A REMARK ON THICKNESS OF FREE-BY-CYCLIC GROUPS

MARK HAGEN

Abstract. Let F be a free group of positive, finite rank and let $\Phi \in \mathrm{Aut}(F)$ be a polynomial-growth automorphism. Then $F \rtimes_\Phi \mathbb{Z}$ is *strongly thick* of order η, where η is the rate of polynomial growth of ϕ. This fact is implicit in work of Macura [Mac02], but [Mac02] predates the notion of thickness. Therefore, in this note, we make the relationship between polynomial growth of and thickness explicit. Our result combines with a result independently due to Dahmani-Li, Gautero-Lustig, and Ghosh to show that free-by-cyclic groups admit relatively hyperbolic structures with thick peripheral subgroups.

1. Definitions, statement, discussion

There has been significant interest in the geometry of mapping tori of polynomial-growth automorphisms of finite-rank free groups (see e.g. [But15, BK16, Mac02, BFH05]). There is also a considerable literature on hyperbolicity, relative hyperbolicity, and acylindrical hyperbolicity of mapping tori of general automorphisms of free groups. For example, $F \rtimes_\Phi \mathbb{Z}$ is word-hyperbolic exactly when $\Phi \in \mathrm{Aut}(F)$ is atoroidal [Bri00, BF92], and recent work of Dahmani-Li [DL19] and Ghosh [Gho18] characterises nontrivial relative hyperbolicity of $F \rtimes_\Phi \mathbb{Z}$: it is equivalent to exponential growth of Φ. Even in the polynomial-growth case, where nontrivial relative hyperbolicity is impossible (by combining [Mac02, Theorem 7.2] and [Sis12, Theorem 1.3]), recent results show that virtual acylindrical hyperbolicity holds provided Φ has infinite order [Gho18, BK16]. In this note, we show that when Φ has polynomial growth, $F \rtimes_\Phi \mathbb{Z}$ is non-relatively hyperbolic in a strong way: $F \rtimes_\Phi \mathbb{Z}$ is *thick* in the sense of [BDM09].

There is a general question of which classes $\mathcal{C}$ of groups have the property that each $G \in \mathcal{C}$ is either relatively hyperbolic or thick, and, more strongly, which $\mathcal{C}$ have the property that each $G \in \mathcal{C}$ exhibits a (possibly trivial) relatively hyperbolic structure in which the peripheral subgroups are thick. This property is interesting because such a relatively hyperbolic structure is quasi-isometry invariant and "minimal": each peripheral subgroup is peripheral in any relatively hyperbolic structure on G, by [BDM09, Corollary 4.7] or [DS05, Theorem 1.7].

Classes of groups that have (possibly trivial) relatively hyperbolic structures with thick peripherals include Coxeter groups [BHS17], fundamental groups of "mixed" 3–manifolds (consider the geometric decomposition and apply [BDM09, Theorem 1.2] to the graph manifold pieces), and Artin groups (combine [BDM09, Lemma 10.3] with [CP14, Theorem 1.2]).

Our main result combines with a theorem established independently by Dahmani-Li, Gautero-Lustig, and Ghosh to yield:

Corollary 1.1 (Relatively hyperbolicity with thick peripherals)**.** *Let F be a free group of finite positive rank, let $\Phi \in \mathrm{Aut}(F)$, and let $G = F \rtimes_\Phi \mathbb{Z}$. Then either G is thick, or G is hyperbolic relative to a finite collection of proper subgroups, each of which is thick.*

Proof. If Φ is polynomially growing or of finite order, then G is thick by Theorem 1.2. Otherwise, Φ is exponentially growing. Theorem 3.9 of [DL19] implies that G is hyperbolic relative to a finite collection $\mathcal{P}'$ of peripheral subgroups, each of which is the mapping torus of a

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polynomial-growth free group automorphism and therefore thick by Theorem 1.2. (One can also use Corollary 3.12 of [Gho18] in conjunction with [BDM09, Theorem 4.8, Remark 7.2] in place of [DL19, Theorem 3.9].) □

Combining Corollary 1.1 with [BDM09, Theorem 4.8] shows that if G' is a group quasi-isometric to a free-by-$\mathbb{Z}$ group, then G' is hyperbolic relative to thick subgroups.

We now turn to our main theorem. Fix a finite-rank free group F. Given $\phi \in \mathrm{Out}(F)$, by a *lift* Φ of ϕ we mean an automorphism $\Phi : F \to F$ whose outer class is ϕ. Fix a free basis $\mathcal{S}$ for F. Recall that the *growth function* $\mathrm{GR}_{\Phi,\mathcal{S}} : \mathbb{N} \to \mathbb{N}$ is defined by $\mathrm{GR}_{\Phi,\mathcal{S}}(n) = \max_{s\in\mathcal{S}} \|\Phi^n(s)\|$, where $\|g\|$ denotes word length. Recall also that the asymptotic behaviour of $\mathrm{GR}_{\Phi,\mathcal{S}}$ is independent of the choice of generating set, and that the growth function is either exponential or polynomial of degree $\eta \leq |\mathcal{S}|$. In the latter case, we say Φ (and its outer class ϕ) are *polynomially growing* and refer to η as the *polynomial growth rate*.

We now recall the notion of a *thick* group, which was introduced in [BDM09] as both an obstruction to the existence of a nontrivial relatively hyperbolic structure and a "structural" version of the property of having a polynomial divergence function. The definition of thickness is inductive, and, if G is a thick group, there is an associated invariant $n \geq 0$, the *order of thickness*. The reader is referred to [BDM09] and [BD14] for a more detailed discussion of the several closely-related notions of thickness. Here, we just restate the facts about thickness needed for most of our discussion; see [BD14, Section 4].

- A finitely generated group G is *strongly thick of order* 0 if no asymptotic cone of G has a cut-point. For example, if $G \cong A \times B$, where A, B are infinite groups, then G is strongly thick of order 0.
- Let G split as a finite graph of groups where the edge groups are infinite and the vertex groups are thick of order n. Suppose, moreover, that the vertex groups are *quasi-convex*, in the sense that there exist constants C, L so that for each vertex group A, any two points in A can be connected by an (L, L)–quasigeodesic in $\mathcal{N}_C(A)$. Then G is strongly thick of order $\leq n + 1$. (This is Proposition 4.4 in [BD14].)

We will need the full definition of strong thickness in the case $n = 1$, in the proof of Lemma 2.1, so we give the definition in that proof. Our main theorem is:

Theorem 1.2. *Let F be a free group of finite rank at least* 1*. Let $\phi \in \mathrm{Out}(F)$ be polynomially-growing, with polynomial growth rate $\eta \geq 0$, and let $\Phi \in \mathrm{Aut}(F)$ be a lift of ϕ. Then $F \rtimes_\Phi \mathbb{Z}$ is strongly thick of order η.*

If a group G is thick of order n, then the divergence function of G (see [DMS10, BD14, Ger94a, Ger94b]) is bounded above by a polynomial of degree $n + 1$, although lower bounds are more difficult to establish in general (see e.g. [DT15, BHS17, Lev18, Mac02]). In [Mac02], Macura gave upper and lower bounds on the divergence function of $F \rtimes_\Phi \mathbb{Z}$, both polynomial of degree $\eta + 1$. Macura's result uses the decomposition of $F \rtimes_\Phi \mathbb{Z}$ as graph of groups with $\mathbb{Z}$ edge groups coming from a relative train track representative for Φ, and implies that η distinguishes quasi-isometry types of mapping tori of polynomial-growth automorphisms.

To an extent, thickness is implicit in Macura's argument, but her work predates the formal definition by several years. Our proof of Theorem 1.2 follows Macura's strategy, relying on the same splitting coming from a relative train track representative.

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2. Proof of Theorem 1.2

Throughout, we adopt the notation from Theorem 1.2.

Proof. Adopt the notation of the statement; in particular, Φ has polynomial growth rate of order η. Since the order of strong thickness is a quasi-isometry invariant (see [BDM09, Remark 7.2] and [BD14, Definition 4.13]), and the polynomial growth rate of any positive power of Φ coincides with that of Φ, it suffices to prove the theorem for $G = F \rtimes_{\Phi^k} \mathbb{Z}$, for any $k > 0$. By [BFH00, Theorem 5.1.5], we can choose $k > 0$ with the property that Φ^k admits an *improved relative train track representative.* In particular, there exists a finite connected graph Γ, with $\pi_1\Gamma$ identified with F, and a cellular map $f : \Gamma \to \Gamma$, inducing the map Φ^k on $\pi_1\Gamma$, so that the following hold:

(A) There is a filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$, with each Γ_i an f–invariant subgraph. Each vertex is fixed by f.
(B) For $1 \leq i \leq n$, the graph Γ_i is obtained from Γ_{i-1} by adding an (oriented) edge e_i.
(C) For each $i \geq 1$, we have $f(e_i) = e_i p_i$, where p_i is a closed edge-path whose edges belong to Γ_{i-1}.

We can take each p_i to be immersed. The latter two properties on the above list rely on the fact that our automorphism has polynomial growth rate.

An edge e_i is *invariant* if $f(e_i) = e_i$, i.e. if p_i is trivial.

The graphs Γ_i need not be connected when $i < n$. More precisely, if e_i is non-invariant, then e_i necessarily shares a vertex with Γ_{i-1}. However, if e_i is invariant, then e_i can be disjoint from Γ_{i-1}. Hence, let Λ_i index the set of components $\Gamma_i^\alpha, \alpha \in \Lambda_i$, of Γ_i.

Note that since $\Gamma_n = \Gamma$ is connected, Γ_{n-1} has at most two components.

The base case: Observe that the edge e_1 is necessarily invariant, since $\Gamma_0 = \emptyset$. Suppose Γ_1 has a single vertex. Then $\pi_1\Gamma_1 \cong \mathbb{Z}$, i.e. the mapping torus of $f|_{\Gamma_1}$ is a torus. On the other hand, if Γ_1 has two vertices, then e_2 is invariant, because $p_1 \to \Gamma_1$ is immersed and therefore trivial. Continuing in this way, we eventually find that there exists $i_0 \geq 1$ so that, for some $\alpha \in \Lambda_{i_0}$, the component $\Gamma_{i_0}^\alpha$ is non-simply-connected and each edge of $\Gamma_{i_0}^\alpha$ is invariant.

Writing $G = \langle \pi_1\Gamma, t \mid \{tft^{-1} = \Phi^k(f) : f \in \pi_1\Gamma\}\rangle$, we thus have a subgroup $G_0 \cong \pi_1\Gamma_{i_0}^\alpha \times \langle t\rangle$ in G. Since $\pi_1\Gamma_{i_0}^\alpha \neq \{1\}$, the group G_0 decomposes as the direct product of two infinite groups, so G_0 is strongly thick of order 0. Hence, if $i_0 = n$, then on one hand, Φ has polynomial growth of order 0, and on the other hand, $G_0 = \pi_1\Gamma_{i_0} \times \langle t\rangle$ is thick of order 0, as required.

The iterated splitting: Suppose that $n > i_0$. By construction, $\Gamma_n = \Gamma_{n-1} \cup e_n$. Since Γ_{n-1} is f–invariant and $f(e_n) = e_n p_n$, we have an associated splitting of G_n as a graph of groups with the following properties:

- The underlying graph is a single edge.
- The vertex groups have the form $\pi_1\Gamma_{n-1}^\alpha \rtimes_{\Phi^k} \mathbb{Z}$. Note that $|\Lambda_{n-1}|$ is 1 or 2 according to whether the edge e separates Γ_n. (Recall that $\Gamma_n = \Gamma$ is connected, and the open edge e has 1 or 2 complementary components.)

  Moreover, at most one component of Γ_{n-1} is simply connected. (If not, $\pi_1\Gamma_n \cong \{1\}$, contradicting that $\mathrm{rk}(F) \geq 1$.)
- The edge-groups are conjugate to $\langle t\rangle$.

Viewed as a graph of spaces, the mapping torus M_n of f has vertex spaces which are mapping tori of the restriction of f to components of Γ_{n-1}. We are attaching a cylinder as follows. First, in M_{n-1}, every edge not belonging to Γ_{n-1} is a *horizontal* edge that joins some $v \in \Gamma_{n-1}$ to itself and, viewed as a loop in M_n, represents a conjugate of t. Our cylinder is attached on one side along a horizontal edge. On the other side, it is attached along a path of the form $p_n t_n$, where p_n is as above, and t_n is a horizontal edge.

(This splitting is discussed in detail in [Mac02, Section 3], where it is called the *topmost edge decomposition*. The only difference is that, for the moment, we are just removing the single edge e_n, rather than many edges, as Macura does in producing the topmost edge decomposition.)

For each component Γ^α_{n-1} in Λ_{n-1}, there is an induced filtration of Γ^α_{n-1} so that Γ^α_{n-1} and the restriction of f to Γ^α_{n-1} satisfy properties (A),(B),(C) above. By induction on the number of edges, either the vertex group $\pi_1\Gamma^\alpha_{n-1} \rtimes_{\Phi^k} \mathbb{Z}$ is thick of order at most $n-1-i_0$, or, if Γ^α_{n-1} is simply connected, then $\pi_1\Gamma^\alpha_{n-1} \rtimes_{\Phi^k} \mathbb{Z}$ is isomorphic to an incident edge group. There is at least one Γ^α_{n-1} so that the former holds.

We now check that the vertex group is quasi-convex in the sense of [BD14], described above. To this end, note that the map $F \rtimes_\Phi \langle t\rangle \to \langle t\rangle$ is a coarsely Lipschitz retraction to $\langle t\rangle$, so that, regarding a Cayley graph of G as a tree of spaces associated to the above splitting, we have that each edge-space is a coarsely Lipschitz retract. Fix a vertex space V, and fix $x, y \in V$. Let γ be a geodesic of G joining x, y. Then either γ lies in V, or we can write $\gamma = \alpha_0\beta_1\alpha_1 \cdots \beta_k\alpha_k$, where each α_i lies in V and each β_i starts and ends in some edge space E_i incident to V. Replacing each β_i by its projection to E_i gives a path in V that joins x to y and has length bounded by a linear function of $\mathsf{d}_G(x, y)$. Hence $V \hookrightarrow G$ is a quasi-isometric embedding, so any geodesic in V (which is a connected graph) from x to y maps to a (uniform-quality) quasigeodesic of G that lies in V.

Hence, by the inductive hypothesis and [BD14, Proposition 4.4], G_n is strongly thick of order $\tau_n \leq n - i_0$. This completes the proof that G_n is thick and thus proves Corollary 1.1; we now bound the order of thickness independently of the relative train track representative.

Upper bound on order of thickness: We now analyse related splittings of G to bound the order of thickness τ_n in terms of the polynomial growth rate η_n. For $n = i_0$, we saw that $\tau_n = \eta_n = 0$, and we are done.

Suppose $n > i_0$. Recall that each edge e_b has an associated *polynomial growth rate* [Mac02, Definition 2.11]. Let d_i be the polynomial growth rate of the edge e_i. At this point, we will also apply Proposition 2.7 from [Mac02] in order to assume that f is a *Kolchin map*. The exact definition is not important, but this assumption will enable us to use facts from [Mac02] about growth rates of edges.

There are two cases. First, suppose that $\eta_n \geq 2$. Then by Lemma 2.16 of [Mac02], there is a nonempty set $\mathcal{E}$ containing exactly the edges e_i with $d_i = \eta_n$. By the same lemma, each $e_i \in \mathcal{E}$ maps over some edge $e_{j(i)}$ with $d_{j(i)} = \eta_n - 1$, and conversely any edge mapping over some edge of growth rate $\eta_n - 1$ belongs to $\mathcal{E}$. An edge e_i is *doomed* if either $e_i \in \mathcal{E}$, or the following holds: $d_i < \eta_n$, and the largest connected subgraph of Γ that contains e_i and consists of edges not in $\mathcal{E}$ is simply-connected. A vertex is doomed if each of its incident edges is doomed.

Observe that if e is a doomed edge not in $\mathcal{E}$, then e is invariant. Indeed, the image of e has the form ep, where p is an immersed path consisting of edges in the subgraph of Γ described above, which is simply-connected since e is doomed. Thus p is trivial, i.e. e is invariant.

Let Γ' be obtained from Γ by removing each doomed vertex and removing the interior of each doomed edge. Then each component of Γ' is f–invariant (note that its constituent edges need not be invariant). Indeed, let e_i be an edge of Γ'. Then $d_i \leq \eta_n - 1$, so by Lemma 2.16 of [Mac02], the path p_i cannot traverse any edge in $\mathcal{E}$. Now, suppose that p_i has a subpath q lying in a connected subgraph C that is maximal with the property that none of its edges is in $\mathcal{E}$. Write $p_i = aqb$. Then since a, b cannot contain edges in $\mathcal{E}$, maximality of C implies that a, b lie in C, so p_i is an immersed closed path in C. Hence C is not simply-connected, so its edges are not doomed. Thus p_i, and hence $f(e_i) = e_ip_i$, lies in Γ'.

Thus, removing the edges of $\mathcal{E}$ induces a splitting of G as a graph of groups whose edge groups are infinite cyclic and whose vertex groups are (necessarily quasi-convex) subgroups which are either: (a) mapping tori of the restriction of Φ^k to subgroups on which Φ^k has growth rate at most $\eta_n - 1$; or (b) conjugate to edge groups. Hence, by induction and [BD14, Proposition 4.4], we have that G is thick of order at most $\eta_n - 1 + \tau_1$: here, τ_1 is the maximal order of thickness of $\pi_1\Lambda \rtimes_{\Phi^k} \mathbb{Z}$, where Λ is a connected, non-simply connected f–invariant subgraph of Γ consisting entirely of edges whose polynomial growth rates are at most 1.

Hence it remains to consider the case where $\eta_n = 1$. Here the situation is somewhat more complicated because [Mac02, Lemma 2.16] does not apply: linearly-growing edges can map over other linearly-growing edges. This case is handled in Lemma 2.1, which shows that $\eta_n = 1$ in the linearly growing case. So, $\tau_1 = 1$ and G is thick of order at most η.

Lower bound on order of thickness: By [Mac02, Theorem 8.1], the divergence function of G is polynomial of degree at least $\eta + 1$. On the other hand, if G is strongly thick of order τ, then by [BD14, Corollary 4.17], G has divergence function that is polynomial of degree at most $\tau + 1$. This gives a contradiction unless $\tau \geq \eta$. We thus conclude that G is strongly thick of order η.

More precisely, we have the following: given $x \in G$ and $r \geq 0$, and $y, z \in G$ with $\mathsf{d}_G(x, y), \mathsf{d}_G(x, z) \leq r$, let $\mu_x(y, z)$ be the infimum of $|P|$, where P varies over all paths in G from y to z that avoid the ball of radius $r/2$ about x. Let $\chi_x(r)$ be the supremum of $\mu_x(y, z)$ over all such y, z, and let $\chi(r)$ be the supremum of $\chi_x(r)$ over all $x \in G$. Theorem 8.1 of [Mac02] shows that $\chi(r)$ is bounded below by a polynomial of degree $\eta + 1$.

On the other hand, applying Theorem 4.9 of [BD14] inductively, exactly as in the proof of [BD14, Corollary 4.17], shows that $\chi(r)$ is bounded above by a polynomial of degree $\tau + 1$. The only difference between our situation and that in [BD14] is the base case. Specifically, in order to apply [Mac02, Theorem 8.2], we defined $\chi(r)$ using paths that avoid balls of radius $r/2$. When applying [BD14, Theorem 4.9], we are taking advantage of the fact that the constant δ in that statement can be any element of $(0, 1)$; we are using the case $\delta = \frac{1}{2}$.

In the base case, we cannot rely on [BD14, Proposition 4.12], as is done in the proof of [BD14, Corollary 4.17], because that statement requires $\delta \in (0, \frac{1}{54})$. Instead, we use a simple special case of [BD14, Proposition 4.12]. Specifically, we need to show that there is a fixed linear function ρ so that for each component Λ of Γ_{i_0}, the mapping torus M_Λ of the restriction of f to Γ_{i_0} satisfies $\chi(r) \leq \rho(r)$, where χ is now defined in $\pi_1 M_\Lambda$. But this is clear since $M_\Lambda = \Lambda \times \mathbb{S}^1$, and we can take $\rho(r) = 4r$. □

It remains to prove thickness of order 1 in the linear growth case:

Lemma 2.1. *Suppose that the automorphism ϕ has linear growth. Then G is strongly thick of order at most* 1.

Proof. We will induct on the number of edges in the filtration of a graph Γ associated to a relative train track representative of ϕ. The goal is to construct a *tight network* $\mathcal{W}$ of *uniformly wide* subspaces of G; as in [BD14], this is sufficient to prove the claim. (The definitions of *tight network* and *uniformly wide* are given below.)

The relative train track representative: Exactly as in the proof of Theorem 1.2, we can assume that ϕ is represented by a Kolchin map $f : \Gamma \to \Gamma$ with Γ equipped with a filtration $\Gamma_0 \subset \cdots \subseteq \Gamma_n = \Gamma$ exactly as before.

We are going to use the following properties of f:

(i) For each i, we have $f(e_i) = e_i p_i$ where p_i is either trivial (i.e. e_i is an invariant edge) or p_i is an immersed closed (possibly trivial) *Nielsen path*, i.e. the tightening of $f(p_i)$ is p_i. This occurs when e_i is a linearly-growing edge by [BFH00].

(ii) If e_i is an invariant edge, and e_j is an edge with $j < i$, then we can reverse the order of e_i, e_j in the filtration, because p_i cannot map over e_j, since e_i is invariant. Hence we can assume that $d_n = 1$, where e_n is the topmost edge in the filtration.

Sub-mapping tori: Let M_n be the mapping torus of f. For each $i \leq n$, and each component Γ_i^α of Γ_i, we have that Γ_i^α is f–invariant, and we take M_i^α to be the mapping torus of the restriction of f to Γ_i^α. Note that $M_i^\alpha \subset M_j^{\alpha'}$, for some α', whenever $i < j$.

Collapsing cyclic sub-mapping tori to circles: Let $i_0 \leq n$ be maximal such that each component $\Gamma_{i_0}^\alpha$ of Γ_{i_0} consists of f–invariant edges. So, property (ii) guarantees that $d_i = 1$ if and only if $i > i_0$. We may assume that $n > i_0$, for otherwise G is thick of order 0, and we are done.

We now do some collapsing; this step isn't necessary, but makes later parts of the proof easier to picture. We would like to assume that every component $\Gamma_{i_0}^\alpha$ consisting of invariant edges is a core graph, i.e. has no valence–1 vertex. To this end, collapse each free face of $\Gamma_{i_0}^\alpha$, yielding a new graph $\bar{\Gamma}_n$ which is a deformation retract of Γ_n. The map f descends to a homotopy equivalence $\bar{f} : \bar{\Gamma}_n \to \bar{\Gamma}_n$ inducing the map ϕ on fundamental group, because we collapsed invariant edges. There is an obvious filtration of $\bar{\Gamma}_n$ induced by the filtration of Γ_n, and it is easily verified that $\bar{f}$ satisfies properties (i),(ii) above.

So, we can assume that the components $\Gamma_{i_0}^\alpha$ of Γ_{i_0} are either single points or non-simply connected core graphs; as in the proof of Theorem 1.2, there exists at least one component of the latter type.

The sub-network $\mathcal{W}_0$: Let $\rho : \widetilde{M}_n \to M_n$ be the universal cover. Let $\mathcal{W}_0$ be the set of components of $\rho^{-1}(M_{i_0}^\alpha)$, where $\alpha \in \Lambda_{i_0}$ is such that the component $\Gamma_{i_0}^\alpha$ of Γ_{i_0} is non-simply-connected. So, each $W \in \mathcal{W}_0$ is isometric to the product of $\mathbb{R}$ with one of finitely many trees.

The sub-network $\mathcal{W}_1$ of tori: For each $i_0 < j \leq n$, consider the immersed closed Nielsen path $p_j \to \Gamma_{j-1}$. Let v_j be the initial vertex of p_j and let t_j be the (unique) edge of M_{j-1} joining v_j to itself to produce a loop representing a conjugate of $\langle t \rangle$ in $\pi_1 M_n$. Situating the basepoint of M_n at v_j, we see that the elements of $\pi_1 M_n$ represented by p_j and t_j commute, so we have a torus T_j and a π_1–injective cellular map $T_j \to M_n$ so that the image of the induced map $\pi_1 T_j \to \pi_1(M_n, v_j)$ is the $\mathbb{Z}^2$ subgroup generated by t_j and p_j. We can choose T_j to lie in some component M_{j-1}^α of M_{j-1}, namely the component containing v_j, and that the paths $t_j \to M_n$ and $p_j \to M_n$ factor through $T_j \to M_n$.

Remark 2.2 (A T_j example)**.** Here is an example of a torus of the type just described. Let $F = \langle e_0, e_1, e_2, e_3 \mid \rangle$ and let f be defined by: $f(e_0) = e_0, f(e_1) = e_1 e_0, f(e_2) = e_2 e_0, f(e_3) = e_3 e_0 e_1 e_2^{-1}$. So,

$$G = \langle e_0, e_1, e_2, e_3, t \mid t e_0 t^{-1} = e_0, t e_1 t^{-1} = e_1 e_0, t e_2 t^{-1} = e_2 e_0, t e_3 t^{-1} = e_3 e_0 e_1 e_2^{-1} \rangle.$$

The tori T_1, T_2 correspond to the subgroup $\langle t, e_0 \rangle$. The torus T_3 corresponds to the subgroup $\langle t, e_0 e_1 e_2^{-1} \rangle$. Each of these is a $\mathbb{Z}^2$ subgroup. For example, note that $t e_0 e_1 e_2^{-1} t^{-1} = f(e_0 e_1 e_2^{-1}) = e_0 e_1 e_0 e_0^{-1} e_2^{-1} = e_0 e_1 e_2^{-1}$.

Consider the topmost edge splitting obtained by removing e_3. This is an HNN extension with vertex group $\langle e_0, e_1, e_2, t \rangle$ and stable letter e_3; we have $e_3^{-1} t e_3 = e_0 e_1 e_2^{-1} t$. Note that $\langle e_0 e_1 e_2^{-1} t \rangle$ is contained in the subgroup $\langle t, e_0 e_1 e_2^{-1} \rangle$ corresponding to T_3. This phenomenon will be important later. This concludes the example, and we resume the proof.

The candidate network: Let $\mathcal{W}_1$ be the set of components of $\rho^{-1}(T_j)$ for $i_0 < j \leq n$. Let $\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1$ and note that the elements of $\mathcal{W}$ coarsely cover $\widetilde{M}$.

Quasiconvexity of the elements of $\mathcal{W}$: Arguing as in the proof of Theorem 1.2 shows that $\widetilde{M}_i^\alpha$ is (uniformly) quasiconvex in $\widetilde{M}_{i+1}^{\alpha'}$ whenever $M_i^\alpha \subset M_{i+1}^{\alpha'}$, so each element of $\mathcal{W}_0$ is (r, r)–quasiconvex in $\widetilde{M}_n$ for some fixed constant r. (Here quasiconvexity is in the sense of [BD14].)

Let $\widetilde{T}_j \in \mathcal{W}_1$. Then $\widetilde{T}_j$ uniformly coarsely coincides with the orbit of an $\mathbb{Z}^2$ subgroup of $\pi_1 M^\alpha_{j-1}$, which is a free-by-cyclic group. By Corollary 6.9 of [But17], abelian subgroups of free-by-$\mathbb{Z}$ groups are undistorted, so any two points in $\widetilde{T}_j$ can be joined by a uniform-quality discrete quasigeodesic of $\pi_1 M^\alpha_{j-1}$ (hence a quasigeodesic of $\pi_1 M_n$) that lies in $\widetilde{T}_j$. Since there are only finitely many orbits of subspaces in $\mathcal{W}$, we conclude that there exists s such that each $W \in \mathcal{W}$ is (s,s)–quasiconvex in $\widetilde{M}_n$ (quasiconvexity in the sense of [BD14]).

$\mathcal{W}$ is uniformly wide: $\mathcal{W}$ contains finitely many isometry types of spaces, each of which is quasi-isometric to $F \times \mathbb{Z}$ for some finitely generated free group F. So, any ultralimit of rescaled spaces in $\mathcal{W}$ has no cut-point, i.e. $\mathcal{W}$ is *uniformly wide* in the sense of [BD14, Definition 4.11].

Induction: Fix $i \geq i_0$ and $r \geq 0$. By induction on the number of edges in Γ_i, there exists $\ell(i-1,r)$ such that the following hold for each M^α_{i-1} which is not a circle, i.e. where Γ^α_{i-1} is not a point (the first property is the defining property of a tight network, and the second property will be needed to make the induction work):

(1) Let $W, W' \in \mathcal{W}$ be contained in $\widetilde{M}^\alpha_{i-1}$. Suppose that, for some $x \in \widetilde{M}^\alpha_{i-1}$, each of W, W' intersects $\mathcal{N}_r(x)$. Then there is a sequence $W = W_1, \ldots, W_{\ell(i-1,r)} = W'$ such that each $W_t \in \mathcal{W}$, each W_t lies in $\widetilde{M}^\alpha_{i-1}$, and for all $t \leq \ell(i-1,r)-1$, the spaces W_t, W_{t+1} have unbounded, coarsely connected coarse intersection.

(2) Let $\rho : \widetilde{M}^\alpha_{i-1} \to M^\alpha_{i-1}$ be the universal covering map. Let M^β_i be the component of M_i containing M^α_{i-1}. Suppose that $e_i \subset M^\beta_i$. Consider the graph of spaces decomposition of M^β_i induced by removing e_i from Γ^β_i (i.e. the topmost edge decomposition). Then the edge space is a circle intersecting Γ^i_β in the midpoint of e_i. Let $E \subset \widetilde{M}^\alpha_{i-1}$ be a component of the ρ–preimage of this circle. Then E is uniformly coarsely contained in some $W \in \mathcal{W}$ that lies in $\widetilde{M}^\alpha_{i-1}$.

In the base case, where $i = i_0$, the former statement holds since $\widetilde{M}^\alpha_{i_0} \in \mathcal{W}$.

Fix M^β_i, the mapping torus of the restriction to f of some non-simply connected Γ^β_i. We will verify that the same properties hold for $\widetilde{M}^\beta_i$. First, as before, removing the edge e_i from Γ^β_i decomposes $\widetilde{M}^\beta_i$ as a tree $\mathcal{T}$ of spaces whose vertex spaces are various translates of various $\widetilde{M}^\alpha_{i-1}$ and whose edge spaces are two-ended.

Consider a splitting of M^γ_{i+1} that arises from the deletion of e_{i+1} from Γ_{i+1} and has M^β_i as a vertex space. The incoming edge space is attached via a circle in M^β_i homotopic into the ρ–image of some element of $\mathcal{W}$ contained in $\widetilde{M}^\beta_i$.

Indeed, let C be such an attaching circle in M^β_i. There are two cases. First, we could have $C = p_{i+1}t_{i+1}$, where t_{i+1} is an edge of M^β_i that forms a loop representing a conjugate of t, and p_{i+1} is the Nielsen path such that $f(e_{i+1}) = e_{i+1}p_{i+1}$. In this case, the torus T_{i+1} contains t_{i+1} and the image of p_{i+1}, by construction.

Otherwise, C traverses a single horizontal edge t' in M^β_i joining some vertex $v \in M^\beta_i$ to itself. If v is contained in some nontrivial $\Gamma^\delta_{i_0} \subset M^\beta_i$, then t' is contained in $M^\delta_{i_0}$, and we are done, because then $\widetilde{M}^\delta_{i_0} \in \mathcal{W}$. Otherwise, every edge of Γ^β_i containing v has the form e_j for some $j \leq i$ (here we have used that nontrivial components of Γ_{i_0} are core graphs). Applying property (2) inductively, $\widetilde{C}$ lies at Hausdorff distance at most 1 from a line that is coarsely contained in some element of $\mathcal{W}$ lying in $\widetilde{M}^\beta_i$. This verifies property (2) for M^β_i. (The constant implicit in "coarsely contained" has increased, but this happens at most n times.)

Now we verify property (1). Let $x \in \widetilde{M}^\beta_i$ lie at distance at most r from $W, W' \in \mathcal{W}$. Let v, v' be vertices of $\mathcal{T}$ so that the corresponding vertex spaces V, V' have unbounded intersection with W, W' respectively. Let γ be a geodesic of $\mathcal{T}$ from v to v', so that $|\gamma| \leq 2r$. For each valence-2 vertex of γ whose corresponding vertex space is the mapping torus of the restriction of f to a non-simply-connected graph, let $A, B \in \mathcal{W}$ lie in the associated vertex space and respectively coarsely contain the incoming and outgoing edge spaces along γ. By the inductive hypothesis, A, B can be joined by a sequence of at most $\ell(i-1, Kr)$ elements of $\mathcal{W}$ that lie in the given vertex space, where Kr is the distance in the vertex space from A to B (the constant K depends only on s). Consecutive elements of the sequence have unbounded, coarsely connected coarse intersection.

Similarly, if v is the initial vertex of γ, then we can choose $B \in \mathcal{W}$ that lies in the associated vertex space, contains the initial edge space, and can be joined to W by a sequence of at most $\ell(i-1, Kr)$ elements of $\mathcal{W}$, with the same intersection properties. The same holds for the terminal vertex of γ, with W' replacing W. Hence property (1) holds, with $\ell(i, r) = \ell(i-1, Kr)(2r+1)$.

Conclusion: We have shown that $\mathcal{W}$ is a uniformly wide *tight network* in $\widetilde{M_n}$, so, according to [BD14, Definition 4.13], $\widetilde{M_n}$, and hence G, is strongly thick of order at most 1. (Indeed, to prove that $\mathcal{W}$ is a tight network, it sufficed to exhibit the constant $\ell(n, 3s)$.) $\square$

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School of Mathematics, University of Bristol, Bristol, United Kingdom
E-mail address: `markfhagen@posteo.net`