pi_1 M$ is not hyperbolic, so the class of subgroups guaranteed by specialness to be separable — namely those that are represented by local isometries — is more restricted than in the hyperbolic case. Meanwhile, canonical completion *really needs* the domain of the local isometry to be compact. □

3.1. **How to cubulate.**

Definition 3.10. G a finitely generated group, $G = \Gamma^{(0)}$ with Γ the Cayley graph. A subgroup $H \leqslant G$ is *codimension*–1 if there exists $r < \infty$ such that $\Gamma - N_r(H)$ has at least 2 H–orbits of components C such that C contains points arbitrarily far from Γ. □

Theorem 3.11 (Sageev 1995)**.** *If G has a codimension–1 subgroup H, then G acts on a CAT(0) cube complex with no global fixed point and H is commensurable with the stabiliser of a hyperplane.*

Theorem 3.12 (Sageev 1997, Hruska-Wise 2009)**.** *If G is hyperbolic and $H_1, \ldots, H_k$ are quasiconvex codimension–1 subgroups, then G acts cocompactly on a CAT(0) cube complex whose hyperplane stabilisers are commensurable with $H_1, \ldots, H_k$.*

Theorem 3.13 (Bergeron-Wise 2009)**.** *Let G be hyperbolic. Suppose that for all distinct $p, q \in \partial G$ there is a quasiconvex codimension–1 subgroup $H \leqslant G$ such that p, q are in H–distinct components of $G - \partial G$. Then G acts properly and cocompactly on a CAT(0) cube complex.*

4. Some exercises and problems

Some of these are from the talks, plus a few extra. Some of them can be done in more generality than stated here (e.g. one can sometimes weaken compactness in the statements about special cube complexes). Please let me know (`markfhagen@posteo.net`) of errors, impossible exercises, etc. Most of these are aimed at someone who is new to cube complexes, has seen some "general geometric group theory", whatever that means, and is happy to read stuff/look stuff up/ask people things in order to do them.

(1) If G is finitely presented and residually finite, it has solvable word problem.
(2) If G is virtually linear (resp. $\mathbb{Z}$–linear) then it is linear (resp. $\mathbb{Z}$–linear).
(3) Prove "by hand" that $SL_n(\mathbb{Z})$ is residually finite.
(4) **Profinite topology.** Fix a group G. Let $\mathcal{N}$ be the set of all finite-index normal subgroups of G. For each $N \in \mathcal{N}$, let $q_N : G \to G/N$ be the natural quotient. Let $q : G \to \prod_{N \in \mathcal{N}} G/N$ be $q(g) = (q_N(g))_{N \in \mathcal{N}}$. Equip each G/N, $N \in \mathcal{N}$ with the discrete topology, and $\prod_{N \in \mathcal{N}} G/N$ with the product topology, and let $\mathcal{T}_G$ be the smallest topology on G making q continuous.
   (a) Check that $(G, \mathcal{T}_G)$ is a topological group, and that if H is some other group, then any homomorphism $\phi : (G, \mathcal{T}_G) \to (H, \mathcal{T}_H)$ is continuous.
   (b) Check that $\mathcal{T}_{G \times G}$ is the same as the product topology on $G \times G$ where each factor is topologised using $\mathcal{T}_G$.
   (c) Show that G is residually finite if and only if q is injective if and only if $\mathcal{T}_G$ is Hausdorff. Deduce that G is residually finite if and only if $\Delta = \{(g, g) : g \in G\}$ is closed in $G \times G$.
   (d) Show that $S \subset G$ is separable if and only if S is closed in $(G, \mathcal{T}_G)$.
   (e) Let $f : H \to G$ be a surjective homomorphism and suppose that G is not residually finite. Prove that $(f \times f)^{-1}(\Delta)$ is a non-separable subgroup of $H \times H$.

(f) Give a (quick) topological proof that, if $G' \leqslant_{fi} G$ and $H \leqslant G$, then H is separable if and only if $H \cap G'$ is separable in G'. And that if G is residually finite, then retracts of G are separable. (See Exercise 1.15 and Example 1.17.)

(5) **Salvetti complex.** Let Γ be a graph and let $X(\Gamma)$ be the Salvetti complex of the right-angled Artin group $A(\Gamma)$. Verify that $X(\Gamma)$ is nonpositively curved and $\pi_1 X(\Gamma) \cong A(\Gamma)$. Try to construct something analogous for right-angled Coxeter groups.

(6) **Salvetti hyperplanes.** Let Γ be a graph and let $X(\Gamma)$ be the Salvetti complex of the right-angled Artin group $A(\Gamma)$. Let $h \to X(\Gamma)$ be a hyperplane, and recall that h inherits a cubical structure whose cubes are midcubes in $X(\Gamma)$. Find a graph Ω with the property that h, equipped with this cubical structure, is isomorphic to $X(\Omega)$, and relate Ω to Γ.

(7) **Compact immersed hyperplanes.** Let X be a nonpositively-curved cube complex and let $h \to X$ be an immersed hyperplane. Check that h is compact provided X is. Slightly more complicated and more general: if G acts on the CAT(0) cube complex $\widetilde{X}$ cocompactly (but not necessarily properly), show that each hyperplane-stabiliser acts on its hyperplane cocompactly.

(8) **Torsion.** Let $\widetilde{X}$ be a CAT(0) cube complex. The goal of this exercise is to prove, without CAT(0) geometry, that finite groups acting on $\widetilde{X}$ fix points.

(a) Let F act on $\widetilde{X}$ by cubical automorphisms, and suppose that F is a finite group. Show that $\widetilde{X}$ contains an F–invariant subcomplex C such that C has finitely many cubes, and C is *convex* in the sense that the inclusion $C \to \widetilde{X}$ is a local isometry. (Hint: consider combinatorial halfspaces containing an F–orbit.)

(b) Show that C is itself a CAT(0) cube complex, and conclude that we can therefore assume that $\widetilde{X}$ is compact.

(c) In $\widetilde{X}$, what should it mean for a hyperplane to be "outermost"? Come up with a definition of "outermost hyperplane" that you can use to find an F–invariant convex subcomplex of $\widetilde{X}$ with strictly fewer hyperplanes. Now iterate until you find a fixed point.

(d) Conclude that if G is the fundamental group of a nonpositively-curved cube complex, then G is torsion-free.

(9) **Quasi-tree cube complexes.** Let G be a finitely generated group acting freely [resp. properly] and cocompactly on a CAT(0) cube complex $\widetilde{X}$. Suppose that every hyperplane of $\widetilde{X}$ is compact. Prove that G is free [resp. virtually].

(10) **Non-residually finite examples and self-crossing.** Let Y be a compact connected nonpositively curved cube complex, let S be a circle subdivided so that it is a graph, and suppose that $\gamma : S \to Y$ is a combinatorial local isometry with the following property (situating basepoints appropriately): the element $[\gamma] \in \pi_1 Y$ belongs to every finite-index subgroup of $\pi_1 Y$.

Such Y and γ exist, and the original examples (Burger-Mozes [BM97, Wis07], see [Cap19] for a survey) arise as quotients of the product of two trees by a free cocompact action.

(a) Why isn't $\pi_1 Y$ virtually special?

(b) If T_1, T_2 are trees, and $G \leqslant \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acts freely and cocompactly, and $Y = G \backslash T_1 \times T_2$, then which of the "hyperplane pathologies" can occur in Y?

(c) Construct a compact nonpositively-curved cube complex Z with an immersed hyperplane $h \to Z$ such that for all finite covers $\widehat{Z} \to Z$, if $\hat{h} \to h$ is any cover so that $\hat{h} \to h \to Z$ lifts to $\hat{h} \to \widehat{Z}$, then this lift is not an embedding.

(11) **Tree embeddings.** Let $\widetilde{X}$ be a CAT(0) cube complex.

(a) Let $\mathcal{H}$ be the set of all hyperplanes in $\widetilde{X}$. Let $\mathcal{H}_0 \subset \mathcal{H}$ be a set of disjoint hyperplanes. By considering components of $\widetilde{X} - \bigcup_{h \in \mathcal{H}_0} h$, produce a tree T and a quotient map $q : \widetilde{X} \to T$ such that the preimage of each vertex in T is a convex subcomplex and the preimages of edges are exactly carriers of hyperplanes in $\mathcal{H}_0$.

(b) Suppose that G is virtually compact special. Produce trees $T_1, \ldots, T_n$ and an action of G on $T_1 \times \cdots \times T_n$ by cubical automorphisms.

(c) Generalise both of the above in some way you find compelling.

(12) **Special QVH.** Let G be a hyperbolic group and suppose that $G = \pi_1 X$ where X is a compact special cube complex. Show that $G \in$ QVH.

(13) **Maps to cubes.** Let G act freely and with finitely many orbits of hyperplanes on the CAT(0) cube complex $\widetilde{X}$. Suppose that for each hyperplane h, the stabiliser G_h of h is separable in G.

(a) Show that for each $r \in \mathbb{N}$, there is a natural number N_r and a cubical map $f_r : \widetilde{X} \to [-\frac{1}{2}, \frac{1}{2}]^{N_r}$ such that if $x, y \in \widetilde{X}^{(0)}$ satisfy $\mathsf{d}_1(x,y) \leqslant r$, then $\mathsf{d}_1(f_r(x), f_r(y)) = \mathsf{d}_1(x,y)$.

(b) Show that there is a G–action on $[-\frac{1}{2}, \frac{1}{2}]^{N_r}$ with respect to which f_r is G–equivariant.

(c) How is this related to stuff like the abelianisation of a right-angled Coxeter group?

(14) **Surfaces.** Let S be a closed connected orientable surface of genus $g \geqslant 1$. (I don't know the answer to either of the following.)

(a) What is the minimum number of hyperplanes in a special 2–cube complex homeomorphic to S?

(b) What is the minimum number of hyperplanes in a special cube complex X with $\pi_1 X \cong \pi_1 S$?

(15) **Guided meditation on cubical quasiconvexity.** Let X be a compact special cube complex with $\pi_1 X$ hyperbolic. Suppose that $H \leqslant \pi_1 X$ is a finitely generated subgroup which is quasiconvex.

Let $\delta \in \mathbb{R}$ be such that the metric space $(\widetilde{X}, \mathsf{d}_1)$ is δ–hyperbolic (if this sort of thing is new to you, it might be helpful to look up an argument explaining why any Cayley graph of $\pi_1 X$ is quasi-isometric to $(\widetilde{X}, \mathsf{d}_1)$ — this doesn't involve cube complexes; it's the Milnor-Svarc lemma — and why hyperbolicity is preserved by quasi-isometries).

Let Y_0 be some H–orbit in $\widetilde{X}^{(0)}$, and fix κ so that Y_0 is κ–quasiconvex in $\widetilde{X}$. Let $\widetilde{Y}$ be the smallest subcomplex containing the intersection of all the halfspaces in $\widetilde{X}$ that contain Y_0.

(a) Show that $\widetilde{Y}$ is a convex, H–invariant subcomplex of $\widetilde{X}$, let $Y = H \backslash \widetilde{Y}$, and deduce that there is a local isometry $Y \to X$ inducing $H \hookrightarrow \pi_1 X$.

(b) Let $x \in \widetilde{X}$ and let $\pi(x)$ be the set of all $y \in Y_0$ such that $\mathsf{d}_1(x,y) \leqslant \mathsf{d}_1(x, Y_0) + 1$. Show that $\mathsf{diam}(\pi(x))$ can be bounded in terms of δ and κ. (*This is really a general fact about quasiconvex subsets of a hyperbolic space, nothing to do with cubes.*)

(c) Deduce that there is some $R < \infty$ such that any geodesic in $\widetilde{X}$ that starts at x and ends in Y_0 must intersect the R–neighbourhood of $\pi(x)$. (*This is also a general hyperbolicity/quasiconvexity thing.*)

(d) Deduce that there exists $R' < \infty$ such that for all $x \in \widetilde{X}$, either $\mathsf{d}_1(x, Y_0) \leqslant R'$, or there is a hyperplane separating x from Y_0.

(e) Conclude that Y is compact, i.e. H is represented by a compact local isometry $Y \to X$.

(f) Strengthen the argument to show that you can choose the compact Y and the local isometry $Y \to X$ representing H with the property that for all bi-infinite geodesics L of $\widetilde{X}$ that are contained in some neighbourhood of $\widetilde{Y}$, we actually have $L \subset \widetilde{Y}$.

(g) Look up what a *median algebra* is and see if you can find or cook up a version of the above arguments involving fewer inequalities.

(16) **Guided meditation on fibre products/height.** See also [Wis21, Lem. 8.9] for something similar to the following. Let X be a connected nonpositively-curved cube complex and let $x \in X$. Let Y, Z be connected nonpositively curved cube complexes and let $y \in Y, z \in Z$ be vertices. Suppose that we have base-point-preserving local isometries $f : (Y, y) \to (X, x)$ and $g : (Z, z) \to (X, x)$. Let $G = \pi_1(X, x)$ and let $H_Y, H_Z \leqslant G$ be the images of f_*, g_* (induced maps on fundamental group) respectively.

(a) Let $p_Y : (\widehat{X}_Y, \hat{x}_Y) \to (X, x)$ and $p_Z : (\widehat{X}_Z, \hat{x}_Z) \to (X, x)$ be based covers corresponding to H_Y, H_Z respectively. Note that f, g lift to injective local isometries $\hat{f} : (Y, y) \to (\widehat{X}_Y, \hat{x}_Y)$, $\hat{g} : (Z, z) \to (\widehat{X}_Z, \hat{x}_Z)$.

(b) Let C be the component of $\widehat{X}_Y \otimes_X \widehat{X}_Z$ (fibre product defined using the maps p_Y, p_Z) containing $(\hat{x}_Y, \hat{x}_Z)$ (check that such a component exists). Let D be the component of $Y \otimes_X Z$ containing (y, z) (check that this component exists, too, where the fibre product is with respect to the maps f, g). Produce a based local isometry $\iota : (D, (y, z)) \to (C, (\hat{x}_Y, \hat{x}_Z))$ such that the image of

$$\pi_1(D, (y, z)) \xrightarrow{\iota_*} \pi_1(C, (\hat{x}_Y, \hat{x}_Z)) \xrightarrow{p_Y \otimes p_Z} \pi_1(X, x)$$

is contained in $H_Y \cap H_Z$, where $p_Y \otimes p_Z : \widehat{X}_Y \otimes_X \widehat{X}_Z \to X$ is the composition of p_Y (resp. p_Z) with the projection from the fibre product to $\widehat{X}_Y$ (resp. $\widehat{X}_Z$).

(c) Use the definition of fibre products to show that the image of $\pi_1(C, (\hat{x}_Y, \hat{x}_Z)) \xrightarrow{p_Y \otimes p_Z} \pi_1(X, x)$ is all of $H_Z \cap H_Y$.
(Note: one might want to pick a based loop $\gamma \to X$ representing an arbitrary element of $H_Z \cap H_Y$, lift it to $\widehat{X}_Y$ and $\widehat{X}_Z$, and use the universal property to find a lift to the fibre product, but that's not quite right, since $\gamma \to X$ probably isn't a local isometry. What to do instead?)

(d) Show that ι_* is π_1–surjective.
Now you've shown that the component D of $Y \otimes_X Z$ containing (y, z) corresponds to the subgroup $H_Z \cap H_Y$ of $\pi_1(X, x)$, in the sense that the map $\pi_1(D, (y, z)) \to \pi_1(X, x)$ is injective and has image $H_Z \cap H_Y$.)

(e) Check that your construction of ι actually gives a map $\iota : Y \otimes_X Z \to \widehat{X}_Y \otimes_X \widehat{X}_Z$. Letting y vary in $f^{-1}(x)$ and z vary in $g^{-1}(x)$, deduce the following. Given $\alpha, \beta \in \pi_1 X$ such that $H_Y^\alpha \cap H_Z^\beta \neq \{1\}$, the following are equivalent:
- there is a based component (D, d) of $Y \otimes_X Z$ such that the map $Y \otimes_X Z \to X$ restricts to a based map $(D, d) \to (X, x)$ such that the image of the induced homomorphism on fundamental group is $H_Y^\alpha \cap H_Z^\beta$, and
- the component C of $\widehat{X}_Y \otimes_X \widehat{X}_Z$ corresponding to $H_Y^\alpha \cap H_Z^\beta$ has nonempty intersection with the image of ι.

(f) Let C and ι be as in the previous item. How does Wise's argument cited above use *superconvexity* to ensure that C has nonempty intersection with the image of ι, so that $H_Y^\alpha \cap H_Z^\beta$ arises from a component of $Y \otimes_X Z$?

(g) Deduce that if $\pi_1 X$ is hyperbolic, and $H \leqslant \pi_1 X$ is a quasiconvex subgroup, then there exists n such that the intersection of any $n+1$ distinct conjugates of H in

$\pi_1 X$ is trivial. (Previous exercise is useful here; this statement doesn't actually need cube complexes, but this is an exercise!)

(h) Find a (non-hyperbolic) compact nonpositively-curved cube complex X and a compact connected Y and a local isometry $Y \to X$ such that $\widehat{X}_Y \otimes_X \widehat{X}_Y$ has (lots of) non-simply-connected components disjoint from $Y \otimes_X Y$.

(17) **Products.** Let A and B be CAT(0) cube complexes and let $\widetilde{X} = A \times B$. Let G act freely and cocompactly on $\widetilde{X}$, preserving the product decomposition, and K_A and K_B be the kernels of the maps $G \to \mathrm{Aut}(A)$ and $G \to \mathrm{Aut}(B)$ given by these actions.

(a) For each $a \in A, b \in B$, let $\widetilde{Y}_a = \{a\} \times B$ and $\widetilde{Z}_b = A \times \{b\}$. Note that $\widetilde{Y}_a, \widetilde{Z}_b$ are convex subcomplexes of $\widetilde{X}$; check this if unsure. Convince yourself that the stabilisers $\mathrm{Stab}(\widetilde{Y}_a), \mathrm{Stab}(\widetilde{Z}_b)$ act cocompactly on $\widetilde{Y}_a, \widetilde{Z}_b$ respectively, and that they respectively contain K_A and K_B.

(b) Let $H = \langle K_A, K_B \rangle \leqslant G$. Show that H is the internal direct product of K_A and K_B.

(c) Given $b, b' \in B^{(0)}$, explain why $\mathrm{Stab}(\widetilde{Z}_b) \cap \mathrm{Stab}(\widetilde{Z}_{b'})$ has finite index in $\mathrm{Stab}(\widetilde{Z}_b)$; denote this index $I^B(b, b')$. This gives a function $I^B : B^2 \to \mathbb{N}$, and we similarly have $I^A : A^2 \to \mathbb{N}$.

(d) Show that I^B is bounded on B^2 if and only if K_B has finite index in $\mathrm{Stab}(\widetilde{Z}_b)$ for all $b \in B$.

(e) Deduce that I^B and I^A are both bounded if and only if H acts cocompactly on $\widetilde{X}$, i.e. if and only if H has finite index in G.

(f) Suppose that there is a finite-index subgroup $G' \leqslant G$ such that $G' \backslash \widetilde{X}$ is special. Show that I^A and I^B are bounded, and in particular G virtually splits as a direct product.

(18) **Surfaces in stuff.** Let Γ be a 5–cycle with vertices $v_0, \ldots, v_4$ and v_i, v_{i+1} adjacent for $i \in \mathbb{Z}/5$. Find a nonpositively-curved square complex Y that is homeomorphic to a closed hyperbolic surface and admits a local isometry $Y \to X(\Gamma)$.

(19) **Largeness.** Let X be a compact special cube complex.

(a) Using [KMT17] or [Hag22] or otherwise, show that either $\pi_1 X$ is virtually abelian, or there is a compact connected nonpositively-curved cube complex Y such that $\pi_1 Y \cong F_2$, and a local isometry $Y \to X$.

(b) Using the above, deduce that if G is a virtually compact special group, then either G is virtually abelian or there is a finite-index $G' \leqslant G$ with a surjection $G' \to F_2$.

(20) **Arranging free actions.** Let G be a torsion-free hyperbolic group and suppose that for each $g \in G$, there is a CAT(0) cube complex X_g such that G acts cocompactly on X_g, there is one G–orbit of hyperplanes h, and $\mathrm{Stab}_G(h)$ is quasiconvex in G and, moreover, $\langle g \rangle$ acts on X_g with unbounded orbits. Does G admit a free action on a CAT(0) cube complex? It G virtually compact special?

5. References and further reading

Summary of the references mentioned in the lectures, and some further reading:

5.1. RAAGs and RACGs.

- Koberda's notes on RAAGs: [Kob]
- Charney's survey on RAAGs: [Cha07]
- Davis's book on Coxeter groups: [Dav08]
- Cashen's notes on Coxeter groups: [Cas]

- Embedding RAAGs in RACGs: [DJ00]
- Linearity of RAAGs and some more graph products: [HW99, Gre90, Hum94]

5.2. **Residual properties.**

- Mal'cev's result on residual finiteness of linear groups: [Mal40]
- Marshall Hall's theorem on free groups: [Hal49].
- Non-linear residually finite 1–relator groups, Druţu and Sapir: [DS05]
- Non-linear mapping tori: [DS04]
- Borisov and Sapir, about (non)linearity/residual finiteness of mapping tori of endomorphisms: [BS05]
- Scott's paper on separability in surface groups: [Sco78]
- Mihailova, $F_2 \times F_2$ is not subgroup separable: [Mih66]
- Baumslag-Solitar groups, non-residual finiteness: [Mes72]

5.3. **Special cube complexes and canonical completion.**

- Stallings' paper on graph immersions and covers etc., prefiguring special cube complex viewpoint in dimension 1: [Sta83]
- Wise's expository book: [Wis12]
- Sageev's lecture notes: [Sag14]
- Haglund and Wise, special cube complexes: [HW08]
- CAT(0) cube complexes, medians and metrics: [Che00, Mie14]
- Hyperplane theorem: [Sag95]
- Haglund papers with many useful cube complex statements: [Hag23, Hag08]
- Early application of 1–dimensional canonical completion and retraction: [Wis00]
- Sample canonical completion application, with some explanation of the Haglund-Wise canonical completion construction: [BRHP15]
- Wise's monograph, Section 6+7 deals with special cube complexes: [Wis21]
- Work of Minasyan and Gitik on double coset separability: [Min06, Git99]
- Some non-special examples (non-exhaustive): [She22, Cap19, Wis07, BM97].

5.4. **Cubulating and specialising; applications.** The following is very incomplete, both when it comes to examples and when it comes to applications:

- Agol's theorem on virtual specialness: [Ago13] and simplified account of the proof due to Shepherd: [She21]
- Wise's quasiconvex hierarchy theorem, 3–manifold, one-relator group applications: [Wis21]
- Sageev's construction: [Sag95, Nic04a, Nic04b]
- Bergeron-Wise, boundary cubulation criterion: [BW12]
- Dufour, cubulating fibred hyperbolic manifolds: [Duf12]
- Kahn-Markovic, immersed walls in hyperbolic 3–manifolds: [KM12]
- Hagen-Wise, cubulating hyperbolic mapping tori: [HW16, HW15]
- Some non-hyperbolic 3–manifold results: [PW14, PW18, Liu13, HP15, Tid18]
- Cubulating more general hyperbolic mapping tori: [DM22, DMM23]
- Linton and Kielak-Linton, one-relator and free-by-cyclic applications: [Lin22b, Lin22a, KL23]
- Useful modern survey/reference on 3–manifolds, Aschenbrenner-Friedl-Wilton: [AFW15]
- Agol's virtual fibring criterion: [Ago08]
- Giralt's cubulation of Gromov-Thurston manifolds: [Gir17]

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