REAL CUBINGS AND ASYMPTOTIC CONES OF HIERARCHICALLY HYPERBOLIC GROUPS

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Abstract. We first introduce the class of $\mathbb{R}$–*cubings*, which specialise the class of complete geodesic median metric spaces of finite rank in roughly the same way that hierarchically hyperbolic spaces specialise coarse median spaces.

The first of our main results says that, if $\mathcal{X}$ is a hierarchically hyperbolic space, then any asymptotic cone of $\mathcal{X}$ is bilipschitz equivalent to an $\mathbb{R}$–cubing. This generalises the fact that asymptotic cones of hyperbolic groups are $\mathbb{R}$–trees, and also generalises and strengthens a result of Behrstock-Druţu-Sapir on asymptotic cones of mapping class groups. This makes essential use of a result of Bowditch about medians on asymptotic cones of coarse median spaces, as well as Fioravanti's work on *measured halfspaces*. We then introduce the notion of a *universal* $\mathbb{R}$–*cubing*, which is a homogeneous $\mathbb{R}$–cubing whose structure is completely determined by the *local* $\mathbb{R}$–*cubing* structure at any point. We show that asymptotic cones of G are bilipschitz equivalent to universal $\mathbb{R}$–cubings.

This reduces the problem of studying asymptotic cones of G to that of understanding the local structure. Under algebraic conditions on a hierarchically hyperbolic group G — satisfied by the motivating examples of mapping class groups of surfaces, fundamental groups of compact special cube complexes, and hyperbolic groups —, we prove that (up to bilipschitz equivalence), the asymptotic cones of G are independent of the ultrafilter and rescaling sequence.

We also include a (mostly) self-contained exposition of hierarchically hyperbolic groups, and establish various properties of real cubings beyond what is needed for the application to asymptotic cones, with an eye to future applications.

Note: *(Sub)sections marked ☺ are of independent interest but do not contain material needed in our application to asymptotic cones.*

Contents

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1. Introduction

Background. In 1872, Klein proposed group theory as a means of formulating and understanding geometrical constructions. Geometric group theory embraces this view and also reverses it by using geometrical ideas to give new insights into central problems in group theory. More precisely, a guiding line in geometric group theory is to deduce properties of a group via the topological and geometric properties of the spaces on which it acts.

A classical example of this philosophy is Bass-Serre Theory — the theory of groups acting on simplicial trees. In his foundational book *Trees*, [Ser80], Serre proved that a group acts on a simplicial tree (without a global fixed point or inversion of edges) if and only if the group splits as a graph of groups (with vertex groups conjugates of the vertex stabilisers). As an easy consequence, one obtains that a group acts freely on a simplicial tree if and only if it is free. Bass-Serre theory quickly became a standard tool of geometric group theory and geometric topology, particularly in the study of 3-manifolds.

The theory of groups acting on simplicial trees developed in three important directions — one, by considering more general classes of trees leading to the theory of groups acting on *real trees*; two, to groups whose geometry is coarsely like a tree, including the influential theory of hyperbolic groups; and finally to the theory of groups acting on higher-dimensional generalisations of simplicial trees, namely CAT(0) cube complexes.

In his famous paper [Gro87], Gromov noticed that many results of Dehn concerning the fundamental group of a hyperbolic surface do not rely either on it having dimension two or even on being a manifold and hold in much more general context for groups with a geometry which is "coarsely like a tree". More precisely, Gromov defined hyperbolic groups as finitely generated (fg, for short) groups with a geometric property abstracting and coarsifying a basic property of trees: trees are discrete geodesic metric spaces where each geodesic triangle is a tripod; considering geodesic metric spaces where triangles are "coarse tripods", namely δ-thin triangles, one arrives to the notion of a hyperbolic space. Groups whose Cayley graphs are hyperbolic are precisely hyperbolic groups. Equivalently, hyperbolic groups are precisely fg groups that have a geometric (proper discontinuous and cobounded) action on a proper hyperbolic space.

Originally, the work of Serre on groups acting on simplicial trees had the aim of describing the structure of subgroups of the group $SL_2(F)$, where F is a field with an integer valuation. Considering the case when F is a field with a real valuation, one arrives to a natural generalisation of simplicial trees - the notion of a *real tree*. In fact, real trees were formally introduced in the mid 1970s by Chiswell and Tits in this context. So if one removes the requirement of the metric on the tree to be discrete and allows for a real metric one obtains the notion of a real tree - a geodesic metric space in which geodesic triangles are tripods. The theory of real trees came into prominence with the work of Morgan and Shalen [MS84], who viewed real trees as "degeneration" of hyperbolic spaces and established connections between the theory of real trees, hyperbolic geometry and Thurston's theory of measured laminations. Namely, they showed that if G is the fundamental group of a closed acyclic n-manifold, $n > 2$, then the space of discrete, faithful representations of G (as conjugacy classes of orientation-preserving isometries of the hyperbolic space) has a compactification, in which the ideal points are obtained from certain actions of G on real trees, hence generalising Thurston's compactification of the Teichmüller space.

The concept that formalises the "degeneration" of a space, that is, the space looked from "infinitely far away", is that of an asymptotic cone. The idea of asymptotic cone was first used by Gromov in his proof of the polynomial growth theorem, formalised by van den Dries and Wilkie in 1984 [vdDW84] and has proven to be extremely fruitful in analysing the structure of a group via its action on the cone as well as to study the large scale geometry of the group.

In this language, hyperbolic groups are characterised by their asymptotic cones - a group is hyperbolic if and only if it is fg and all of its asymptotic cones are bilipschitz equivalent to the universal real tree.

Once real trees are introduced and related to "limiting" actions of hyperbolic groups, it is natural to study the structure of groups acting on them. In the summer of 1991, in a series of lectures, Rips exposited a breakthrough method (based on the Makanin-Razborov process) for analysing the structure of groups acting on real trees. Rips' ideas were formalised by Gaboriau-Levitt-Paulin and further developed by Bestvina-Feighn, see [BF95], who proved structural results for groups acting stably on real trees, thereby generalising Bass-Serre theory. This way, the theory of groups acting on real trees grew into a rich theory which deeply impacted Group Theory in many different ways: by providing tools to attack new problems, by simplifying proofs of classical results and by establishing new connections between group theory and geometry, topology, dynamical systems and model theory, see [Bes02]. More precisely, Rips' theory was applied to the study of compactifications of spaces of geometric structures and was a key tool in the understanding of hyperbolic groups: their boundary, automorphisms, the isomorphism problem and their model theory, see [Bow98, Sel95, RS94, Sel09].

As already noticed by Gromov in [Gro87], the class of hyperbolic groups has its limitations, since many important families of groups, such as isometry groups of manifolds and singular spaces with non-positive curvature, are not hyperbolic, and the class of hyperbolic groups is not closed under basic group-theoretic operations, such as the direct product. Hence, from the very beginning, the need for developing a more general theory was apparent.

As simplicial trees have a special role in geometric group theory, it is only natural to look at higher dimensional generalisations. If trees are contractible CW-complexes built from 1-cubes (edges), one can consider similar complexes built from n-cubes, called CAT(0) cube complexes (or *cubings*).

The theory of groups acting on cubings has been developed since the 1980s. Intuitively, one expects that if a group acts on a tree in such a way that it captures its geometry, then the group should be relation-free, that is a free group. In the same way, if a group acts nicely on the direct product of trees — one of the simplest possible example of CAT(0) cube complex — one would like to understand which type of action captures the structure of the direct product of the space and reflects it onto the group. Somewhat surprisingly, as shown by examples of Wise and Burger-Mozes, even free co-compact actions on the simplest possible cubings fail one's intuition of a good action. In [BM97], the authors construct a series of finitely presented simple groups acting freely and co-compactly on the direct product of regular trees; in [Wis07], similar examples are constructed which are not, for instance, residually finite. Hence, the structure of such groups can be extremely far away from (subgroups) of direct products of free groups.

Haglund and Wise [HW08] described a further property of a good action on a CAT(0) cube complex, called specialness, which removes the "unexpected" examples and transfers the geometry of the space back to the group. In this context, Haglund and Wise show that a fg group acts freely, co-specially and co-compactly on the direct product of two trees if and only if it is virtually the direct product of two free groups. More generally, a fg group acts freely and co-specially on a CAT(0) cube complex if and only if it is a subgroup of a right-angled Artin group (RAAG, for short).

In this vein, we have that just as free groups (and their subgroups) are precisely the groups acting freely on trees, RAAGs and their subgroups are the groups acting freely and co-specially on CAT(0) cube complexes.

The theory of special cube complexes has since become central in group theory, particularly in the resolution of the virtual Haken and virtual fibering conjectures about 3–manifolds [Ago13, Wis21].

Having established CAT(0) cube complexes as the reference for higher-dimensional trees, it is natural to ask what the corresponding generalisations of hyperbolic spaces and real trees are.

As in the case of trees, one would like to relax the definition of a CAT(0) cube complex, both in terms of allowing coarseness and not restricting the metric to be simplicial, to obtain a family of spaces so that the groups with that geometry enjoy good algebraic properties.

CAT(0) cube complexes can be charecterised via their one-skeleton: a graph is median if and only if it is the one skeleton of a CAT(0) cube complex [Che00]. From this point of view, a coarsification of a median graph leads to the notion of coarse median space/group and the analogue of a real tree is simply a median space.

Coarse median spaces seem to have been introduced (by Bowditch in [Bow13]) as a means of extracting the essence of questions related to quasi-isometric rigidity of mapping class groups and Teichmüller space, but they have become objects of geometric and group-theoretic interest in their own right, featuring strong geometric results like Bowditch's quasiflats theorem [Bow19] and Fioravanti's very recent work on coarse-median preserving automorphisms [Fio21].

Median spaces (of which $\mathbb{R}$–trees, CAT(0) cube complexes, and simplicial trees are examples), have a very rich structure and arise naturally in geometry and group theory. Their ubiquity partially derives from the fact that they arise naturally as "duals" of collections of bipartitions of some underlying set or space. More precisely, there is an equivalence of categories between discrete spaces with walls and CAT(0) cube complexes [HP98, Nic04, CN05, Rol16, Sag95]. Discrete spaces with walls were generalized by Cherix, Martin and Valette in [CMV04] to spaces with measured walls. It turns out that in some sense the category of spaces with measured walls is equivalent to the category of general median spaces, see [CDH10, Fio20, Fio19]) and this connection was key to relate properties (T) and Haagerup (a-T-menability) to actions on median spaces and on spaces with measured walls. This has inspired much recent work, see e.g. [Fio20, Fio19, Fio18, CD17], on group actions on median spaces.

Another motivation for the study of median spaces comes from the work of Bowditch, who shows that any asymptotic cone of a coarse median space is bilipschitz equivalent to a median space [Bow18b, Bow13]. In the context of mapping class groups, median structures on asymptotic cones were also studied in [BDS11b, BDS11a].

Although there is a variety of structural results for finite rank (coarse) median spaces and groups, this class is too wide to admit a robust structural theory of groups acting on them, see [CD17]. Notice that for instance, the metric space $L^1(X,\mu)$ is median for any measured space (X,B,μ).

This indicates that more restrictive notions for (coarse) median were necessary for developing a good theory of group actions on them.

A more controlled way of "coarsifying" a CAT(0) cube complex leads to the notion of a hierarchically hyperbolic space (HHS for short).

Roughly speaking an HHS is a coarse median space that has sufficiently many coarsely lipschitz projections to hyperbolic spaces to recover the geometry of the space from these projections. For example, in a CAT(0) cube complex, one can always cone off certain convex subspaces to obtain a graph that is hyperbolic (in fact quasi-isometric to a tree) [Hag14, BHS17b, Gen19]. In reasonable circumstances (e.g. when the cube complex is the universal cover of a compact special cube complex), these convex subspaces have less complexity than the original CAT(0) cube complex, so recursively doing this process finitely many times, one obtains a family of hyperbolic spaces that collectively capture the coarse geometry of the original cube complex.

Similarly, if S is a finite-type hyperbolic surface then, fixing a word-metric on the mapping class group $MCG(S)$, one obtains a hyperbolic space by coning off left cosets of multicurve stabilisers [MM99] and again one can repeat this process for each multicurve stabiliser and this process will stop in finitely many steps giving a family of hyperbolic spaces. The key point is that the metric on the MCG(S) can be approximated by the metric on the family of hyperbolic spaces (via the coarse projections) [MM00, Beh04, BKMM12].

This is the motivation behind *hierarchically hyperbolic spaces* (introduced in [BHS17b, BHS19] and generalised via a slightly different set of axioms in in [Bow18a]).

The main goals of this paper are to introduce and study *real cubings* and to relate them to *asymptotic cones* of HHSes and prove that hierarchically hyperbolic groups with suitable algebraic properties have unique asymptotic cones (up to bilipschitz equivalence). We next turn to these two main themes: real cubings and asymptotic cones.

Real cubings. The classes of spaces discussed thus far are in Figure 1. At the intersection of all of the classes of spaces in the picture, we have simplicial trees. On the left are shown the various "coarse" ways to generalise trees that we have discussed: hyperbolic spaces and coarse median spaces, with hierarchically hyperbolic spaces sitting in between. CAT(0) cube

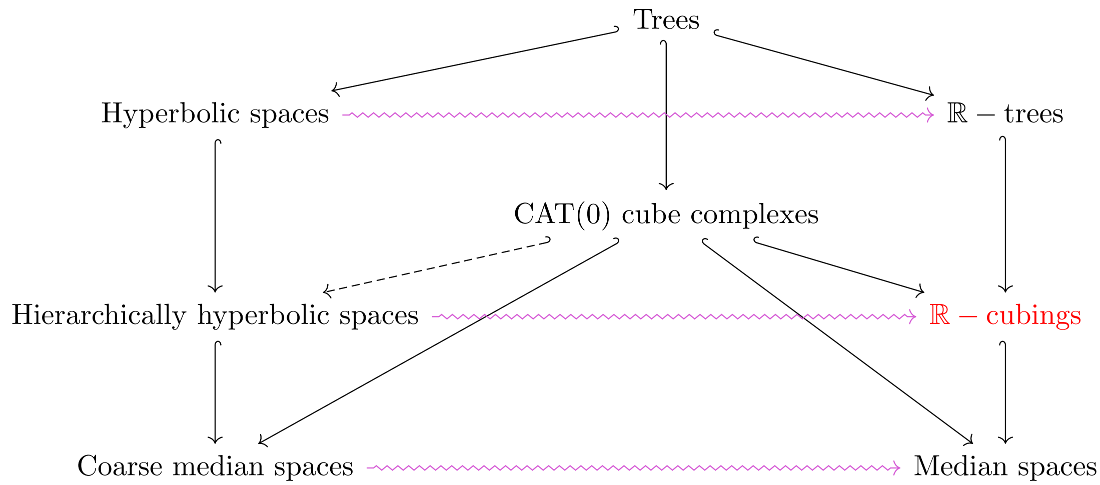


Figure 1. Some classes of spaces generalising trees. Inclusions of classes are shown. The dashed arrow is meant to communicate that CAT(0) cube complexes are not all hierarchically hyperbolic spaces (see [HS20]), but most examples considered in this paper are. The wavy horizontal arrows indicate passage to asymptotic cones (up to possibly modifying the metric on the cone in its bilipschitz equivalence class). Not shown are many other important classes of "nonpositively curved" spaces, graphs, and groups: Helly graphs and coarsely Helly spaces, injective spaces, CAT(0) spaces, systolic complexes, Morse local-to-global spaces, spaces with quasicubical intervals, quasimedian graphs, etc.

complexes also fit into this picture (we regard them as basically "fine-geometric" objects whose coarse geometry is of great interest due to results like those in [BKS16, Hua17, Hua18, HK18]). On the right, we have the "fine-geometric" generalisations, ℝ–trees and median spaces.

The question is: what is a useful notion of a ℝ–*cubing*[1]? What class of spaces includes both ℝ–trees and CAT(0) cube complexes, but is less general than the class of (finite-rank, geodesic) median spaces? There are several things we want from such a class of spaces.

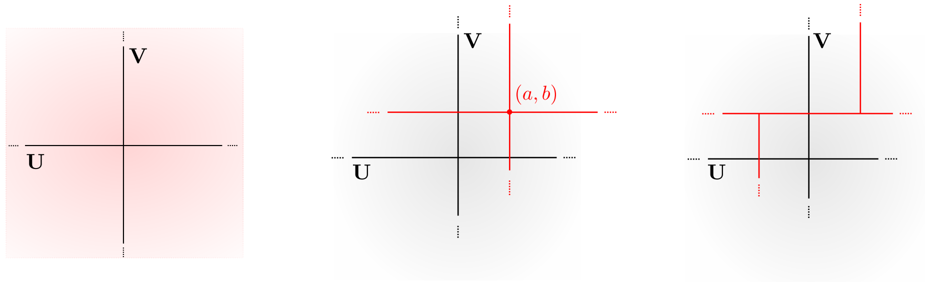
Firstly, we want the class of ℝ–cubings to have a more controlled structure and sit between ℝ–trees and median spaces in the same way that hierarchically hyperbolic spaces sit between hyperbolic spaces and coarse median spaces. So, this suggests that an ℝ–cubing should be a median space, equipped with many projections to ℝ–trees, in such a way that the space isometrically and median-preservingly embeds in the product of these projections, which are real trees. We introduce the following

Definition 1.1 (See Definition 7.5)**.** Fix a set $\mathfrak{F}$, for each $\mathbf{U} \in \mathfrak{F}$, let $I(\mathbf{U})$ be a connected subset of $\mathbb{R}$, and let $\ell_1(\mathfrak{F})$ be the subset of $\prod_{\mathbf{U}} I(\mathbf{U})$ consisting of countably-supported ℓ_1–functions $f : \mathfrak{F} \to \mathbb{R}$ with $f(\mathbf{U}) \in I(\mathbf{U})$.

A *real cubing* is a connected median subalgebra of $\ell_1(\mathfrak{F})$ of finite rank, finite *depth* and with projections into each $I(\mathbf{U}) \times I(\mathbf{V})$ of one of the following forms:

□

[1]I.e. a "ℝ–CAT(0) cube complex". Some authors call CAT(0) cube complexes "cubings" and we adopt that name here since "real cubing" rolls off the tongue nicely



$\mathbf{U} \perp \mathbf{V}$ $\mathbf{U} \pitchfork \mathbf{V}$ $\mathbf{U} \subsetneq \mathbf{V}$

The rank can defined as a bound on the cardinality of subsets of $\mathfrak{F}$ in which any two elements are related as in the first picture. If the projection in a plane is of the third type, we say that $\mathbf{U} \subsetneq \mathbf{V}$, and this turns out to partially order $\mathfrak{F}$. The finite depth asks for a finite bound on the length of chains in this partial ordering.

The definition can be formulated in such a way that it more closely resembles the definition of a hierarchically hyperbolic space from [BHS19]; see Definition 4.2. The condition of the type of projection into the different planes can be reformulated in terms of solutions of equations reminiscent of the Behrstock inequalities for the mapping class group. The above description is easier to state; the reformulated version is convenient for working with real cubing structures on asymptotic cones of hierarchically hyperbolic spaces.

We also characterise real cubings inside the class of median spaces. More precisely, we introduce the notion of a *poset-colouring* of the set of walls in a median space and show that median spaces always admit a canonical poset-colouring. This poset-colouring has a partial order which defines the *depth* of the median. In these terms, we show that real cubings are precisely complete, connected median spaces of finite rank which admit a (tangible) poset-colouring of finite depth. We also show that the finite depth of the canonical poset-colouring is equivalent to the median space having a real cubing structure with some additional useful properties (*wedges* and *clean containers*). Poset-colourings are introduced in Section 3.1. The characterisation of median spaces that are real cubings in terms of poset-colourings is in Section 5 and the stronger properties of wedges and clean containers are addressed in Section 6.

This should suggests that ℝ–cubings are very organised, structured spaces where one has different tools to study them — both the tools of median geometry, and the "hierarchically hyperbolic philosophy" of projecting a complicated space (the ℝ–cubing) to a family of tractable spaces (in this case, the ℝ–trees) and working there. Furthermore, the finite depth and rank defines a finite complexity on the real cubing that allows for inductive arguments, being the base case, real trees.

The second point we want from real cubings is a connection with HHGs via asymptotic cones. More precisely, as asymptotic cones of hyperbolic spaces are ℝ–trees and, by an important result of Bowditch, asymptotic cones of coarse median spaces are bilipschitz equivalent to median spaces (see Section 23.2), the definition of an ℝ–cubing is aimed at ensuring that asymptotic cones of hierarchically hyperbolic spaces are bilipschitz equivalent to ℝ–cubings. Therefore, this result allows to move from the left to the right side of Figure 1, by passing to asymptotic cones.

Yet another motivation is generalising Thurston's theory of measured laminations, namely if G is a fg group, then the space of discrete, faithful representations of G (as conjugacy classes of orientation-preserving isometries of an HHS) has a compactification, in which the ideal points are obtained from certain actions of G on real cubings, hence generalising Thurston's compactification of the Teichmüller space.

Finally, one of the key motivations, is that we expect that real cubings will provide a platform for a high-rank generalisation of Rips' machine.

A key feature of Rips' theory is that finitely presented groups acting stably on real trees admit induced actions on simplicial trees — *stable* actions on real trees are *approximated* by simplicial actions. Already in the 1-dimensional case is already apparent that only *some* good actions can be approximated by simplicial ones and these are precisely the ones that transfer a good structure theory to the group.

This brings to the natural question of what are the good actions that can be approximated by simplicial actions on CAT(0) cube complexes. Chatterji-Druțu-Haglund asked this question [CDH10, Question 1.11] in the context of median spaces, namely when a nontrivial action of a group G on a median space can be promoted to an action on a CAT(0) cube complex. Chatterji and Druțu have shown that irreducible uniform lattices in the product of finitely many copies of $SO(n, 1)$ act properly and coboundedly on median spaces [CD17], while results of Chatterji-Fernós-Iozzi [CFI16] strongly limit the possible actions of the lattice on CAT(0) cube complexes (including ruling out proper cocompact actions). However, as pointed out in [CD17, Remark 4.8], the above median spaces have infinite rank and these type of finitely generated examples are unknown to exist for finite rank median spaces.

We believe that the analogous question for real cubings is more approachable. We show that if are real cubing admits a discrete metric then it is a CAT(0) cube complex. Since real metrics can be approximated by (rescaled) discrete ones, this indicates that real cubings have natural approximations by CAT(0) cube complexes.

In [CRK15], the authors introduced precisely real cubings as ultralimits of sequences of *well-behaved* CAT(0) cube complexes. The condition on well-behaved CAT(0)-cube complexes relates directly to the finite depth of the real cubing and it played a key role in the dynamical analysis of the action and the structure of a fg group acting on the real cubing as a limit of co-special actions on the CAT(0) approximations. For instance, if the limiting action on the real cubing is in addition free, it was proven that the group is a subgroup of the graph product of cyclic and (non-exceptional) surface groups. One should contrast this result with Rips' theorem: if a fg group acts freely on a real tree then it is a (subgroup of a) free product of free abelian and (non-exceptional) surface groups.

In this sense, the definition of real cubing gives a geometric, intrinsic description of the spaces defined in [CRK15] and any real cubing defined there is a real cubing in the definition given in this paper. This answers Problem 1 in [CRK15].

This result establishes a strategy and indicates that a good action on a real cubing should retain the features of co-special actions of groups on CAT(0)-cube complexes. We expect to use the machinery developed in [CRK15] in order to analyse the structure of groups having co-special actions on real cubings and, in particular, the structure of groups having a limiting action coming from homomorphisms to *algebraic hierarchically hyperbolic groups* (see Section 35). This would open the door to the model-theoretic study of the mapping class groups and more generally, of an algebraic HHG.

We will discuss some questions, test cases, and possibilities more precisely in Section **??**.

Asymptotic cones. Asymptotic cones have played an important role in geometric group theory since its emergence as a distinct area of mathematics — the idea essentially emerged in Gromov's proof of the polynomial growth theorem [Gro81], before being formalised by van den Dries and Wilkie in 1984 [vdDW84].

Roughly speaking, an asymptotic cone of a metric space is the picture of the space one sees when standing infinitely far away. Since passing to an asymptotic cone "converts coarse geometry into fine geometry" — for example, quasi-isometries are converted into bilipschitz maps — it is not surprising that asymptotic cones have played an important role in the study of quasi-isometric rigidity and related questions, see e.g. [BKMM12, Beh04, Bow18b, Bow16a, Bow19, Dru00, BHS17c, KL97a, KL95, KL97b, Ham05]. Although passing to an asymptotic cone involves a loss of information, some classes of group can be characterised by properties of their asymptotic cones. For example, among finitely generated groups, being virtually nilpotent is equivalent to having all asymptotic cones locally compact [Gro81, vdDW84]; being virtually abelian is equivalent to having all asymptotic cones isometric to Euclidean space [Gro81, Pan83], and being Gromov-hyperbolic is equivalent to having all asymptotic cones isometric to real trees [Gro93].

The latter fact motivates the main question addressed by our work. The construction of the asymptotic cone of a space involves choices — a non-principal ultrafilter on $\mathbb{N}$, a rescaling sequence, and an observation point. For groups, the observation point is immaterial, but the construction depends in an essential way on the choice of ultrafilter and rescaling sequence. Indeed, in [TV00], Thomas and Velicovic constructed a finitely generated group for which distinct choices of ultrafilters yield non-homeomorphic asymptotic cones, using an infinite small-cancellation presentation to create "holes" in the Cayley graph visible on some scales but not others. So, some ultrafilters yield non-simply connected asymptotic cones, while other asymptotic cones are real trees. These groups are hence *lacunary hyperbolic* in the sense of Ol'shanskii-Osin-Sapir; many interesting constructions involving such groups appear in [OOS09]. In [DS05], Druţu-Sapir construct many examples of finitely generated groups, each of which has "many" different asymptotic cones as one varies the ultrafilter/rescaling; for example, there is a group G with continuously many non-homeomorphic asymptotic cones. Even stranger is a result of Kramer-Shelah-Tent-Thomas: for a uniform lattice G in an appropriately chosen Lie group, G has continuously many homeomorphism types of asymptotic cones *provided the Continuum Hypothesis fails* [KSTT05]. On the other hand, Thornton had earlier shown that, *assuming the Continuum Hypothesis*, the same lattices have a single bilipschitz class of asymptotic cones [Tho02]. Finitely presented groups with non-unique asymptotic cones (irrespective of the Continuum Hypothesis) were constructed by Ol'shanskii-Sapir [OS07], and this was extended by Osin-Ould Houcine in [OH11]. In short, among finitely generated, and even finitely presented, and even some familiar groups, asymptotic cones depend in a serious way on the choice of ultrafilter/rescaling sequence.

And yet, if G is word-hyperbolic, then every asymptotic cone of G is isometric to a (universal) real tree: a point if G is finite, a line if G is virtually $\mathbb{Z}$, or the unique complete homogeneous $2^{\aleph_0}$–valent real tree constructed in [DP01]. Together with the aforementioned result about virtually abelian groups, it is then natural to ask whether uniqueness of asymptotic cones (up to homeomorphism or even bilipschitz equivalence) is a natural feature of groups satisfying appropriate "generalised hyperbolicity" properties.

In fact, there are various results in this direction. First, Osin-Sapir proved in [OS11, Corollary 1.6] that if G is hyperbolic relative to subgroups P_i, and each P_i has the property that all of its asymptotic cones are bilipschitz equivalent, then the same is true of G. Sisto proved the same result in [Sis13]. Using this, plus geometrisation, and his own analysis of asymptotic cones of fundamental groups of graph manifolds, Sisto proved in [Sis11] that if M is a compact, connected, orientable 3–manifold with toral boundary, then any two asymptotic cones of $\pi_1 M$ are bilipschitz equivalent.

Uniqueness of the asymptotic cones for fundamental groups of non-geometric graph manifolds is of particular interest since these groups are not nontrivially relatively hyperbolic; in fact they are *thick* in the sense of [BDM09]. Roughly, this means that they contain a highly connected network of subspaces quasi-isometric to products with unbounded factors, and these "product regions" can have large coarse intersection. In the case of graph manifolds, these pieces are arranged in a tree-like fashion, because they are vertex spaces associated to vertices in the Bass-Serre tree of the JSJ splitting.

There are other natural examples with similar geometry. If Γ is a finite simplicial graph and A_Γ is the associated right-angled Artin or Coxeter group, then A_Γ again consists of product regions — cosets of subgroups generated by subgraphs of Γ splitting as proper joins — arranged in a "quasi-tree-like" fashion (which can be made precise by considering the *extension graph* from [KK14] or the *contact graph* of the Salvetti or Davis complex [Hag14]). Likewise, if S is a finite-type oriented hyperbolic surface and $\mathcal{MCG}(S)$ is its mapping class group, then the "product regions" in $\mathcal{MCG}(S)$ — cosets of multicurve stabilisers — are quasi-isometric to products of simpler mapping class groups with $\mathbb{Z}^n$, and are arranged in a "hyperbolic" fashion, in the sense that coning off the product regions yields a space quasi-isometric to the curve graph $\mathcal{C}S$, which is hyperbolic [MM99].

The geometric setup alluded above is formalised in the notion of *hierarchical hyperbolicity* from [BHS17b, BHS19], and this is a key ingredient in our main theorem. Rather than involve hierarchical hyperbolicity at this point in the discussion, we state the following special case of our main result:

Theorem A (Uniqueness of asymptotic cones, special case). *Let G be a finitely generated group that is quasi-isometric to one of the following:*

- *$\pi_1 X$, where X is a compact special cube complex (in the sense of* [HW08]*);*
- *$MCG(S)$, where S is an oriented hyperbolic surface of finite genus, with finitely many punctures and boundary components.*

Then any two non-principal asymptotic cones of G are bilipschitz equivalent, and in particular homeomorphic.

Theorem A was not previously known even for the classes of right-angled Artin or right-angled Coxeter groups, or braid groups (thought of as mapping class groups of punctured discs). It also applies to the (non-right-angled) Coxeter groups for which the Niblo-Reeves action on a cube complex, from [NR03], is cocompact (these are characterised by Williams in [Wil98]), using a result of Haglund-Wise [HW10].

The seemingly strange hypothesis on G — mapping class groups and special cube complexes are not obviously related — comes from the fact that we actually prove the above theorem for any G in the class of *algebraic hierarchically hyperbolic groups.* Hierarchically hyperbolic groups are a well-studied class about which we will say more in this introduction; our theorem does not apply to all such groups, but requires that the hierarchically hyperbolic structure (a geometric feature) arises in a particular way from algebraic properties of G. While hierarchical hyperbolicity is a natural notion — it is easy to build new examples from old, and many groups of interest to geometric group theorists are hierarchically hyperbolic — the algebraic conditions we impose are more ad hoc and are designed to navigate between specific twin perils: on one hand, the goal is a proof that covers compact special groups and mapping class groups, and on the other, we need certain geometric properties to imply algebraic ones. [2]

There are other examples to which we expect one can apply Theorem A. For instance, we expect (but have not shown) that the Artin groups shown in [HMS21] to be hierarchically hyperbolic are in fact (virtually) algebraically hierarchically hyperbolic — these are the Artin groups of *large hyperbolic type* — and hence have a unique bilipschitz class of asymptotic cones.

[2]For the HHG enthusiasts, we need standard product regions to be cosets of subgroups, and we need orthogonality to imply commutation of certain elements, though we stop short of insisting that the coarse product structure of product regions comes from an actual direct product of groups, since that isn't true in the mapping class group.

Summary of results. We now summarise the main results of the paper.

In the first part, we introduce and develop a systematic study of the theory of real cubings.

We further characterise real cubings inside the class of median spaces. More precisely, we introduce the notion of a poset-colouring of the set of walls associated to a median space and show that median spaces always admit a canonical poset-colouring of the set of walls. This poset-colouring has a partial order which defines the depth of the median. In this terms, we show that real cubings are precisely complete, connected median spaces of finite rank which admit a (tangible) poset-colouring of finite depth. We also show that the finite depth of the canonical poset-colouring is equivalent to the median space having a real cubing structure with wedges and clean containers.

In Part 2, we review and complement the theory of HHS needed for the paper.

In part 3, we show that any asymptotic cone of an HHS is a real cubing generalising the fact that an asymptotic cone of a hyperbolic group is a real tree.

In part 4, we introduce the notion of universal real cubing. The local structure of a universal real tree (germs at the identity) is a sheaf of lines and so the universal real tree is uniquely determined by the cardinality of the sheaf. In the case of real cubings, the local structure is defined as a the quotient of a union of Euclidean spaces (with global bound on the dimension and endowed with the l_1-metric and a base point 0) where the identifications are over convex subspaces (and all the based points are identified), for instance a sheaf of planes and lines. In this case, the local structure does not only depend on the cardinality and it is far from being unique. For such a local structure, we construct a universal real cubing.

In part 5, we introduce the notion of algebraic HHG. As we discussed, free cocompact actions on CAT(0) cube complexes are not rigid enough to encode the geometry of the space in the algebraic properties of the group and so the notion of co-special actions/groups was introduced. In this vein, we introduce the notion of algebraic HHG as a generalisation of a special group and a hyperbolic group.

In part 6, we concentrate our attention on the question of the uniqueness of asymptotic cones for algebraic HHGs. We then show that the asymptotic cone of an algebraic HHG is unique, i.e. it is a universal real cubing and the local structure is unique. In particular, we establish uniqueness of asymptotic cones of RAAGs and MCGs.

1.1. **Real cubings in the world of generalised negative curvature.** Since the basic innovation in this paper is the introduction of the class of *real cubings*, we first discuss in a high-level way how these spaces fit into the existing world of spaces exhibiting "generalised negative curvature".

Simplicial trees are among the basic objects of mathematics, and group actions on simplicial trees are vital to geometric group theory in a wide variety of ways — Bass-Serre theory [Ser80] is crucial as both a tool for studying "natural" groups and constructing new ones, and even free actions on trees provide an entire rich universe (consider outer space). So we start our discussion of hierarchical hyperbolicity and real cubings with a discussion of trees.

A simplicial tree T is characterised by geometric features that generalise in various ways, and one can think of the resulting definitions as notions of *generalised negative curvature*. Some of the classes of spaces that generalise trees are naturally thought of from a "fine-geometric" point of view: one cares about specific points, exact distances, geodesics, local properties matter, etc. Some of the generalisations are "coarse-geometric": one cares more about large scales and less about local phenomena.

1.1.1. *Median graphs and cube complexes.* For example, given three vertices $a, b, c \in T$, there is a unique vertex m — the *median* of a, b, c — that lies on each of the three geodesics joining pairs of distinct points in $\{a, b, c\}$. In view of the fact that any two vertices are joined by a unique geodesic in T, this means that geodesic triangles in T are tripods.

One can abstract just the existence of medians, dropping uniqueness of geodesics, to obtain the class of *median graphs*. Part of the great importance of median graphs comes from an important result of Chepoi [Che00], which says that the class of median graphs is exactly the class of 1–skeletons of *CAT(0) cube complexes*, ubiquitous objects that have arisen at many times and in multiple guises (see, for example, [Sag14] for an introduction to cube complexes geared toward geometric group theory).

Median graphs/CAT(0) cube complexes are a higher-dimensional generalisation of trees (and the median property is only one of several properties of trees with an analogue in the class of CAT(0) cube complexes). After their introduction into geometric group theory in [Gro87, Bri91], it was recognised by Sageev [Sag95] that group actions on CAT(0) cube complexes arise from the presence of so-called *codimension*–1 subgroups, and that these "actions on high-dimensional trees" can under certain circumstances be promoted to actual splittings [Sag97]. This sort of notion matured into the *special* cube complexes of Haglund-Wise [HW08], which played a fundamental role in Agol's resolution [Ago13], relying on work of Wise [Wis21], of the virtual Haken conjecture. The class of groups known to act properly on CAT(0) cube complexes is now very large, and significant information about a group — coarse geometric, algebraic, algorithmic, etc. — can often be gleaned from the construction of such an action.

1.1.2. *Real trees.* The property of trees, "every geodesic triangle is a geodesic tripod" (more formally, given any $a, b, c \in T$, any geodesic joining two distinct points in $\{a, b, c\}$ is contained in the union of the other two geodesics determined by those three points) can be generalised in another way. Specifically, we keep the requirement about uniqueness of geodesics, but no longer insist that our geodesic space is a graph. We then obtain the class of 0–hyperbolic geodesic metric spaces, i.e. $\mathbb{R}$–trees/real trees.

1.1.3. *Median spaces.* Moving up in dimension, we get the class of geodesic median metric spaces. In such spaces, any three points determine *some* geodesic tripod, but geodesics are no longer unique.

Median spaces (of which $\mathbb{R}$–trees, CAT(0) cube complexes, and simplicial trees are examples), have a very rich structure and arise naturally in geometry and group theory. Their ubiquity partially derives from the fact that they arise naturally as "duals" of collections of bipartitions of some underlying set or space, via a collection of closely related constructions (wallspaces or *poc-sets*, in the discrete case [CN05, Nic04, Rol16, Sag95] or, more generally, measured walls/halfspaces [CDH10, Fio20, Fio19]).

A primary reason why median spaces are geometrically tractable is the existence of a notion of convexity of subspaces that can be defined in terms of the median, without (explicit) reference to the metric. This is because median spaces are a particular case of *median algebras*, which have a long history in order theory [Isb80, Sho54, BH83]. A median algebra is a set equipped with a ternary operator, the *median*; a geodesic median metric space is one where the median operator detects the "central point" of each of the above-mentioned geodesic tripods (see Definition 2.1 and Definition 2.2).

Median spaces play several important roles in geometric group theory. In work of Bowditch, who seems to have been motivated by quasi-isometric rigidity of mapping class groups and Teichmüller space, median spaces arise as asymptotic cones of *coarse median spaces*, discussed below; in the context of mapping class groups, median structures on asymptotic cones were also studied in [BDS11b, BDS11a]. Work of Chatterji-Druţu-Haglund on *measured wallspaces* established the relationship between actions on median spaces and a-T-mendability, and introduced *measured wallspaces*. This has inspired much recent work, notably by Fioravanti [Fio20, Fio19, Fio18], on group actions on median spaces. All of the above will be very important for us later.

1.1.4. *Gromov-Rips hyperbolicity.* If we again start with trees, but now "coarse-ify" the geodesic tripods property, we get the familiar class of hyperbolic (quasi)geodesic spaces. The notion of a hyperbolic group seems to have its roots in small-cancellation theory (among other things) and was defined by Gromov in [Gro87]. The theory of hyperbolic groups and spaces is extensive. In this paper, we will mainly not be concerned with hyperbolic groups, but rather with (non-proper) hyperbolic geodesic spaces arising from Cayley graphs of finitely-generated groups by "coning off" appropriate subgroups and their left cosets. This is also routine practice in geometric group theory, beginning with relatively hyperbolic groups [Far98, Bow12]. The range of groups that are not hyperbolic but for which such cone-off procedures, or similar constructions, yield useful (e.g. *acylindrical* [Bow08, Osi16]) actions on hyperbolic spaces is vast.

1.1.5. *Coarse median spaces.* Just as geodesic median spaces can be viewed as "higher rank $\mathbb{R}$–trees", and CAT(0) cube complexes can be viewed as "higher-dimensional simplicial trees", Bowditch has introduced *coarse median spaces* [Bow13]. In such spaces, triples of points determine quasigeodesic "coarse tripods", and, more generally, finite sets can be (fairly loosely) approximated by CAT(0) cube complexes. One can define a notion of *rank* for coarse median spaces, and geodesic coarse median spaces of rank 1 are exactly hyperbolic spaces.

Coarse median spaces seem to have been introduced as a means of extracting the essence of questions related to quasi-isometric rigidity of mapping class groups and Teichmüller space, but they have become objects of geometric and group-theoretic interest in their own right, featuring very strong geometric results like Bowditch's quasiflats theorem [Bow19] and Fioravanti's very recent work on coarse-median preserving automorphisms [Fio21].

1.1.6. *Hierarchical hyperbolicity.* Although they include a variety of important examples (mapping class groups, CAT(0) cube complexes), and the seemingly quite inclusive definition is restrictive enough to enable one to prove surprisingly strong coarse-geometric results, it is sometimes useful to work with more restrictive sub-classes of coarse median spaces.

In many motivating examples of coarse median spaces, actual hyperbolicity lurks. For example, if X is a CAT(0) cube complex, one can always cone off certain convex subspaces to obtain a graph that is hyperbolic (in fact quasi-isometric to a tree) [Hag14, BHS17b, Gen19].

Similarly, if S is a finite-type hyperbolic surface then, fixing a word-metric on the mapping class group $MCG(S)$, one obtains a hyperbolic space by coning off left cosets of multicurve stabilisers [MM99].

So, it would appear that some coarse median spaces admit many coarsely lipschitz projections to hyperbolic spaces (in the above examples, these are just inclusions of the original space in the coned-off space). Perhaps, since hyperbolic spaces are particularly tractable, one can study these coarse median spaces via these projections?

This is the motivation behind *hierarchically hyperbolic spaces* (introduced in [BHS17b, BHS19] and generalised via a slightly different set of axioms in in [Bow18a]).

Extant discussions of hierarchically hyperbolic spaces focus on two intuitive ways to think about them. First, as alluded to above, they are spaces where one can obtain a hyperbolic space by coning off certain subspaces, and these subspaces are coarsely products of "simpler" spaces, each with the same property; the "complexity" decreases at each stage, and at the bottom one finds hyperbolic spaces. (See Sisto's expository article [Sis19] for an elaboration of this viewpoint.) Here we offer two other intuitive viewpoints.

First, one of the most important facts about hierarchically hyperbolic spaces is the *cubical approximation theorem* from [BHS17c], which generalises a standard fact about hyperbolic spaces and gives a specific sense in which hierarchical hyperbolicity is a strictly stronger condition than being coarse median. Very roughly, finite sets in coarse median spaces are approximated by finite CAT(0) cube complexes; in a hierarchically hyperbolic space, these approximations need to have better convexity properties. So hierarchically hyperbolic spaces are coarse median spaces that are "coarsely locally cubical" in a particularly strong way. (In fact, recent work [HP21, Pet21] shows that in many cases, there is even a global quasi-isometry to a CAT(0) cube complex.)

Second, formally, a hierarchically hyperbolic space $\mathcal{X}$ comes equipped with a set $\mathfrak{F}$ and, for each $U \in \mathfrak{F}$, a coarsely lipschitz projection to a hyperbolic space, $\pi_U : \mathcal{X} \to \mathcal{C}U$ (the coarse lipschitz constants and the hyperbolicity constants are uniform over $\mathfrak{F}$). The product of these projections gives a map $\mathcal{X} \to \prod_{U\in\mathfrak{F}} \mathcal{C}U$. There are further axioms whose job is to ensure one can prove two key theorems: the *distance formula* and the *realisation theorem*. The first says that the metric on $\mathcal{X}$ can be recovered, up to quasi-isometry, from the projections (it does not say that the map to the product of hyperbolic spaces is a quasi-isometric embedding, but it is roughly in that spirit). The second says that the image of the map $\mathcal{X} \to \prod_U \mathcal{C}U$ is characterised by some *consistency conditions*, which compare projections onto certain pairs of hyperbolic spaces. Very roughly, one can think of $\mathcal{X}$ as the "solution set of a system of coarse equations" in $\prod_U \mathcal{C}U$, with equations associated to certain pairs $U, V \in \mathfrak{F}$.

(In the mapping class group context, the "consistency conditions" are sometimes known as the *Behrstock inequalities* after [Beh04], and the fact that they coarsely characterise points in the mapping class group among points in the product of the curve graphs of the subsurfaces is the *realisation theorem* of [BKMM12].)

1.2. **Hierarchically hyperbolic groups.** Our main motivation for working in the context of *hierarchically hyperbolic spaces (HHS)* and *groups (HHG)* is that these provide a common geometric framework for studying mapping class groups of hyperbolic surfaces of finite type and fundamental groups of compact *special cube complexes* (in the sense of Haglund-Wise [HW08]). Hierarchically hyperbolic spaces were introduced in [BHS17b, BHS19], inspired by the cases of mapping class groups [MM99, MM00, Beh06, BKMM12] and CAT(0) cube complexes [Hag14, KK14].

There is now a fairly extensive literature on HHS/HHG, part of which is devoted to expanding the class of known examples (e.g. [BHS17a, BR20a, RS20, Vok17, HMS21, BHMS20, Ber21, BR20b, Mil20, DDLS20, DDLS21]) and part of which is to applying the theory to obtain conclusions about these spaces/groups (e.g. [ABD21, BHS17a, BHS17c, DMS20, DHS17, RST18, HHP20, HP21, Pet21, Pet21, PS20, Bow18a]).

Rather than refer the reader to that literature, we give a fairly self-contained account of the parts of the theory that we shall need, in Section 2. Many of the results of that section (like the realisation theorem and distance formula) are from [BHS19], but we also discuss tools — notably the *cubical approximation theorem* — from [BHS17c] and [DHS17], and relate hierarchically hyperbolicity to the work of Bowditch on coarse median spaces [Bow13].

For the moment, we just recall the data that is needed to define a hierarchically hyperbolic space. An HHS is a pair $(\mathcal{X}, \mathfrak{F})$, where $\mathcal{X}$ is a quasigeodesic metric space (the reader should just picture a graph) and $\mathfrak{F}$ is a set. To each $U \in \mathfrak{F}$, we associate a δ–hyperbolic space $\mathcal{C}U$ and a projection $\pi_U : \mathcal{X} \to \mathcal{C}U$ (which the reader can assume is coarsely surjective).

This data yields a map $\pi : \mathcal{X} \to \prod_{U\in\mathfrak{F}} \mathcal{C}U$, just by taking products of projections. One of the HHS axioms — the exact definition is Definition 10.1, and the relevant part is the *uniqueness* axiom — guarantees in particular that this map will be coarsely injective.

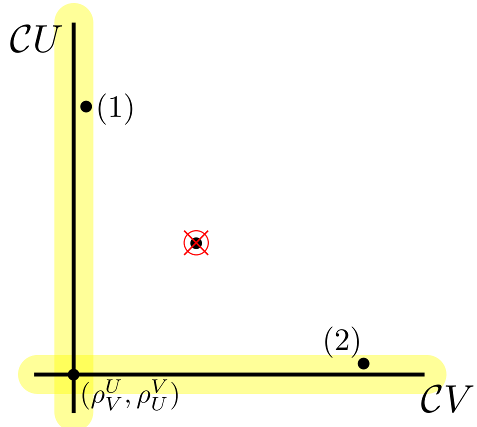


FIGURE 2. The consistency condition for $U \pitchfork V$. Some possible pairs (a, b) are labelled (1) and (2); an illegal pair is crossed out.

Some more data is needed to say which points in $\prod_{U \in \mathfrak{F}} \mathcal{C}U$ are in the image of π. The idea is similar to the earlier discussion of real cubings arising from (infinite) products of intervals.

Specifically, for every distinct $U, V \in \mathfrak{F}$, we will define a subspace $Cons(U, V) \subset \mathcal{C}U \times \mathcal{C}V$. (The notation is short for "consistent".)

The consistency conditions in Definition 10.1 say that

$$\pi(\mathcal{X}) \subseteq \bigcap_{U \neq V} (p_U \times p_V)^{-1}(Cons(U, V)),$$

where $p_U : \prod_{W \in \mathfrak{F}} \mathcal{C}W \to \mathcal{C}U$ is natural projection.

Definition 10.1 is designed to enable the proof of the *realisation theorem* (see Theorem 10.5), which says that any $y \in \bigcap_{U \neq V}(p_U \times p_V)^{-1}(Cons(U, V))$ satisfies

$$\sup_W \mathsf{d}_{\mathcal{C}W}(y, \pi_W(x)) \leqslant K,$$

for some $x \in \mathcal{X}$ and some global constant K independent of x and y.

The subspace $Cons(U, V)$ will be chosen according to how U and V are related.

First, they can be *orthogonal*, denoted $U \perp V$. In this case,

$$Cons(U, V) = \mathcal{C}U \times \mathcal{C}V,$$

so the pair U, V puts no constraint on which points in $\prod_W \mathcal{C}W$ lie in $\pi(\mathcal{X})$. Another word for orthogonality could therefore have been *independence*.

There are two other possibilities. The first is where U, V are *transverse*, denoted $U \pitchfork V$. We fix (coarse) points $\rho^U_V \in \mathcal{C}V$ and $\rho^V_U \in \mathcal{C}U$, i.e. we pick a basepoint $(\rho^V_U, \rho^U_V) \in \mathcal{C}U \times \mathcal{C}V$. Here $Cons(U, V)$ is the set of all (a, b) such that a is E–close to ρ^V_U or b is E–close to ρ^U_V, where E is a global constant. This is shown in Figure 2.

(In the setting where $\mathcal{X}$ is the mapping class group of a surface and U, V are overlapping subsurfaces, this condition is a restatement of [Beh06, Theorem 4.3].)

There is a final way in which U, V can be related: U can be *nested*, denoted $U \sqsubseteq V$. This relation is anti-symmetric (and in fact a partial order). In this case, we put a basepoint ρ^U_V in $\mathcal{C}V$, and define a map $\rho^V_U : \mathcal{C}V \to \mathcal{C}U$. One of the axioms, *bounded geodesic image*, governs this map by asking that, if $\pi_V(x), \pi_V(y)$ can be joined by a geodesic in $\mathcal{C}V$ avoiding the E–ball about ρ^U_V, then $\rho^V_U(\pi_V(x))$ and $\rho^V_U(\pi_V(y))$ coarsely coincide. The set $Cons(U, V)$ consists of those $(a, b) \in \mathcal{C}U \times \mathcal{C}V$ such that either $\mathsf{d}_{\mathcal{C}V}(b, \rho^U_V) \leqslant E$, or $\mathsf{d}_{\mathcal{C}U}(a, \rho^V_U(b)) \leqslant E$. A heuristic picture is Figure 3.

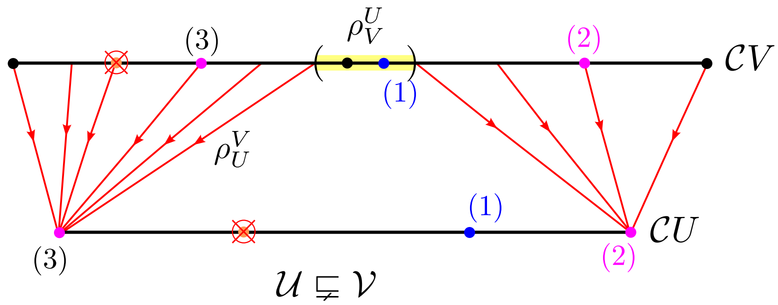


FIGURE 3. A simple situation with $U \subsetneq V$. The map ρ^V_U collapses each component of the segment $\mathcal{C}V - \mathcal{N}_E(\rho^U_V)$ to one of the endpoints of $\mathcal{C}U$, and does whatever it likes on the yellow neighbourhood of ρ^U_V. The red arrows indicate this map. The numbered pairs of points, viewed as points in $\mathcal{C}U \times \mathcal{C}V$, satisfy the consistency condition, but the ordered pair crossed out in red does not.

We impose two conditions on the partial order $\sqsubseteq$: first, if $\mathfrak{F} \neq \emptyset$, then it contains a unique $\sqsubseteq$–maximal element, and second, there is a uniform bound on the length of $\sqsubseteq$–chains. There are some additional rules relating nesting and orthogonality.

So, coarsely, $\mathcal{X}$ is the subset of an (infinite) product of hyperbolic spaces obtained by "solving" some coarse consistency conditions on transverse and nested pairs in $\mathfrak{F}$. That this characterisation is geometrically faithful is the content of the *distance formula* (Theorem 10.7). This does not quite say that $\pi : \mathcal{X} \to \prod_{W \in \mathfrak{F}} \mathcal{C}W$ is a quasi-isometric embedding when the latter is given the ℓ_1 metric. Instead, it says that, up to additive and multiplicative error, for any $x, y \in \mathcal{X}$, the distance $\mathsf{d}_{\mathcal{X}}(x, y)$ is obtained by summing the quantities $\mathsf{d}_{\mathcal{C}V}(\pi_V(x), \pi_V(y))$ over the V for which those terms exceed some fixed threshold (independent of x, y). A useful feature is that one can enlarge the threshold and obtain the same conclusion, at the expense of modifying the additive and multiplicative constants.

One important feature of a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{F})$ is the *standard product region* $P_U \subset \mathcal{X}$ associated to each $U \in \mathfrak{F}$. One way to define P_U is as the set of $x \in \mathcal{X}$ for which $\mathsf{d}_V(\pi_V(x), \rho^U_V) \leqslant E$ for all $V \in \mathfrak{F}$ for which the bounded set ρ^U_V is defined, i.e. $U \pitchfork V$ or $U \subsetneq V$. As explained in Section 15, P_U is quasi-isometric to a product $F_U \times E_U$, where F_U is roughly the set of points obtainable from a fixed basepoint in P_U by varying the projections to $\mathcal{C}V, V \sqsubseteq U$, and E_U is the set of points obtainable by varying π_V for $V \perp U$.

Let G be a finitely generated group equipped with a word metric. For G to be a *hierarchically hyperbolic group (HHG)* means that there is an HHS structure $(G, \mathfrak{F})$, but it means somewhat more than this. First, G must act on $\mathfrak{F}$ cofinitely, preserving the three relations $\perp, \sqsubseteq, \pitchfork$ mentioned above. Moreover, for each $g \in G$ and $U \in \mathfrak{F}$, we have an associated isometry $g : \mathcal{C}U \to \mathcal{C}gU$, and these isometries compose in the expected way. We ask that $g(\rho^U_V) = \rho^{gU}_{gV}$ wherever the bounded set ρ^U_V is defined, and that $\pi_{gU}(gx) = g(\pi_U(x))$ for all $U \in \mathfrak{F}$ and $g, x \in G$. In other words, G "acts on the HHS structure" in a way that is compatible with left-multiplication.

In an HHG $(G, \mathfrak{F})$, we have that $P_{gU} = gP_U$ for $g \in G, U \in \mathfrak{F}$, and in practice we often impose an additional assumption (satisfied by our target examples) that $\mathrm{Stab}_G(U)$ acts coboundedly on P_U. Because we are free to replace P_U by any subspace at uniformly bounded Hausdorff distance from the subspace P_U defined above, and since there are finitely many G–orbits in $\mathfrak{F}$, this lets us assume that there are finitely many $U_1, \ldots, U_k \in \mathfrak{F}$ such that $1 \in P_{U_i} = \mathrm{Stab}_G(U_i)$, and the other product regions are left cosets of the various P_{U_i}. So $(G, \mathfrak{F})$ is weakly hyperbolic relative to $\{P_{U_1}, \ldots, P_{U_k}\}$ — coning off the P_{U_i} and their left cosets yields something quasi-isometric to the hyperbolic space $\mathcal{C}S$, where $S \in \mathfrak{F}$ is the

$\sqsubseteq$–maximal element. In general, the product regions can have large overlaps, so this is not a relatively hyperbolic structure, but the definitions give significant control over how the product regions can coarsely intersect.

1.3. **Real cubings.** One way of defining a real cubing — Definition 4.2 — is modelled on hierarchical hyperbolicity. Specifically, we have a complete path-connected metric space $\mathcal{X}$, equipped with a collection $\mathfrak{F}$, with a real tree $\mathcal{T}U$ associated to each $U \in \mathfrak{F}$. Exactly as for hierarchically hyperbolic spaces, $\mathfrak{F}$ has nesting, orthogonality, and transversality relations. These satisfy slightly weaker combinatorial constraints than those for HHSes — for example, we don't ask for the existence of a unique $\sqsubseteq$–maximal element — in order to accommodate arbitrary finite-dimensional CAT(0) cube complexes. However, when $U \subsetneq V$ or $U \pitchfork V$, we have a point $\rho_V^U \in \mathcal{T}V$, as before, and we have projections $\pi_U : \mathcal{X} \to \mathcal{T}U$. Another key difference with the hierarchically hyperbolic setup is that we ask for the product map $X \to \prod_U \mathcal{T}U$ determined by the π_U to be an isometric embedding, when the product is given the ℓ_1–metric.

The axioms imply that the image of π is an embedding, and we can characterise the image in much the same way as in the HHS case: it is the solution set of a collection of conditions on transverse and nested pairs in $\mathfrak{F}$ — these are illustrated in Figure 14.

In HHSes, there is a constant E such that for any x, y, the set of $U \in \mathfrak{F}$ with $\mathsf{d}_{\mathcal{C}U}(\pi_U(x), \pi_U(y)) \geqslant E$ is finite; we no longer require this in real cubings, which just need the sum of all such projection distances to be finite. Another key difference is that in an HHS, if two points are far apart, their projections on some $\mathcal{C}U$ are far apart (quantitatively), but we also no longer require this for real cubings. For example, in the standard real cubing structure on a CAT(0) cube complex, all of the associated real trees are copies of $[0, 1]$.

A significant part of Part 1 is devoted to alternate characterisations of real cubings. These are not needed in the proof of Theorem A, but we believe them to be of independent interest.

First, a standard trick embeds any real tree $\mathcal{T}U$ in the normed vector space $\ell_1(U)$ of countably supported real functions on some set, with the ℓ_1 metric. Using this, we show in Section 7 that any real cubing is isometric to a *semialgebraic subset* — i.e. the solution to a system of equations and inequalities, each in one or two variables — of an ℓ_1 space. We also prove a converse. The equations and inequalities are closely modelled on the consistency conditions for HHSes, but the underlying spaces are just copies of $\mathbb{R}$, not arbitrary hyperbolic spaces.

In a way, this is as expected. Definition 4.2 turns out to imply that any real cubing is a complete geodesic median space of finite rank, and the projections to the underlying real trees are median-perserving. Since median subalgebras of a product of copies of $\mathbb{R}$ with the product median are obtained by deleting intersections of pairs of halfspaces, the semialgebraic characterisation of real cubings says in particular that the consistency conditions built into Definition 4.2 specify a median subalgebra of an ℓ_1 space, which turns out to coincide with the original real cubing.

The fact that real cubings are median raises the obvious question of which median spaces are real cubings. Real cubings are complete and path-connected by definition, and it follows fairly easily from the definition that they have finite rank as median algebras. So the question is really about which complete, connected, finite-rank median spaces are real cubings. We answer this in terms of *colourings* of the walls in a median space. Specifically, we prove Theorem 5.1, which gives sufficient conditions on a median space ensuring that it is a real cubing. This is what we will use for asymptotic cones. We also generalise the notion of a *factor system* in a CAT(0) cube complex, introduced in [BHS17b], to the context of median spaces, and use this to characterise median spaces admitting "well-behaved" real cubing structures (where "well-behaved" means that the nesting and orthogonality relations satisfy some combinatorial conditions reminiscent of cube complexes). This is done in Section 6, which is not needed for asymptotic cones but which we think significantly aids understanding of real cubings.

1.4. **Real cubing structures on asymptotic cones of HHS.** Our first main result about asymptotic cones is Theorem 26.3, which we restate as:

Theorem B. *Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space. Then any asymptotic cone of $\mathcal{X}$ is bilipschitz equivalent to an $\mathbb{R}$–cubing.*

Theorem B generalises a theorem of Behrstock-Druţu-Sapir [BDS11b], which says that any asymptotic cone of a mapping class group admits a median structure, and a median-preserving bilipschitz embedding into an (infinite) product of $\mathbb{R}$–trees. Our result generalises this because mapping class groups are hierarchically hyperbolic spaces (see [BHS19] or Section 36.1). Our theorem also strengthens the result of [BDS11a], even in the mapping class group case: we don't merely embed the asymptotic cone in a product of real trees, we characterise (from the consistency conditions in the definition of a real cubing) exactly which points in the product are in the image of this embedding.

Our approach is quite different from that of [BDS11b], with some conceptual overlap. For them, constructing a median on the asymptotic cone is a *consequence* of the embedding in the product of $\mathbb{R}$–trees. For us, the median on the cone, and a bilipschitz-equivalent metric making the cone an honest median metric space, is the *starting point* — this median and metric are provided by results of Bowditch [Bow13, Bow18b]. Once we know that we have a median metric space, replete with median-convex product regions arising as ultralimits of standard product regions, we can use the *measured halfspace* structure from the work of Fioravanti [Fio20] to construct the $\mathbb{R}$–cubing structure.

The idea is as follows. Any asymptotic cone $\mathrm{Cone}^\omega(\mathcal{X})$, viewed as a median metric space, contains many convex subspaces $\mathbf{P_U}$ arising as ultralimits of sequences of standard product regions; passing to ultralimits converts the coarse product structure into an actual product structure, $\mathbf{P_U} = \mathbf{F_U} \times \mathbf{E_U}$. As a median space, $\mathrm{Cone}^\omega \mathcal{X}$ contains walls, and a wall can be assigned a "colour", namely the $\sqsubseteq$–minimal sequence $\mathbf{U}$ of elements of $\mathfrak{F}$ such that the wall crosses $\mathbf{F_U}$. Given two points $\mathbf{x}, \mathbf{y} \in \mathbf{F_U}$, we can consider all of the positive-measure sets of walls that separate $\mathbf{x}, \mathbf{y}$ and have colour properly nested in $\mathbf{U}$. After removing these sets, the measure of what remains in a quantity $\mathbf{D_U}(\mathbf{x}, \mathbf{y})$. This defines a pseudometric, and the metric quotient turns out to be a real tree. In this way, we obtain a family of real trees $\mathcal{T}\mathbf{U}$ and projections $\mathrm{Cone}^\omega(\mathcal{X}) \to \mathcal{T}\mathbf{U}$. Intuitively, what he have done is start with the median space $\mathbf{F_U}$, and collapsed its nontrivial intersections with product regions to points, yielding a quotient median space which has rank one and is therefore a real tree. We let $\mathfrak{F}_\infty$ denote the set of real trees. Since each real tree came from an ω–class of sequences in $\mathfrak{F}$, the set $\mathfrak{F}_\infty$ inherits nesting, orthogonality, and transversality relations from $\mathfrak{F}$ (these relations depend in principle on the ultrafilter). Using these, we verify the axioms from Definition 4.2.[3]

In the remainder of Section 26, we turn to the special case of asymptotic cones of an HHG $(G, \mathfrak{F})$. We consider the action of the ultrapower G^* on the real cubing structure provided by Theorem B and show that this is an action by *real cubing automorphisms*. We use this to begin describing the local structure of the asymptotic cone.

By a *real cubing automorphism*, what we mean is that elements of G^* act as median-preserving isometries on $\mathrm{Cone}^\omega(G)$ (with the median metric), which is the restriction of an

[3]Formally, we show in Part 1 that any complete connected finite rank median space whose walls admit something we call a "finite-depth tangible poset-colouring" is a real cubing, and then in the setting of asymptotic cones of HHSes, we use the subspaces $\mathbf{F_U}$ to construct a poset-colouring with the desired properties.

action on the product of all of the underlying real trees (permuting the factors and inducing isometries on them).

At any point $\mathbf{x} \in \mathrm{Cone}^\omega(G)$, one can consider the set of points $\mathbf{y}$ that can be reached from $\mathbf{x}$ by changing only those real tree "coordinates" belonging to some pairwise-orthogonal set. This is a *local real cubing*, is a real cubing in its own right, and is median convex in $\mathrm{Cone}^\omega(G)$. Homogeneity of $\mathrm{Cone}^\omega(G)$ means that the local structures at any two points are isomorphic as real cubings.

The local structure plays an essential role, because the strategy for proving Theorem A is as follows: we will show that $\mathrm{Cone}^\omega(G)$ (as a median metric space) depends only on the local structure, and then show that the local structure is independent of the ultrafilter and rescaling.

1.5. **Organisation of this document.** In Part 1, we introduce real cubings. Specifically, we first survey the results on median spaces needed throughout the text, and then introduce the notion of a *poset-colouring* of the walls in a median space. In order to be useful, poset-colourings must have some extra properties, namely *finite depth* (a combinatorial property of the set of colours) and *tangibility* (a property of the relationship between the colouring and the measure on the set of halfspaces coming from the median structure). Although it will not be needed for our application to asymptotic cones, we show that poset-colourings can always be constructed, by building the *canonical orthogonal poset-colouring*. For this poset colouring, we show that finite depth implies tangibility. This is somewhat reminiscent of the construction of *factor systems* in cube complexes from [BHS17b]. This material is in Sections 2 and 3.

In Section 4, we introduce the first definition of real cubings, which is what we will use in our application to asymptotic cones. In this section, we establish the main properties we will need, showing in particular that real cubings are complete, finite-rank, geodesic median spaces. We relate this to the preceding material in Section 5, where it is shown that real cubings are exactly the finite-rank complete connected median spaces whose walls admit a tangible, finite-depth poset-colouring. One direction, namely that the existence of such a poset-colouring for a median space give a real cubing structure, will be used in the proof of Theorem B.

In Section 6, which is not used for asymptotic cones, we press a bit further and characterise real cubings with some extra combinatorial properties — *wedges* and *clean containers* — as exactly the finite-rank, complete, connected median metric spaces where the depth of the canonical orthogonal poset-colouring is finite.

Finally, the "slick" definition of a real cubing given earlier in the introduction — in terms of *semialgebraic* subsets of an ℓ_1 space — is introduced in Section 7, where it is also shown to be equivalent to the original definition.

In Part 2, we give an account of hierarchical hyperbolicity. All of the main notions are defined here, and we have stated all of the results from the literature that we will use (with some explanation). Some of this material is not strictly necessary for our applications, but in various places we have provided details not found elsewhere in the literature, or proven small statements to illustrate the techniques. We give particular emphasis to the *cubical approximation theorem* from [BHS17c]. We also discuss briefly the HHS structure on CAT(0) cube complexes from [BHS17b, HS20] and contrast them with real cubing structures, and relate both to the canonical orthogonal poset-colouring.

Part 3 is devoted to the proof of Theorem B, as well as some additional properties of the real cubing structure on the asymptotic cone of an HHS. Specifically, we first discuss a result of Bowditch that allows one to put a median metric on the asymptotic cone. Then we show how to construct a finite-depth tangible poset-colouring of the walls, and invoke

Theorem 5.1 to conclude that the cone is a real cubing. We do an additional bit of fiddling with the real cubing stucture so that the index set can be related to the index set of the original hierarchically hyperbolic space. Then we turn to the HHG case, and analyse the local real cubing one gets from the real cubing structure on the cone (it is really the local structure that will be important in our applications). As usual, we close with questions and remarks, and we direct the reader to Section 34.1 for some examples to illustrate the construction of the real cubing-structure, which has a couple of subtleties that already make themselves felt in fairly simple examples.

Part 4 concerns the notion of an *algebraic HHG*. Here, we introduce several different restrictions on an HHG structure, and compile them into the main definition, Definition 35.9. The main goal is to prove Theorem 35.46, which provides a sufficiently rich collection of hierarchically quasiconvex subgroups with controlled projections (playing the role of the groups K_V in the mapping class group, mentioned above). We then recall the HHG structure on mapping class groups, and verify that it satisfies each part of the definition of an algebraic HHG (here we rely on algebraic results of Bestvina-Bromberg-Fujiwara [BBF15] and Leininger-McReynolds [LM07] as well as the geometric results used in [BHS19] to produce an HHS structure). We then do the same for fundamental groups of compact special cube complexes in the sense of Haglund-Wise [HW08].

Part 5 has two goals. First, we introduce the notion of a *universal real cubing*, and show how it is constructed deterministically from the data provided by a local real cubing. Then, given an HHG, we start with the local real cubing structure on an asymptotic cone, constructed earlier, and show that the resulting universal real cubing coincides with the original real cubing structure on the cone. This reduces the question of uniqueness of the cone to the question of uniqueness of the local real cubing structure.

To finish the proof of Theorem A, which happens in the short Part 7, we therefore need to pair negligible sequences with respect to one rescaling/ultrafilter with negligible sequences associated to the other, in such a way that (as discussed above), a bijection at the level of local index sets (preserving orthogonality and nesting) is established. The key tools are "normal forms for unbounded sequences" in an algebraic HHG, namely the *straight decomposition* and the *representative decomposition*, both of which are constructed in Part 6. This is where the algebraic hypotheses play a very important role.

As mentioned earlier, the reader only interested in asymptotic cones can safely skip any section marked ☺.

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Part 1. First look at real cubings

The first part of this paper is self-contained modulo its dependence on the theory of median spaces, the relevant parts of which we shall recall.

We introduce the notion of an $\mathbb{R}$–*cubing* (or *real cubing*) — Definition 4.2 — and establish the properties needed in our applications to asymptotic cones — the main result of this type is Theorem 5.1, which gives conditions on a complete, connected, finite-rank median space sufficient to ensure that it is an $\mathbb{R}$–cubing. We will verify these conditions later, in the context of asymptotic cones of hierarchically hyperbolic spaces.

Although it is not necessary for our application to asymptotic cones, we also characterise real cubings among complete, connected, finite-rank median metric spaces, in terms of colourings of walls in such a median space by posets; see Corollary 6.9. This is very much analogous to the construction of *factor systems* in CAT(0) cube complexes, from [BHS17b]. We also give an alternate characterisation of $\mathbb{R}$–cubings as *cubical semialgebraic subsets* of ℓ_1–spaces. We formulate this notion and prove its equivalence with Definition 4.2, in Section 7.

The reader who is familiar with the notion of a hierarchically hyperbolic space will notice some similarities between hierarchically hyperbolic spaces and $\mathbb{R}$–cubings, although there are key differences. However, understanding Part 1 does not require familiarity with hierarchical hyperbolicity.

The class of $\mathbb{R}$–cubings is defined in such a way as to include the following examples:

- CAT(0) cube complexes;
- $\mathbb{R}$–trees, and finite products of $\mathbb{R}$–trees;
- asymptotic cones of mapping class groups, special groups, and other hierarchically hyperbolic spaces.

In previous work [CRK15], the first and third authors introduced a class of spaces also called *real cubings*, namely rescaled ultralimits of sequences of CAT(0) cube complexes of bounded width. Using the definition of bounded width in conjunction with results in [BHS17b] and Section 26 of the present paper, one can check that real cubings in the sense of [CRK15] are real cubings in our sense. It appears that Salvetti complexes of certain infinitely-generated right-angled Artin groups provide examples of spaces that are real cubings in the sense of this paper but not in the sense of [CRK15].

Finally, it will be very important in our applications to introduce the notion of a *universal* $\mathbb{R}$–cubing. Universal $\mathbb{R}$–trees, in the sense of [DP01], are the simplest examples of universal $\mathbb{R}$–cubings. However, universal $\mathbb{R}$–trees are uniquely determined by the germs of directions at a point, which is a sheaf of lines of a given cardinality, but universal $\mathbb{R}$–cubings admit a much more complex structure of germs of directions. So, the universal $\mathbb{R}$–cubing and its uniqueness will be determined by local data in a way made later in the paper, where we introduce these objects. For now, we focus on the fundamentals of $\mathbb{R}$–cubings.

2. Background on median spaces

The theory of median spaces, median algebras, and related structures is extensive. We now recall just the facts needed in our study of $\mathbb{R}$–cubings. Our treatment relies on [Bow20, Bow13, Bow16b, CDH10, Fio20, Fio18]. We refer the interested reader to, for instance, [Ver93, Rol16, Isb80, BH83] for more background.

The fundamental notion is that of a median algebra (there are various equivalent formulations of the axioms; see [BH83, Section 1] and the references therein):

Definition 2.1 (Median algebra, topological median algebra)**.** A *median algebra* (M, μ) is a set M equipped with a map $\mu : M^3 \to M$ satisfying:

- $\mu(x, y, y) = y$ for all $x, y \in M$;
- $\mu(x, y, z) = \mu(z, x, y)$ and $\mu(x, y, z) = \mu(x, z, y)$ for all $x, y, z \in M$;
- $\mu(\mu(x, w, y), w, z) = \mu(x, w, \mu(y, w, z))$ for all $w, x, y, z \in M$.

If M is also equipped with a topology making μ continuous when M^3 is given the product topology, then (M, μ) is a *topological median algebra.*

A *homomorphism* $f : (M, \mu) \to (N, \eta)$ of median algebras is a map $f : M \to N$ such that $\eta(f(a), f(b), f(c)) = f(\mu(a, b, c))$ for all $a, b, c \in M$. In this situation, we sometimes say f *preserves the median.* □

Often, we will be interested in a subclass of median algebras, namely *median metric spaces.* (We will often use the term *median space* to mean *median metric space*, but later in the paper we will sometimes insist on the word "metric" when there is potential for confusion, namely when we have a metric space which is also a median algebra, but the combination of the median and metric do not yield a median metric space.)

Definition 2.2 (Median (metric) space)**.** A *median metric space* is a triple (M, d, μ), where (M, d) is a metric space and $\mu : M^3 \to M$ is a function such that, for all $x, y, z \in M$, the point $m = \mu(x, y, z)$ is the unique point satisfying

$$\mathsf{d}(a, b) = \mathsf{d}(a, m) + \mathsf{d}(b, m),$$

for all distinct $a, b \in \{x, y, z\}$ (we think of $\{x, y, z\}$ as a multiset, so if $x = y$, we are requiring $m = x = y$). If (M, d, μ) is a median metric space, then (M, μ) is a topological median algebra, when M is given the metric topology [Sho54]. □

Definition 2.3 (Subalgebra, median-convexity)**.** Given a median algebra (M, μ), a subset $N \subset M$ is a *median subalgebra* if $\mu(x, y, z) \in N$ whenever $x, y, z \in N$. If N satisfies the stronger property that $\mu(x, y, z) \in N$ whenever $x, y \in N$ (and $z \in M$), then N is *median-convex.* When we are working in a median metric space in which median-convexity is equivalent to all other notions of convexity in the discussion, we will sometimes just say "convex" to mean "median-convex".

The (*median*) *convex hull* of a subset A of M is the intersection of all median-convex subsets containing A. Similarly, the *median subalgebra generated by* A is the intersection of all median subalgebras containing A (which is a median subalgebra). □

One very important example of a convex subset is a *median interval*:

Definition 2.4 (Median interval)**.** Given a median algebra (M, μ) and $a, b \in M$, the *median interval* $I(a, b)$ is the set of $c \in M$ such that $\mu(a, b, c) = c$. □

The median interval $I(a, b)$ is the convex hull of the set $\{a, b\}$, and should be thought of as the set of $c \in M$ that "lie between" a and b. This intuition becomes more concrete in the setting of geodesic median metric spaces, as we will see shortly. See Figure 4.

An extremely important fact about median-convex subsets is [Rol16, Theorem 2.2]:

Theorem 2.5 (Helly property for convex sets)**.** *Let (M, μ) be a median algebra and let $N_1, \ldots, N_k \subset M$ be finitely many median-convex subsets such that $N_i \cap N_j \neq \varnothing$ for all i, j. Then $\bigcap_i N_i \neq \varnothing$.*

The analogue of dimension for a median algebra is the *rank.* First observe that for $n \geqslant 0$, the set $\{0, 1\}^n$ becomes a median algebra by viewing it as the 0–skeleton of an n–cube with the ℓ_1–metric, which is a median metric. (The reader may think about it in the following

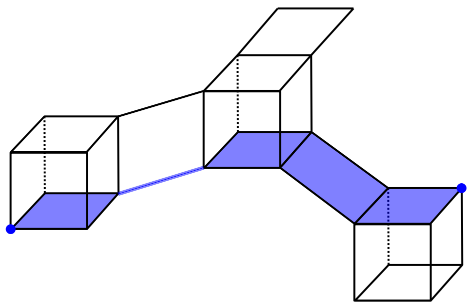

FIGURE 4. The median interval between the blue vertices in this CAT(0) cube complex is shaded.

alternate way: if $f, g, h : \{0, \ldots, n\} \to \{0, 1\}$ are functions, then $\mu(f, g, h)$ is the function sending $i \leqslant n$ to whichever of 0 or 1 is chosen by the majority of f, g, h.)

The *rank* of the median algebra (M, μ) is the supremum of the values of n for which M contains a median subalgebra isomorphic to $\{0, 1\}^n$. We will usually restrict our attention to finite-rank median algebras.

Definition 2.6 (Locally convex)**.** A topological median algebra (M, μ) (for example, a median metric space) is *locally convex* if at each point there is a neighbourhood basis consisting of convex sets. □

We will not work with the definition of local convexity directly, but it will be hypothesised in some of our statements whose proofs use foundational results in [Fio20] that hypothesise local convexity. Typically, we will actually assume finite rank, which implies local convexity by [Fio20, Lemma 2.10].

2.1. **Complete geodesic median spaces.** Let $(\mathbf{X}, \mathsf{d}, \boldsymbol{\mu})$ be a median metric space. We will use the following facts freely:

Lemma 2.7 (Lemma 4.6, [Bow16b])**.** *If $(\mathbf{X}, \mathsf{d})$ is complete and connected, then it is a geodesic metric space. Moreover, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, the interval $I(\mathbf{x}, \mathbf{y})$ is the union of all geodesics in $\mathbf{X}$ joining $\mathbf{x}$ to $\mathbf{y}$.*

In any median space,

$$\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \bigcap_{a,b \in \{\mathbf{x},\mathbf{y},\mathbf{z}\},\ a \neq b} I(a, b).$$

(We allow the possibility that, say, $\mathbf{x} = \mathbf{y}$, in which case we think of $\{\mathbf{x}, \mathbf{x}, \mathbf{z}\}$ as containing two distinct copies of $\mathbf{x}$, so that the trivial interval $I(\mathbf{x}, \mathbf{x})$ is a factor in the above intersection and $\boldsymbol{\mu}(\mathbf{x}, \mathbf{x}, \mathbf{z}) = \mathbf{x}$.)

In particular, if $\mathbf{X}$ is complete and connected, then there is a geodesic triangle with vertices $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the unique point lying on all three sides of this triangle.

Lemma 2.8 (Corollary 2.20 [Fio20])**.** *If $(\mathbf{X}, \mathsf{d})$ is complete and $(\mathbf{X}, \boldsymbol{\mu})$ has finite rank, then $I(\mathbf{x}, \mathbf{y})$ is compact for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.*

Lemma 2.9. *Let $(\mathbf{X}, \mathsf{d}, \boldsymbol{\mu})$ be a complete, connected median metric space. Let $\mathbf{Y} \subset \mathbf{X}$ be a subspace. Then $\mathbf{Y}$ is median-convex if and only if $\mathbf{Y}$ contains every geodesic whose endpoints lie in $\mathbf{Y}$.*

Proof. Suppose that $\mathbf{Y}$ is median-convex. Let $\mathbf{x}, \mathbf{y} \in \mathbf{Y}$. Then for any $\mathbf{z} \in I(\mathbf{x}, \mathbf{y})$, we have $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{z} \in \mathbf{Y}$, i.e. $I(\mathbf{x}, \mathbf{y}) \subset \mathbf{Y}$. By Lemma 2.7, every geodesic from $\mathbf{x}$ to $\mathbf{y}$ lies in $\mathbf{Y}$.

Conversely, suppose that $\mathbf{Y}$ contains every geodesic whose endpoints lie in $\mathbf{Y}$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{Y}$ and $\mathbf{z} \in \mathbf{X}$. Since $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ lies on a geodesic from $\mathbf{x}$ to $\mathbf{y}$, it lies in $\mathbf{Y}$. This proves median-convexity. □

An extremely important notion is that of a *gate*. Gates can be defined more generally — in fact tautologically — for *gate-convex subsets* of median algebras, but we will only need the notion of a gate map in a median metric space. In that context, the gate map to a closed, convex subset is exactly the closest-point projection. More precisely:

Lemma 2.10 (Gate map)**.** *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a median metric space with compact intervals, or which is complete. Let $\mathbf{Y}$ be a closed, median-convex subspace. Then there exists a unique retraction $\mathfrak{g} : \mathbf{X} \to \mathbf{Y}$ (the* gate map*) such that for all $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{Y}$, we have*

$$\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathfrak{g}(\mathbf{x})) = \mathfrak{g}(\mathbf{x}).$$

Moreover, $\mathfrak{g}$ is a 1–lipschitz median algebra homomorphism. If $(\mathbf{X}, \mathsf{d})$ is complete and connected, then for all $\mathbf{y} \in \mathbf{Y}$, the point $\mathfrak{g}(\mathbf{x})$ lies on some geodesic from $\mathbf{x}$ to $\mathbf{y}$.

A set $\mathbf{Y} \subset \mathbf{X}$ is *gated* if there is a map $\mathfrak{g} : \mathbf{X} \to \mathbf{Y}$ with the first property mentioned in the lemma. It is not hard to see that such a map, if it exists, is unique.

Proof. See Lemma 2.6 of [Fio20] for an explanation of why closed, median-convex subsets are gated under the compact intervals hypothesis. Lemma 2.13 of [CDH10] implies the same conclusion under the hypothesis that $\mathbf{X}$ is complete. Lemma 2.13 of [CDH10] says that $\mathfrak{g}$ is 1–lipschitz and [Fio20, Proposition 2.1] says it is a median homomorphism. The defining property of the gate map, together with Lemma 2.7, implies the last statement. □

2.2. **Halfspaces and walls.** In a median algebra (M, μ), a *halfspace* is a subset $h \subset M$ such that h and $M - h$ are both nonempty and median-convex.

Letting $\mathcal{H}$ be the set of all halfspaces in M, we have an involution $^* : \mathcal{H} \to \mathcal{H}$ sending each halfspace to its complement. For each $h \in \mathcal{H}$, the pair $\hat{h} = \{h, h^*\}$ is the *wall associated to* h, and given a wall $\hat{h} = \{h, h^*\}$, the halfspaces h and h^* are *associated to* $\hat{h}$. We denote by $\mathcal{W}$ the set of walls.

Here is some terminology. If $\hat{w}, \hat{h} \in \mathcal{W}$, we say that $\hat{h}$ and $\hat{w}$ *cross* if the sets $h \cap w, h \cap w^*, h^* \cap w, h^* \cap w^*$ are all nonempty. If $A \subset M$, we say that $\hat{w}$ *crosses* A to mean that $A \cap h$ and $A \cap h^*$ are both nonempty. We say that $\hat{h}$ *separates* the subsets A, B if $A \subset h, B \subset h^*$ or vice versa. We say that $\hat{h}$ *separates* the walls $\hat{u}, \hat{v}$ if, up to possibly replacing halfspaces with their complements, we have $u \subset h \subset v$.

In a geodesic median metric space, halfspaces are (geodesically) convex, but they may not be gated since they may not be closed. In fact, if $\mathbf{X}$ is a complete median metric space of finite rank, then each halfspace is either open or closed, by [Fio20, Corollary 2.23].

2.3. **Parallelism.** Fix a median metric space $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$. Given a convex subset $\mathbf{Y}$, we write $\mathcal{W}(\mathbf{Y})$ for the set of walls crossing $\mathbf{Y}$, and we write $\mathcal{H}(\mathbf{Y})$ for the set of halfspaces associated to walls crossing $\mathbf{Y}$.

We will need the following list of facts, which is Lemma 2.2 in [Fio20]:

Lemma 2.11. *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete median metric space. Let $\mathbf{Y}, \mathbf{Y}'$ be closed median-convex subsets, and let $\mathfrak{g}, \mathfrak{g}' : \mathbf{X} \to \mathbf{Y}, \mathbf{Y}'$ denote the gate maps. Then:*

1. *$\mathfrak{g}(\mathbf{Y}')$ is convex. If $\mathbf{Y} \cap \mathbf{Y}' \neq \emptyset$, then $\mathfrak{g}(\mathbf{Y}') = \mathbf{Y}' \cap \mathbf{Y}$.*
2. *The gate map to $\mathfrak{g}(\mathbf{Y}')$ is $\mathfrak{g} \circ \mathfrak{g}'$. In particular, if $\mathbf{Y} \subset \mathbf{Y}'$, then $\mathfrak{g} = \mathfrak{g} \circ \mathfrak{g}'$.*
3. *$\mathfrak{g} \circ \mathfrak{g}' \circ \mathfrak{g} = \mathfrak{g} \circ \mathfrak{g}'$.*

Gates relate to halfspaces and walls via the following, which is Proposition 2.3 in [Fio20]:

Lemma 2.12. *Let* $\mathbf{X}, \mathbf{Y}$ *be as in* Lemma 2.11*. Then the assignment* $h \mapsto h \cap \mathbf{Y}$ *defines a bijection from* $\mathcal{H}(\mathbf{Y})$ *(the set of halfspaces associated to walls crossing* $\mathbf{Y}$*) to the set of halfspaces in the median space* $(\mathbf{Y}, \mathsf{d}_1, \boldsymbol{\mu})$*. The inverse map is given by* $h \mapsto \mathfrak{g}^{-1}(h)$*. Moreover, if* $\hat{u}, \hat{v}$ *are walls crossing* $\mathbf{Y}$*, then* $u \subseteq v$ *if and only if* $u \cap \mathbf{Y} \subseteq v \cap \mathbf{Y}$*.*

Lemma 2.13. *Let* $\mathbf{X}, \mathbf{Y}$ *be as in* Lemma 2.11*. Let* $\mathbf{x} \in \mathbf{X}$ *and let* $\hat{w} = \{w, w^*\} \in \mathcal{W}$*. Then* $\hat{w}$ *separates* $\mathbf{x}$ *from* $\mathbf{Y}$ *if and only if* $\hat{w}$ *separates* $\mathbf{x}$ *from* $\mathfrak{g}(\mathbf{x})$*.*

Proof. One direction is clear, since $\mathfrak{g}(\mathbf{x}) \in \mathbf{Y}$. The other direction follows from, for example, Lemma 2.4 of [Fio20]. □

Important to us will be the following notion of *parallelism* for convex subspaces:

Definition 2.14 (Parallel)**.** Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete median metric space and let $\mathbf{Y}, \mathbf{Y}'$ be closed, median convex subspaces. Let $\mathfrak{g}, \mathfrak{g}'$ respectively denote the gate maps to $\mathbf{Y}, \mathbf{Y}'$. Then we say that $\mathbf{Y}, \mathbf{Y}'$ are *parallel* if $\mathfrak{g} : \mathbf{Y}' \to \mathbf{Y}$ and $\mathfrak{g}' : \mathbf{Y} \to \mathbf{Y}'$ are isometries. □

Lemma 2.15. *If* $\mathbf{Y}, \mathbf{Y}'$ *are parallel, then the isometries* $\mathfrak{g}|_{\mathbf{Y}'}$ *and* $\mathfrak{g}'|_{\mathbf{Y}}$ *are inverses. Parallelism is an equivalence relation on closed, median-convex subspaces of* $\mathbf{X}$*. Finally, if* $\mathbf{Y}$ *and* $\mathbf{Y}'$ *are parallel, then* $\mathcal{H}(\mathbf{Y}) = \mathcal{H}(\mathbf{Y}')$*.*

Proof. By Lemma 2.11, the gate map to $\mathfrak{g}(\mathbf{Y}')$ is $\mathfrak{g} \circ \mathfrak{g}'$. On the other hand, since $\mathfrak{g}|_{\mathbf{Y}'}$ is an isometry, $\mathfrak{g}(\mathbf{Y}') = \mathbf{Y}$. So the restriction of $\mathfrak{g} \circ \mathfrak{g}'$ to $\mathbf{Y}$ is the same as the restriction of $\mathfrak{g}$ to $\mathbf{Y}$, i.e. the identity. Apply the same argument, reversing the roles of $\mathfrak{g}, \mathfrak{g}'$ and $\mathbf{Y}, \mathbf{Y}'$. This shows that $\mathfrak{g}|_{\mathbf{Y}'}$ and $\mathfrak{g}'|_{\mathbf{Y}}$ are inverses.

If $\mathbf{Y} = \mathbf{Y}'$, then the restriction of $\mathfrak{g}$ to $\mathbf{Y}$ is the identity, so $\mathbf{Y}$ is parallel to itself. Symmetry of the parallelism relation follows directly from the definition. Transitivity follows from Lemma 2.11 since compositions of isomtries are isometries.

Now we prove that parallel closed, convex subspaces cross the same walls, i.e. $\mathbf{Y}, \mathbf{Y}'$ being parallel implies $\mathcal{H}(\mathbf{Y}) = \mathcal{H}(\mathbf{Y}')$.

Suppose that $\hat{h} = \{h, h^*\} \in \mathcal{W}(\mathbf{Y})$. Equivalently, $h, h^* \in \mathcal{H}(\mathbf{Y})$, i.e. $h \cap \mathbf{Y} \neq \varnothing$ and $h^* \cap \mathbf{Y} \neq \varnothing$. These intersections are disjoint and convex.

Since $\mathfrak{g}'$ is a median homomorphism, $\mathfrak{g}'(h \cap \mathbf{Y}), \mathfrak{g}'(h^* \cap \mathbf{Y})$ are nonempty convex subsets of $\mathbf{Y}'$. Since $\mathfrak{g}'|_{\mathbf{Y}}$ is bijective, $\mathfrak{g}'(h \cap \mathbf{Y}), \mathfrak{g}'(h^* \cap \mathbf{Y})$ are disjoint. Since $\mathbf{Y} = (h \cap \mathbf{Y}) \cup (h^* \cap \mathbf{Y})$, we have $\mathbf{Y}' = \mathfrak{g}'(h \cap \mathbf{Y}) \cup \mathfrak{g}'(h^* \cap \mathbf{Y})$, again because $\mathfrak{g}'|_{\mathbf{Y}}$ is bijective. So $\mathfrak{g}'(h \cap \mathbf{Y}), \mathfrak{g}'(h^* \cap \mathbf{Y})$ are complementary halfspaces in $\mathbf{Y}'$.

By Lemma 2.12, we have that $w = (\mathfrak{g}')^{-1}(\mathfrak{g}'(h \cap \mathbf{Y}))$ and $w^* = (\mathfrak{g}')^{-1}(\mathfrak{g}'(h^* \cap \mathbf{Y}))$ are halfspaces in $\mathcal{H}(\mathbf{Y}')$. But $w \cap \mathbf{Y} = h \cap \mathbf{Y}$ and $w^* \cap \mathbf{Y} = h^* \cap \mathbf{Y}$, so by Lemma 2.12, $w = h, w^* = h^*$. Hence $h, h^* \in \mathcal{H}(\mathbf{Y}')$. A symmetric argument gives the opposite conclusion, so $\mathcal{H}(\mathbf{Y}) = \mathcal{H}(\mathbf{Y}')$, as required. □

2.4. **Measured halfspaces and Fioravanti's construction.** We now recall the notion of *measured halfspaces* from [Fio20], which allows somewhat stronger conclusions than the (perhaps more intuitively appealing for those whose mental image of a CAT(0) cube complex favours hyperplanes over halfspaces) *measured walls* viewpoint from [CDH10] under the assumptions on a median space under which we will work. Specifically, we work with measured halfspaces partly because [Fio20] has very general statements about which sets of halfspaces are measurable, and, perhaps more importantly, the following reason. The walls in a median metric space $\mathbf{X}$ define a *measured wallspace* in the sense of [CDH10], and it is shown in that paper that $\mathbf{X}$ embeds in the median space *dual* to the measured wallspace structure. Using measured halfspaces, one obtains the stronger result that, under reasonable conditions on $\mathbf{X}$, this embedding is surjective (along with related results). This explains the choice to use measured halfspaces rather than measured walls.

Fix a complete connected median metric space $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ of rank $N < \infty$.

Given $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, we let $\mathcal{H}(\mathbf{x}, \mathbf{y})$ be the set of halfspaces h such that $\mathbf{x} \in h$ and $\mathbf{y} \in h^*$ or vice versa. (In other words, $\mathcal{H}(\mathbf{x}, \mathbf{y}) = \mathcal{H}(I(\mathbf{x}, \mathbf{y}))$.)

Let $\mathcal{B}_0 \subset 2^{\mathcal{H}}$ be the σ–algebra generated by the sets $\mathcal{H}(\mathbf{x}, \mathbf{y})$ as $\mathbf{x}, \mathbf{y}$ vary in $\mathbf{X}$. In [Fio20], Fioravanti extends $\mathcal{B}_0$ to a σ–algebra $\mathcal{B}$ of *morally measurable* sets of halfspaces. We just need some of the properties of moral measurability established in [Fio20].

The main property is that the morally measurable sets support a measure fio such that

$$\mathrm{fio}(\mathcal{H}(\mathbf{x}, \mathbf{y})) = \mathsf{d}_1(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ (see [Fio20, Lemma 3.3, Theorem 2.17]). In particular, since d_1 is a metric, any two distinct points in $\mathbf{X}$ must be separated by a wall.

Now we look at some important examples of fio–measurable sets of halfspaces.

First, a set $\mathcal{S}$ of halfspaces is *inseparable* if, for all $u, v \in \mathcal{S}$ and all halfspaces h with $u \subset h \subset v$, we have $h \in \mathcal{S}$. Lemma 3.9 of [Fio20] says that every inseparable set of halfspaces is measurable.

If $\mathbf{Y} \subset \mathbf{X}$ is convex, then $\mathcal{H}(\mathbf{Y})$ is fio–measurable; see [Fio20, Lemma 3.6]. This generalises the fact that $\mathcal{H}(\mathbf{x}, \mathbf{y})$ is measurable. In fact, $\mathcal{H}(\mathbf{Y})$ is inseparable.

The other important examples will be *halfspace filters* and *halfspace ultrafilters*.[4]

Definition 2.16 (Halfspace filter, halfspace ultrafilter)**.** A subset $\sigma \subset 2^{\mathcal{H}}$ is a *halfspace filter* if the following hold:

- for all $h, v \in \sigma$, we have $h \cap v \neq \varnothing$ (i.e. $h \not\subset v^*$);
- if $h \in \sigma$ and $h \subset v$, then $v \in \sigma$.

If the filter σ has the additional property that, for all walls $\{w, w^*\}$, either $w \in \sigma$ or $w^* \in \sigma$, then σ is a *halfspace ultrafilter*.

For example, if $A \subset \mathbf{X}$, then the set σ_A of halfspaces containing A is a filter, and this is an ultrafilter if and only if A is a single point, since any two distinct points are separated by a wall. □

As inseparable sets of halfspaces, halfspace filters (and in particular halfspace ultrafilters) are fio–measurable.

Definition 2.17 (Tangible halfspace filter)**.** The halfspace filter σ is *tangible* if for some, and hence any, $\mathbf{x} \in \mathbf{X}$, we have $\mathrm{fio}(\sigma - \sigma_{\mathbf{x}}) < \infty$. □

For us, the key result from the theory of measured halfspaces is Corollary 3.11 from [Fio20]:

Theorem 2.18. *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete connected median space of finite rank. Then for any tangible halfspace filter σ, there exists a convex subspace C such that* $\mathrm{fio}(\sigma \triangle \sigma_C) = 0$ *(here $\triangle$ denotes symmetric difference).*

In practice, we want C to be gated, but this can be arranged as follows:

Corollary 2.19. *The convex subspace C from* Theorem 2.18 *can always be taken to be closed.*

The proof of the corollary uses the following notation: if $S \subset \mathcal{H}$, then $S^* = \{h^* : h \in S\}$.

Proof. Let C be as provided by Theorem 2.18 and let $\overline{C}$ be its closure. Then $\overline{C}$ is convex. (Indeed, let $\mathbf{x}, \mathbf{y} \in \overline{C}$ and $\mathbf{z} \in \mathbf{X}$. Choose sequences $(\mathbf{x}_n), (\mathbf{y}_n)$ in C converging to $\mathbf{x}, \mathbf{y}$. Then since $\boldsymbol{\mu}$ is 1–lipschitz, $(\boldsymbol{\mu}(\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}))$, which is a sequence in C by convexity, converges to $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, which is therefore in $\overline{C}$, as required.)

Following [Fio20, Section 3], let $Adj(C)$ be the set of halfspaces h such that h intersects $\overline{C}$ but not C. Then $\sigma_C - \sigma_{\overline{C}} = Adj(C)^*$. By [Fio20, Lemma 3.6], $\mathrm{fio}(Adj(C)) = 0$, while by Lemma 2.20, $\mathrm{fio}(Adj(C)^*) = \mathrm{fio}(Adj(C))$. Hence $\mathrm{fio}(\sigma \triangle \sigma_{\overline{C}}) = 0$, as required. □

[4] We use the term "halfspace ultrafilter" where Fioravanti uses "ultrafilter" because later, in our discussions of asymptotic cones, we will often refer to ultrafilters on ℕ, a different notion.

Lemma 2.20. *The measure* fio *is invariant under the involution* *.

Proof. This is essentially contained in [Fio20, Section 2, Section 3]. Specifically, for any measurable set E of halfspaces, $\mathrm{fio}(E)$ is defined in terms of quantities of the form $\eta(E \cap \mathcal{H}(\mathbf{x}, \mathbf{y}))$, where η is a slightly different measure observed in [Fio20, Section 2] (right after Theorem 2.17) to be *–invariant. But $(E \cap \mathcal{H}(\mathbf{x}, \mathbf{y}))^* = E^* \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$, so *–invariance of fio follows from invariance of η and the definition of fio given in [Fio20, Section 3.1]. □

The next lemma supports the following proposition, and is also used in Section 5.1.

Lemma 2.21 (Parallelism criterion)**.** *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete connected median metric space of finite rank. Let A, A' be closed convex subspaces such that*

$$\mathrm{fio}(\mathcal{H}(A) \triangle \mathcal{H}(A')) = 0.$$

Then A and A' are parallel. Hence, in fact, $\mathcal{H}(A) = \mathcal{H}(A')$.

Proof. Once we prove that A and A' are parallel, it follows from the assumption that they are closed and convex, along with Lemma 2.15, that $\mathcal{H}(A) = \mathcal{H}(A')$. So we now verify parallelism.

Let $\mathfrak{g} : \mathbf{X} \to A$ and $\mathfrak{g}' : \mathbf{X} \to A'$ be the gate maps. By definition, it suffices to show that $\mathfrak{g}|_{A'}$ is an isometry (a symmetric argument will apply to $\mathfrak{g}'$).

Let $\mathbf{x}, \mathbf{y} \in A'$. Let $\bar{\mathbf{x}} = \mathfrak{g}(\mathbf{x}), \bar{\mathbf{y}} = \mathfrak{g}(\mathbf{y})$. We have that $\mathsf{d}_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \mathrm{fio}(\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}(A))$. Indeed, the walls separating the gates of $\mathbf{x}, \mathbf{y}$ are precisely those that separate $\mathbf{x}, \mathbf{y}$ and cross A, by Lemma 2.12). Applying the hypothesis that $\mathrm{fio}(\mathcal{H}(A) \triangle \mathcal{H}(A')) = 0$, we thus get that $\mathsf{d}_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \mathsf{d}_1(\mathbf{x}, \mathbf{y})$ since the assumption that $\mathbf{x}, \mathbf{y} \in A'$ implies that $\mathcal{H}(\mathbf{x}, \mathbf{y}) \subset \mathcal{H}(A')$. Hence $\mathfrak{g}$ is an isometric embedding. Similarly, $\mathfrak{g}'$ is an isometric embedding.

On the other hand, the walls separating $\mathfrak{g}'(\mathfrak{g}(\mathbf{x}))$ from $\mathbf{x} = \mathfrak{g}'(\mathbf{x})$ is the set of walls crossing A' and separating $\mathbf{x}$ from $\mathfrak{g}(\mathbf{x})$ and crossing A. Now, no wall crossing A separates $\mathbf{x}$ from $\mathfrak{g}(\mathbf{x})$, so by the hypothesis, $\mathrm{fio}(\mathcal{H}(\mathbf{x}, \mathfrak{g}'(\mathfrak{g}(\mathbf{x}))) = 0$. Hence $\mathfrak{g}'$ is surjective. Similarly, $\mathfrak{g}$ is surjective, so both maps are isometries between A, A', whence A, A' are parallel. □

We will need the following fact about convex product subspaces:

Proposition 2.22 (Product regions)**.** *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete connected median space of finite rank. Let $A, B \subset \mathbf{X}$ be closed, median-convex subspaces. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be* fio*–measurable sets of halfspaces such that the following hold:*

- $\mathrm{fio}(\mathcal{H}_A \triangle \mathcal{H}(A)) = 0$;
- $\mathrm{fio}(\mathcal{H}_B \triangle \mathcal{H}(B)) = 0$;
- *if $h \in \mathcal{H}_A$ and $v \in \mathcal{H}_B$ are associated to walls $\hat{h}, \hat{v}$ respectively, then $\hat{h}$ and $\hat{v}$ cross.*

Then there exist closed, median convex subspaces A', B', respectively parallel to A and B, such that the inclusions A', B' coincide with the restrictions to $A' \times \{b\}$ and $\{a\} \times B'$ of an isometric embedding $A' \times B' \to \mathbf{X}$ with median-convex image (for some $a \in A', b \in B'$).

We give two similar proofs, to illustrate various concepts.

First proof. We will define a filter σ, check that it is tangible, and apply Theorem 2.18 and Corollary 2.19 to produce a closed median-convex subspace P such that, up to a set of measure 0, the halfspaces associated to walls crossing P are exactly those in $\mathcal{H}_A \cup \mathcal{H}_B$. Then we will use a result from [Fio18] to conclude that P decomposes as a product.

Fix an arbitrary basepoint $\mathbf{x} \in \mathbf{X}$, so that $\sigma_{\mathbf{x}}$ denotes the halfspace ultrafilter consisting of halfspaces containing $\mathbf{x}$.

Note that we can assume that $\mathcal{H}_A \subset \mathcal{H}(A)$ and $\mathcal{H}_B \subset \mathcal{H}(B)$, by taking intersections, without affecting our hypothesis. Note that since $\mathcal{H}(A), \mathcal{H}(B)$ are inseparable, and any halfspace separating two halfspaces in $\mathcal{H}_A$ must cross every halfspace in $\mathcal{H}_B$, we can replace

$\mathcal{H}_A, \mathcal{H}_B$ with their inseparable closures without affecting our hypothesis. Finally, we can assume $\mathcal{H}_A, \mathcal{H}_B$ are involution-invariant without affecting our hypothesis.

For each wall $\hat{w}$ that does not cross A or B, we will assign exactly one of the halfspaces w or w^* associated to $\hat{w}$ to our candidate halfspace filter σ. There are cases to consider:

- Suppose $\hat{w}$ separates A from B and crosses all walls in $\mathcal{H}_A$. Then assign to σ whichever of w, w^* contains B.
- Suppose that $\hat{w}$ separates A from B and crosses all walls in $\mathcal{H}_B$ but not all walls in $\mathcal{H}_A$. Then assign to σ whichever of w, w^* contains A.
- Otherwise, $\hat{w}$ cannot separate A from B. In this case, assign to σ whichever of w, w^* contains $A \cup B$.

We claim that σ is a filter. For a wall $\hat{w}$ not crossing A or B, let $\sigma(\hat{w})$ be the associated halfspace belonging to σ. Note that if $\hat{w}, \hat{w}'$ both fall into the same itemised case above, then $\sigma(\hat{w}) \cap \sigma(\hat{w}') \neq \varnothing$ (both halfspaces contain A or both contain B). The same conclusion holds if $\hat{w}$ falls in the third case and $\hat{w}'$ falls in either of the first two.

The only other possibility is thus if $\sigma(\hat{w}) \supset A$ and $\sigma(\hat{w}') \supset B$. This means that $\hat{w}'$ crosses all the walls in $\mathcal{H}_A$, and $\hat{w}$ crosses all those in $\mathcal{H}_B$ but not all those in $\mathcal{H}_A$. But in this case either $\hat{w}, \hat{w}'$ cross, or σ orients them toward each other.

Now, for each wall $\hat{w}$ not crossing A or B, we have chosen an associated halfspace. If u is a halfspace such that $\sigma(\hat{w}) \subset u$, then either $u \in \sigma$, or $\hat{u}$ crosses A or B, and we add u to σ. So, after these additions, σ is a halfspace filter.

Note that $\mathcal{H}_A \cup \mathcal{H}_B$ does not contain any halfspace belonging to σ. Indeed, if $u \in \sigma$, then there exists $\hat{w}$ not crossing A or B with, say, $w \in \sigma$ and $w \subset u$. In each of the three cases above for $\hat{w}$, we see that u cannot contain w if $u \in \mathcal{H}_A \cup \mathcal{H}_B$.

So, σ consists of one halfspace for each wall not crossing A or B, together with some elements of $(\mathcal{H}(A) - \mathcal{H}_A) \cup (\mathcal{H}(B) - \mathcal{H}_B)$.

We now check tangibility. If $h \in \sigma - \sigma_{\mathbf{x}}$, then h separates $\mathbf{x}$ from either A or B, or belongs to the measure 0 set $(\mathcal{H}(A) - \mathcal{H}_A) \cup (\mathcal{H}(B) - \mathcal{H}_B)$, so

$$\operatorname{fio}(\sigma - \sigma_{\mathbf{x}}) \leqslant \mathsf{d}_1(\mathbf{x}, \mathfrak{g}_A(\mathbf{x})) + \mathsf{d}_1(\mathbf{x}, \mathfrak{g}_B(\mathbf{x})),$$

where $\mathfrak{g}_A, \mathfrak{g}_B$ are the gate maps to A and B. Since the latter quantity is finite, the halfspace filter σ is tangible.

Hence Theorem 2.18 and Corollary 2.19 provide a closed, convex P such that, up to a set of measure 0, we have

$$\mathcal{H}(P) = \mathcal{H}_A \sqcup \mathcal{H}_B.$$

By the hypothesis and [Fio18, Proposition 2.10], P decomposes as the product of closed convex subspaces A', B' that cross the same walls as A, B (up to measure-0 sets of halfspaces).

More precisely,

$$\mathcal{H}(P) = (\mathcal{H}(P) \cap \mathcal{H}_A) \sqcup (\mathcal{H}(P) \cap \mathcal{H}_B) \sqcup (\mathcal{H}(P) - (\mathcal{H}_A \cup \mathcal{H}_B)),$$

and all sets involved are fio–measurable. Now, $\mathcal{H}_A$ is $*$–invariant and the same is true for $\mathcal{H}_B$. So to apply [Fio18, Proposition 2.10], we just need to check that

$$\operatorname{fio}((\mathcal{H}(P) - (\mathcal{H}_A \cup \mathcal{H}_B)) = 0.$$

But by construction of σ, we have that $\operatorname{fio}(\mathcal{H}(P) - (\mathcal{H}(A) \cup \mathcal{H}(B)) = 0$, so it suffices to show that

$$\operatorname{fio}((\mathcal{H}(A) \cup \mathcal{H}(B)) - (\mathcal{H}_A \cup \mathcal{H}_B)) = 0.$$

This follows since our hypotheses imply $\operatorname{fio}(\mathcal{H}(A) - \mathcal{H}_A) = \operatorname{fio}(\mathcal{H}(B) - \mathcal{H}_B) = 0$, as required.

Finally, by Lemma 2.21, A and A' are parallel, and B and B' are parallel, completing the proof. □

A variant more conceptual proof of Proposition 2.22 is:

Alternate proof using parallel sets. Let

$$Par(A) = \bigcup_{A''} A'',$$

where A'' ranges over the closed convex subsets parallel to A.

Note that if $\{w, w^*\}$ is a wall crossing $Par(A)$, then by Lemma 2.21, either $w, w^* \in \mathcal{H}(A'')$ for each parallel copy A'' of A, or there exist parallel copies A_1, A_2 of A with $A_1 \subset w, A_2 \subset w^*$. So the set $\mathcal{W}$ of walls crossing $Par(A)$ decomposes as $\mathcal{W} = \mathcal{W}(A) \sqcup \mathcal{O}(A)$, where $\mathcal{O}(A)$ consists of walls crossing every wall in $\mathcal{W}(A)$.

Let P be the intersection of all halfspaces containing $Par(A)$, so P is convex as the intersection of convex subspaces, and $\mathcal{W}(P) = \mathcal{W}(A) \sqcup \mathcal{O}(A)$. So the closure $\overline{P}$ is a closed convex set, and as in Corollary 2.19, $\mathcal{H}(\overline{P})$ is the disjoint union of the (measureable, $*$–invariant) sets of halfspaces associated to walls in $\mathcal{W}(A)$ and $\mathcal{O}(A)$, together with a (necessarily $*$–invariant) measure–0 set. So, by [Fio18, Proposition 2.10] and Lemma 2.21, $\overline{P} = A \times C$, where C is a closed convex set.

Take $B' = \mathfrak{g}_{\overline{P}}(B)$. From our assumptions, $\mathfrak{g}_A(B)$ is crossed by a measure–0 set of halfspaces, and is thus a point, so B' is parallel to a subset of C and hence P contains a product of the form $A' \times B'$ where A' is a parallel copy of A.

So, to conclude, we need to show that B is parallel to B', i.e. that $\mathfrak{g}_{\overline{P}} : B \to B'$ is an isometry. If not, then there exist distinct $\mathbf{b}, \mathbf{b'} \in B$ such that $\mathcal{H}(\mathbf{b}, \mathbf{b'}) \cap \mathcal{H}_B$ contains a positive measure set $\mathcal{S}$ of halfspaces s with $\overline{P} \subset s$. Since the measure is positive, we can choose $s \in \mathcal{S}$ such that $\overline{P} \cap \overline{s^*} = \varnothing$, by Lemma 2.23 below.

Let A_s be the image of A under the gate map to $\overline{s^*}$. Since all of the walls in $\mathcal{H}_A$ cross $\hat{s}$, and hence $\overline{s^*}$, our hypothesis and Lemma 2.21 ensures A, A_s are parallel. So $A_s \subset \overline{s^*} \cap \overline{P}$, a contradiction. Thus B, B' are parallel, as required. □

The next lemma supports the above proof but is not used elsewhere.

Lemma 2.23. *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete connected median space of finite rank. Let $Q \subset \mathbf{X}$ be closed and convex. Let $\mathcal{S}$ be a* fio*–measurable set of halfspaces such that $Q \subset s$ and $Q \cap \overline{s^*} \neq \varnothing$ for all $s \in \mathcal{S}$. Then* $\mathrm{fio}(\mathcal{S}) = 0$.

Proof. By the definition of fio, it suffices to show that $\mathrm{fio}(\mathcal{S} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})) = 0$ for all $\mathbf{x}, \mathbf{y}$, so we can assume $\mathcal{S} \subset \mathcal{H}(\mathbf{x}, \mathbf{y})$ for some $\mathbf{x}, \mathbf{y}$. Hence the elements of $\mathcal{S}$ are partially ordered by inclusion, and two elements are incomparable if and only if they are either disjoint, or their associated walls cross. Now, all elements of $\mathcal{S}$ contain Q, and any pairwise-crossing set of walls has cardinality at most the rank $\mathrm{rk}(\mathbf{X}) < \infty$, so antichains have size at most $\mathrm{rk}(\mathbf{X})$. Dilworth's theorem [Dil50] thus allows us to write $\mathcal{S}$ as the disjoint union of at most $\mathrm{rk}(\mathbf{X})$ chains. Let $s_1 \subsetneq \cdots \subsetneq s_k$ be a chain in $\mathcal{S}$. Choose $\mathbf{q} \in Q \cap \overline{s_1^*}$, which exists by hypothesis. Then $\mathbf{q} \in s_k \cap \overline{s_1^*}$, so $\mathsf{d}_1(s_1^*, s_k) = 0$. Proposition 2.26 in [Fio20] implies $k \leqslant 2\,\mathrm{rk}(\mathbf{X})$, so $|\mathcal{S}| \leqslant 2\,\mathrm{rk}(\mathbf{X})^2$. Finally, since $\mathbf{X}$ is connected, singletons have measure 0 ([Fio20, Lemma 3.5]), so $\mathrm{fio}(\mathcal{S}) = 0$, as required. □

2.5. **ℝ–trees as median spaces, and product medians.** Let $\mathfrak{F}^\bullet$ be an arbitrary set, and for each $\mathbf{U} \in \mathfrak{F}^\bullet$, let $(\mathcal{T}^\bullet\mathbf{U}, \mathbf{D}_\mathbf{U}, \boldsymbol{\mu}_\mathbf{U})$ be a connected median metric space with compact intervals. Then $\mathcal{T}^\bullet\mathbf{U}$ is a geodesic space (we are not assuming it is complete). Fixing a basepoint $\mathbf{1}_\mathbf{U} \in \mathcal{T}^\bullet\mathbf{U}$ for each $\mathbf{U}$, we let $\ell_1(\mathfrak{F}^\bullet)$ be the set of $(\mathbf{x}_\mathbf{U})_{\mathbf{U} \in \mathfrak{F}^\bullet} \in \prod_\mathbf{U} \mathcal{T}^\bullet\mathbf{U}$ such that

$$\mathsf{d}_1(\mathbf{1}, (\mathbf{x}_\mathbf{U})) = \sum_\mathbf{U} \mathbf{D}_\mathbf{U}(\mathbf{1}_\mathbf{U}, \mathbf{x}_\mathbf{U}) < \infty.$$

(Here we use the convention that the above sum is taken over the nonzero terms, so that the condition that the sum is finite implies that it has countably many nonzero terms. The order of summation is then immaterial since the terms are nonnegative.)

Then it is readily checked that $\ell_1(\mathfrak{F}^\bullet)$, with the ℓ_1 metric d_1, is a median metric space, and the median is determined by the media $\boldsymbol{\mu}_{\mathbf{U}}$ in the various coordinates.

The median space $\ell_1(\mathfrak{F}^\bullet)$ is a bit scary, e.g. it can have infinite rank, and therefore has some surprising walls (see e.g. [Fio20, Example 2.24]). We will be working with finite-rank, complete, connected median subspaces of such spaces, which are much better behaved. We will usually be interested in the case where each $\mathcal{T}^\bullet\mathbf{U}$ is an $\mathbb{R}$–tree (i.e. a 0–hyperbolic geodesic metric space), using the following:

Proposition 2.24. *Every $\mathbb{R}$–tree is a geodesic median metric space of rank at most one. Conversely, suppose that (T, d) is a metric space with the following properties:*

- *for all $a, b \in T$, there is a unique geodesic joining a to b;*
- *for all $a, b, c \in T$, there exists a point m such that m lies on the geodesic from x to y, whenever $x, y \in \{a, b, c\}$ are distinct. (We think of $\{a, b, c\}$ as a multiset so that, if $a = b$, we require m to lie on the geodesic from a to a, i.e. $m = a$.)*

Then (T, d) is an $\mathbb{R}$–tree, and the assignment $(a, b, c) \mapsto m$ is the median operator.

A useful feature of the above proposition is that, to check that a uniquely geodesic space is an $\mathbb{R}$–tree, one must produce the point m (given and triple a, b, c), but one need not show that it is unique (as would be required to check straight from the definition that (T, d) is a median space).

Proof of Proposition 2.24. Every $\mathbb{R}$–tree is a geodesic space by definition, and is a median space of rank at most one by, e.g. [Bow14].

Conversely, let (T, d) be a geodesic space satisfying the properties listed in the statement. Let $a, b, c \in T$, and let m be as in the statement. Then m satisfies

$$d(x, y) = d(x, m) + d(y, m)$$

whenever $x, y \in \{a, b, c\}$ are distinct (again, viewing $\{a, b, c\}$ as a multiset), as required for a median. Since the unique geodesic from a to b passes through m, and the same is true with either a or b replaced by c, uniqueness of the geodesics from m to each of a, b, c, imply that T is 0–hyperbolic. Hence T is an $\mathbb{R}$–tree (and thus a median metric space of rank 1, with median m). ☐

This concludes our review of median spaces. We will sporadically use a few facts not covered here, directing the reader to the appropriate place in the literature where needed.

3. Poset-colourings

This section is about colouring the walls in a median space by partially ordered sets of colours; we will use these colourings later to characterise real cubings among median spaces. Roughly, Definition 3.1 is a general definition of such a poset-colouring sufficient to ensure that the median space in question is isometric to a *real cubing* — real cubings will be defined in Section 4 and the sufficiency of a poset-colouring for constructing a real cubing structure is Theorem 5.1. This theorem relies on two technical assumptions on the poset-colouring — *finite depth*, which asks for a bound on the lengths of chains in the poset — and *tangibility*, which asks that the sets of walls whose colour is bounded above by a given colour naturally determine a tangible filter. Theorem 5.1 is what we will use later to show that asymptotic cones of hierarchically hyperbolic spaces are bilipschitz equivalent to real cubings: we will see that such asymptotic cones are median, and then use the hierarchically hyperbolic structure to poset-colour the walls.

Informally, tangibility is what allows us to construct, for each colour $\mathbf{U}$, a closed convex subspace $\mathbf{F}_{\mathbf{U}}$ such that the walls crossing $\mathbf{F}_{\mathbf{U}}$ are, up to a measure-0 set, those whose colour non-strictly precedes $\mathbf{U}$ in the partial order.

In this section, we will also study a canonical colouring of the walls in a median space by a poset, called the *orthogonal poset-colouring*. This is not needed for the applications to asymptotic cones, but it is of independent interest. Indeed, we will show that the finite depth assumption implies the tangibility assumption in the case of the orthogonal poset-colouring, and we will use this to characterise "well-behaved" real cubings among complete, connected, finite-rank median spaces in Corollary 6.9.

3.1. **Poset-colourings and associated filters.** In this section, $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ is a median metric space.

Later, we will impose additional conditions, namely completeness and connectedness of $(\mathbf{X}, \mathsf{d}_1)$ (which guarantee that it is a geodesic space) and local convexity of $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ or the stronger condition of finite rank.

However, we do not need these conditions yet. The reader should have in mind the situation where $\mathbf{X}$ is complete, connected, and of finite rank, but also the situation where $\mathbf{X}$ is a discrete median algebra, i.e. the 0–skeleton of a CAT(0) cube complex.

We let $\mathcal{W}$ denote the set of walls in $\mathbf{X}$, and $\mathcal{H}$ the set of halfspaces. We refer the reader to Section 2.2 for the background on halfspaces needed in this subsection; we will also use the measure fio from Section 2.4.

We next define a *poset-colouring* on $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$.

Definition 3.1 (Poset-colouring, depth)**.** A *poset-colouring* is a map $Col : \mathcal{W} \to (\mathfrak{F}^\bullet, \sqsubseteq)$, where $(\mathfrak{F}^\bullet, \sqsubseteq)$ is a partially ordered set with a unique $\sqsubseteq$–maximal element, denoted $\mathbf{S}$, such that $(\mathfrak{F}^\bullet, \sqsubseteq)$ has the following properties.

First, for a given colour $\mathbf{U} \in \mathfrak{F}^\bullet$, let

$$\mathcal{W}_{\mathbf{U}} = \bigcup_{\mathbf{V} \sqsubseteq \mathbf{U}} Col^{-1}(\mathbf{V})$$

be the set of walls whose colours precede (or coincide with) $\mathbf{U}$ in the partial order. Let $\mathcal{H}_{\mathbf{U}}$ be the set of halfspaces associated to walls in $\mathcal{W}_{\mathbf{U}}$. Given a set $\mathcal{A}$ of walls, we let $Col(\mathcal{A})$ denote the set of colours in $\mathfrak{F}^\bullet$ arising as colours of walls in $\mathcal{A}$. We say that, e.g. $Col(\mathcal{A}) \sqsubseteq \mathbf{V}$ to mean that $Col(a) \sqsubseteq \mathbf{V}$ for all $a \in \mathcal{A}$.

We require the following to hold for all $\hat{v}, \hat{h}, \hat{u} \in \mathcal{W}$:

(I) For each $\mathbf{U} \in \mathfrak{F}^\bullet$, the set $\mathcal{W}_{\mathbf{U}}$ is inseparable. (In particular, if $\hat{u}$ separates $\hat{h}, \hat{v}$, and $Col(\hat{h}) = Col(\hat{v})$, then $Col(\hat{u}) \sqsubseteq Col(\hat{h})$. Note also that inseparability of $\mathcal{W}_{\mathbf{U}}$ implies inseparability, and hence fio–measurability, of $\mathcal{H}_{\mathbf{U}}$ [Fio20, Lemma 3.9].)

(II) Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet$. If there exist nonempty sets $\mathcal{A} \subset \mathcal{W}_{\mathbf{U}}$ and $\mathcal{B} \subset \mathcal{W}_{\mathbf{V}}$ such that $\mathrm{fio}(\mathcal{H}_{\mathbf{U}} - \mathcal{H}_{\mathcal{A}}) = 0$, $\mathrm{fio}(\mathcal{H}_{\mathbf{V}} - \mathcal{H}_{\mathcal{B}}) = 0$ and each wall in $\mathcal{A}$ crosses each wall in $\mathcal{B}$, then $\mathbf{U}$ and $\mathbf{V}$ are $\sqsubseteq$–incomparable.

(III) Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet$. Suppose we have an inseparable set $\mathcal{A}$ of walls such that $Col(\mathcal{A}) \sqsubseteq \mathbf{U}, \mathbf{V}$ and the set $\mathcal{H}_{\mathcal{A}}$ of halfspaces associated to $\mathcal{A}$ has positive fio–measure. Then there exists a family $\{\mathbf{W}_i\}_{i \in I}$ of elements in $\mathfrak{F}^\bullet$ such that
   - $\mathrm{fio}(\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{H}_{\mathcal{A}}) > 0$ for all $i \in I$, and
   - $\mathbf{W}_i \sqsubseteq \mathbf{U}, \mathbf{V}$ for all $i \in I$, and
   - $\mathrm{fio}\left(\mathcal{H}_{\mathcal{A}} - \bigcup_i (\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{H}_{\mathcal{A}})\right) = 0$.

   The third condition says that, up to a subset of $\mathcal{A}$ whose set of associated halfspaces has measure 0, we have that $\hat{a} \in \mathcal{A}$ implies $Col(\hat{a}) \sqsubseteq \mathbf{W}_i$ for some $i \in I$.

(IV) Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet$ (we allow for the possibility that $\mathbf{U} = \mathbf{V}$). Suppose that there exist inseparable sets $\mathcal{A}, \mathcal{B}$ of walls such that $Col(\mathcal{A}) \sqsubseteq \mathbf{U}, Col(\mathcal{B}) \sqsubseteq \mathbf{V}$ and every wall in $\mathcal{A}$ crosses every wall in $\mathcal{B}$. Let $\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}$ be the sets of halfspaces associated to $\mathcal{A}$ and $\mathcal{B}$ and suppose that both sets of halfspaces have positive measure. Then there exist families $\{\mathbf{U}_i\}_{i \in I}$ and $\{\mathbf{V}_j\}_{j \in J}$ of elements in $\mathfrak{F}^\bullet$ such that

- $\mathbf{U}_i \sqsubseteq \mathbf{U}$ for all $i \in I$,
- $\mathbf{V}_j \sqsubseteq \mathbf{V}$ for all $j \in J$,
- for $i \in I, j \in J$, any wall in $\mathcal{W}_{\mathbf{U}_i}$ and any wall in $\mathcal{W}_{\mathbf{V}_j}$ cross,
- $\mathrm{fio}\left(\mathcal{H}_{\mathcal{A}} - \bigcup_i (\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i})\right) = 0$ and $\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i}) > 0$ for all $i \in I$, and
- $\mathrm{fio}\left(\mathcal{H}_{\mathcal{B}} - \bigcup_j (\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j})\right) = 0$ and $\mathrm{fio}(H_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j}) > 0$ for all $j \in J$.

The last two conditions say that, up to subsets of $\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}$ of measure 0, we have $Col(\hat{a}) \sqsubseteq \mathbf{U}_i, Col(\hat{b}) \sqsubseteq \mathbf{V}_j$ for some $i \in I, j \in J$, when $\hat{a} \in \mathcal{A}, \hat{b} \in \mathcal{B}$.

Col is a *depth D* poset-colouring, for $D \in \mathbb{N}$, if every $\sqsubseteq$–chain has length at most D and D is minimal with this property. We will use the term *nesting* to refer to the partial order $\sqsubseteq$ in a poset-colouring. □

Remark 3.2. [Conditions (III), (IV) will be applied to subsets of halfspace-intervals] In practice, we apply condition (III) to a set $\mathcal{A}$ such that $\mathcal{H}_{\mathcal{A}} = \mathcal{H}_{\mathbf{U}} \cap \mathcal{H}_{\mathbf{V}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. The condition assumes that this set has positive measure, and provides $\{\mathbf{W}_i\}_{i \in I}$ such that each $\mathbf{W}_i \sqsubseteq \mathbf{U}, \mathbf{W}_i \sqsubseteq \mathbf{V}$, and each i satisfies $\mathrm{fio}(\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{H}_{\mathcal{A}}) > 0$. The condition says that, up to a measure 0 set, $\mathcal{H}_{\mathcal{A}}$, which is contained in the halfspace interval $\mathcal{H}(\mathbf{x}, \mathbf{y})$, is covered by positive-measure inseparable sets $\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_{\mathbf{W}_i}$. This is how the condition functions in Theorem 5.1, for instance (and condition (IV) has a similar role). □

Fix a poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^{\bullet}$ as in Definition 3.1. We can use $\mathcal{W}_{\mathbf{U}}$ to associate to $\mathbf{U}$ a halfspace filter as follows.

Definition 3.3 (Filter associated to a colour)**.** Fix a basepoint $\mathbf{x}_0 \in \mathbf{X}$. Fix a colour $\mathbf{U} \in \mathfrak{F}^{\bullet}$. Recall that

$$\mathcal{W}_{\mathbf{U}} = \bigcup_{\mathbf{V} \sqsubseteq \mathbf{U}} Col^{-1}(\mathbf{V}).$$

Suppose that $\mathcal{W}_{\mathbf{U}}$ is nonempty. Let $\mathcal{W}_{\mathbf{U}}^{\perp}$ be the set of walls $\hat{w}$ such that $\hat{w}$ crosses $\hat{h}$ for all $\hat{h} \in \mathcal{W}_{\mathbf{U}}$.

Let $\hat{h} \in \mathcal{W} - \mathcal{W}_{\mathbf{U}}$. Then, up to relabelling h and h^*, one of the following holds:

- $\hat{h} \in \mathcal{W}_{\mathbf{U}}^{\perp}$ and h is the halfspace associated to $\hat{h}$ that contains $\mathbf{x}_0$.
- For some $\hat{w} = \{w, w^*\} \in \mathcal{W}_{\mathbf{U}}$ we have $w \subset h$. Moreover, for any $\hat{w}' \in \mathcal{W}_{\mathbf{U}}$ not crossing $\hat{h}$, some halfspace associated to $\hat{w}'$ is contained in h, by the condition on separation in Definition 3.1.

Let $\sigma_{\mathbf{U}}$ be the set of halfspace h thus chosen, as $\hat{h}$ varies in $\mathcal{W} - \mathcal{W}_{\mathbf{U}}$. See Figure 5. □

The next observation is an immediate consequence of the definition of a filter and the definition of a poset-colouring:

Lemma 3.4. *For each colour $\mathbf{U} \in \mathfrak{F}^{\bullet}$, the set $\sigma_{\mathbf{U}}$ is a filter. In particular, $\sigma_{\mathbf{U}}$ is* fio*–measurable, and so is $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$, where $\sigma_{\mathbf{x}_0}$ is the set of halfspaces containing $\mathbf{x}_0$.*

Proof. Once we show that $\sigma_{\mathbf{U}}$ is a filter, then Lemma 3.9 in [Fio20] will imply that it is fio–measurable, and so is $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$.

First observe that if $a, b \in \sigma_{\mathbf{U}}$, then we cannot have $a \subset b^*$. Indeed, let $\hat{a}, \hat{b}$ be the walls respectively associated to a, b. If $\hat{a}, \hat{b} \in \mathcal{W} - (\mathcal{W}_{\mathbf{U}} \cup \mathcal{W}_{\mathbf{U}}^{\perp})$, then this would violate the separation condition in Definition 3.1. If $\hat{a}, \hat{b} \in \mathcal{W}_{\mathbf{U}}^{\perp}$, then a, b both contain $\mathbf{x}_0$. Finally, if $\hat{a} \in \mathcal{W}_{\mathbf{U}}^{\perp}$ and $\hat{b} \notin \mathcal{W}_{\mathbf{U}}^{\perp}$, then there exists $\hat{w} \in \mathcal{W}_{\mathbf{U}}$ with $w \subset b$ and $\hat{a}, \hat{w}$ crossing. So $a \cap w \neq \emptyset$, so $a \cap b \neq \emptyset$.

Next observe that if $a \in \sigma_{\mathbf{U}}$ and $a \subset b$, then $b \in \sigma_{\mathbf{U}}$, by definition of $\sigma_{\mathbf{U}}$. Hence $\sigma_{\mathbf{U}}$ is a filter. □

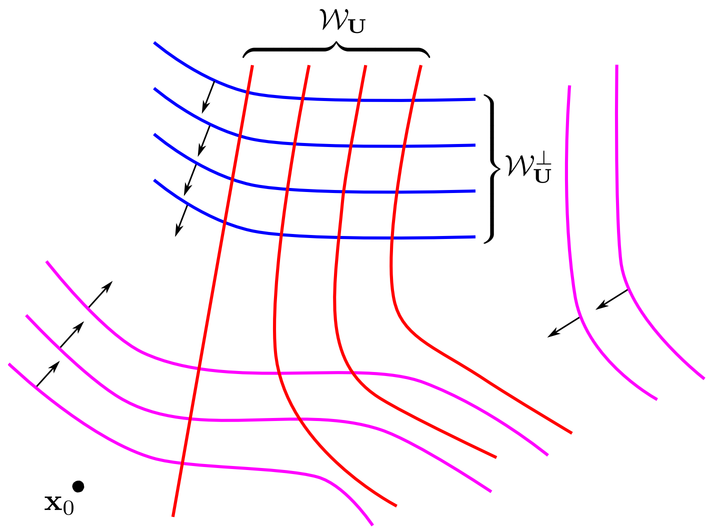

FIGURE 5. Constructing the filter $\sigma_{\mathbf{U}}$. The unoriented walls are in $\mathcal{W}_{\mathbf{U}}$, while the other walls are oriented toward the halfspace in $\sigma_{\mathbf{U}}$.

The point of the following condition is to allow one to construct, for each colour in a poset-colouring, a corresponding closed, median-convex subspace of $\mathbf{X}$, under the hypotheses that $\mathbf{X}$ is complete, connected, and locally convex, using Theorem 2.18 and Corollary 2.19. We have isolated the tangible filter condition because, in our applications to asymptotic cones of hierarchically hyperbolic spaces, there will be a natural poset-colouring for which both finite depth and the below tangible filter condition hold by construction. However, there are poset-colourings that exist more generally, and for these we will check below that the tangible filter condition actually follows from finite depth. This is not strictly needed for the application to asymptotic cones, but it will help with the independently interesting question of which median spaces are ℝ–cubings.

Definition 3.5 (Tangible filter condition)**.** Fix a basepoint $\mathbf{x}_0 \in \mathbf{X}$. The poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^{\bullet}$ satisfies the *tangible filter condition* if for each colour $\mathbf{U} \in \mathfrak{F}^{\bullet}$ such that $\mathcal{W}_{\mathbf{U}} \neq \emptyset$, the filter $\sigma_{\mathbf{U}}$ is *tangible* in the sense of [Fio20, Section 3], i.e. for some (hence any) $\mathbf{x} \in \mathbf{X}$, we have

$$\mathrm{fio}(\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}}) < \infty.$$

The "hence any" bears some explanation since $\sigma_{\mathbf{U}}$ depended on our choice of $\mathbf{x}_0$. If we had defined $\sigma'_{\mathbf{U}}$ in the exact same way, except with reference to a basepoint $\mathbf{x}_1$, then we would have $\sigma_{\mathbf{U}} \triangle \sigma'_{\mathbf{U}} \subset \mathcal{H}(\mathbf{x}_0, \mathbf{x}_1)$, which has measure $\mathsf{d}_1(\mathbf{x}_0, \mathbf{x}_1) < \infty$. The other part of "hence any" — the a priori dependence of the definition of tangibility on $\mathbf{x}_0$ for a *fixed* filter σ, is handled by a similar argument in [Fio20]. □

Example 3.6 (Sector non-example)**.** Let $\mathbf{X}$ be the median metric space whose underlying set is the subset of points $(x, y) \in \mathbb{R}^2$ with $0 \leqslant x$ and $0 \leqslant y \leqslant x$, equipped with the ℓ_1 metric and the median inherited as a median subalgebra of $(\mathbb{R}^2, \ell_1)$. Walls are either (finite-length) vertical line segments or (infinite) horizontal rays. We colour $\mathcal{W}$ with two colours $\mathbf{V}$ (vertical) and $\mathbf{H}$ (horizontal) accordingly. We add an artificial $\sqsubseteq$–maximal colour $\mathbf{S}$ with $\mathbf{H}, \mathbf{V} \sqsubseteq \mathbf{S}$. The colours $\mathbf{H}, \mathbf{V}$ are not $\sqsubseteq$–related. This colouring has all of the properties from Definition 3.1 except (IV), and it fails to have the tangible filter property: there is a subspace, namely the horizontal axis, crossing exactly the vertical walls, but there is no subspace crossing exactly the horizontal walls. □

3.2. ☺ **Orthogonal poset-colouring, tangibility, and finite depth.** Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a median metric space. As usual, we let $\mathcal{W}$ be the set of walls, $\mathcal{H}$ the set of halfspaces, and equip $\mathcal{H}$ with the measure fio; see Section 2.4.

In our application to asymptotic cones of hierarchically hyperbolic spaces, we will be working with a median space with a pre-existing poset-colouring (in fact, even the filters $\sigma_{\mathbf{U}}$ will come from "naturally occurring" subspaces in that setting). However, one can construct a poset-colouring in much more generality, namely an *orthogonal poset-colouring.*

The next definition abstracts the useful properties of an orthogonal poset-colouring. The actual construction occurs in the next subsection.

Definition 3.7 (Orthogonal set, orthogonal poset-colouring)**.** Let $(\mathfrak{F}^{\bullet}, \sqsubseteq)$ be a partially ordered set, and let $\perp$ be a symmetric relation on $\mathfrak{F}^{\bullet}$, and suppose that the following hold for all $\mathbf{U}, \mathbf{V}, \mathbf{W} \in \mathfrak{F}^{\bullet}$:

- $\mathbf{U} \not\perp \mathbf{U}$;
- $\mathfrak{F}^{\bullet}$ has a unique $\sqsubseteq$–maximal element;
- if $\mathbf{U} \sqsubseteq \mathbf{V}$ and $\mathbf{V} \perp \mathbf{W}$, then $\mathbf{U} \perp \mathbf{W}$;
- **(Wedges.)** if $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$, then there exists a unique $\sqsubseteq$–maximal element that $\sqsubseteq$–precedes both $\mathbf{U}$ and $\mathbf{V}$, i.e. there exists $\mathbf{U} \wedge \mathbf{V} \sqsubseteq \mathbf{U}, \mathbf{V}$ such that for all $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$ we have that $\mathbf{W} \sqsubseteq \mathbf{U} \wedge \mathbf{V}$;
- **(Clean containers.)** for all $\mathbf{U} \in \mathfrak{F}^{\bullet}$ such that there exists $\mathbf{V}$ with $\mathbf{V} \perp \mathbf{U}$, there exists $\mathbf{U}^{\perp} \in \mathfrak{F}^{\bullet}$ such that for all $\mathbf{V} \perp \mathbf{U}$ we have that $\mathbf{V} \sqsubseteq \mathbf{U}^{\perp}$ and $\mathbf{U}^{\perp} \perp \mathbf{W}$ if and only if $\mathbf{W} \sqsubseteq \mathbf{U}$;
- **(Nesting is determined by orthogonality.)** $\mathbf{U} \sqsubseteq \mathbf{V}$ (resp. $\mathbf{U} \sqsubsetneq \mathbf{V}$) if and only if the set of $\mathbf{W}$ for which $\mathbf{V} \perp \mathbf{W}$ is contained (resp. properly contained) in the set of $\mathbf{W}'$ for which $\mathbf{W}' \perp \mathbf{U}$. In particular, if nothing is $\perp$–related to $\mathbf{V}$, then $\mathbf{V}$ is the unique $\sqsubseteq$–maximal element, and if there exists $\mathbf{W} \perp \mathbf{V}$, then $\mathbf{V}^{\perp} \sqsubseteq \mathbf{U}^{\perp}$ (resp. $\mathbf{V}^{\perp} \sqsubsetneq \mathbf{U}^{\perp}$) if and only if $\mathbf{U} \sqsubseteq \mathbf{V}$ (resp. $\mathbf{U} \sqsubsetneq \mathbf{V}$).

A set $(\mathfrak{F}^{\bullet}, \sqsubseteq, \perp)$ with the above properties is called *orthogonal.* A poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^{\bullet}$ where:

- $(\mathfrak{F}^{\bullet}, \sqsubseteq, \perp)$ is an orthogonal set;
- for all $\hat{h}, \hat{v} \in \mathcal{W}$, the walls $\hat{v}, \hat{h}$ cross if and only if $Col(\hat{h}) \perp Col(\hat{v})$.

is an *orthogonal poset-colouring* . □

Here are some important combinatorial properties of an orthogonal poset-colouring.

Lemma 3.8. *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a median metric space and denote by $\mathcal{W}$ the set of walls. Let $Col_{\perp} : \mathcal{W} \to (\mathfrak{F}^{\bullet}_{\perp}, \sqsubseteq, \perp)$ be an orthogonal poset-colouring. Then,*

- $\mathbf{U}^{\perp\perp} = \mathbf{U}$ *for all* $\mathbf{U} \in \mathfrak{F}^{\bullet}_{\perp}$*;*
- *if* $\mathbf{U} \sqsubsetneq \mathbf{V}$*, then* $\mathcal{W}_{\mathbf{U}} \subsetneq \mathcal{W}_{\mathbf{V}}$*.*

Proof. By the definition of orthogonal complements, we have $\mathbf{U} \sqsubseteq (\mathbf{U}^{\perp})^{\perp}$. Indeed, $\mathbf{U} \perp \mathbf{U}^{\perp}$. Observe also that $((\mathbf{U}^{\perp})^{\perp})^{\perp} = \mathbf{U}^{\perp}$. Indeed, the right side is orthogonal to $(\mathbf{U}^{\perp})^{\perp}$ and hence nested in the left side. On the other hand, since $\mathbf{U} \sqsubseteq (\mathbf{U}^{\perp})^{\perp}$, we have $\mathbf{U} \perp ((\mathbf{U}^{\perp})^{\perp})^{\perp}$, so $((\mathbf{U}^{\perp})^{\perp})^{\perp} \sqsubseteq \mathbf{U}^{\perp}$.

Suppose that $\mathbf{U} \sqsubsetneq (\mathbf{U}^{\perp})^{\perp}$. Then, taking orthogonal complements and using that orthogonality determines nesting, we get $\mathbf{U}^{\perp} \sqsubsetneq \mathbf{U}^{\perp}$, which is impossible. Hence $\mathbf{U} = (\mathbf{U}^{\perp})^{\perp}$.

Suppose now that $\mathbf{U} \sqsubsetneq \mathbf{V}$. Since orthogonality determines nesting, we have proper containment of orthogonal sets and so there exists $\mathbf{W} \in \mathfrak{F}^{\bullet}_{\perp}$ such that $\mathbf{W} \perp \mathbf{U}$ and $\mathbf{W} \not\perp \mathbf{V}$. Since in an orthogonal poset-colouring, orthogonality of colours is determined by crossing of walls, we have that each wall in $\mathcal{W}_{\mathbf{W}}$ crosses each wall in $\mathcal{W}_{\mathbf{U}}$ and there exists $\hat{h} \in \mathcal{W}_{\mathbf{V}}$ that does not cross a wall in $\mathcal{W}_{\mathbf{W}}$. It follows that $\mathcal{W}_{\mathbf{U}} \subsetneq \mathcal{W}_{\mathbf{V}}$. □

Remark 3.9 (Joins, and wedges of sets)**.** Let $(\mathfrak{F}^\bullet, \sqsubseteq, \perp)$ be an orthogonal set such that any $\sqsubseteq$–chain has a minimal and a maximal element. Suppose that $\{\mathbf{U}_i\}_{i\in I} \subset \mathfrak{F}^\bullet$ has the property that there exists $\mathbf{V}$ such that $\mathbf{U}_i \sqsubseteq \mathbf{V}$ for all i. Then the wedge property and fact that any chain has a minimal element implies there is a unique $\sqsubseteq$–minimal such $\mathbf{V}$, called the *join* $\bigvee_i \mathbf{U}_i$ of the $\mathbf{U}_i$.

On the other hand, if $\mathbf{U}_i$ are such that there is some $\mathbf{V}$ with $\mathbf{V} \sqsubseteq \mathbf{U}_i$ for all i, then let $\mathbf{W}$ be the join of all possible such $\mathbf{V}$, i.e. $\mathbf{W}$ is $\sqsubseteq$–minimal with the property that it contains every $\mathbf{V}$ that is contained in all $\mathbf{U}_i$. Now, since we could replace $\mathbf{W}$ by $\mathbf{W} \wedge \mathbf{U}_i$ for any i and retain the latter property, we see that $\mathbf{W} \sqsubseteq \mathbf{U}_i$ for all i. So $\mathbf{W}$ is the unique maximal $\mathbf{W}$ that is nested into each $\mathbf{U}_i$, and it is sensible to refer to $\mathbf{W}$ as $\bigwedge_i \mathbf{U}_i$. In practice, finite depth will enable us to talk about $\bigvee_i \mathbf{U}_i$ and $\bigwedge_i \mathbf{U}_i$. □

Remark 3.10 (Verifying poset-colourings using wedges and clean containers)**.** Given an orthogonal set $(\mathfrak{F}^\bullet, \sqsubseteq, \perp)$ and a map $Col : \mathcal{W} \to \mathfrak{F}^\bullet$, the "wedges" and "clean containers" property make it easier to verify that Col is a poset-colouring, at least when the wedge operator is defined for arbitrary subsets of $\mathfrak{F}^\bullet$, i.e. for any $\{\mathbf{U}_i\}_i \subset \mathfrak{F}^\bullet$ for which there is some $\mathbf{V}$ with $\mathbf{V} \sqsubseteq \mathbf{U}_i$ for all i, there is a unique $\sqsubseteq$–maximal such $\mathbf{V}$, denoted $\bigwedge_i \mathbf{U}_i$. Remark 3.9 describes one situation in which this occurs.

Suppose that we have verified that each $\mathcal{W}_{\mathbf{U}}$ is inseparable, as demanded by Definition 3.1, and that $\hat{h}, \hat{v}$ cross if and only if $Col(\hat{h}) \perp Col(\hat{v})$, as in Definition 5.7. Then Col is actually an orthogonal poset-colouring. The remaining things to check in order to verify this are conditions (III),(IV) from Definition 3.1.

But the wedge property implies Definition 3.1.(III). Indeed, if $\mathcal{A}$ is an inseparable set of walls with $\mathrm{fio}(\mathcal{A}) > 0$, and $Col(\mathcal{A}) \sqsubseteq \mathbf{U}, \mathbf{V}$, then $Col(\mathcal{A}) \sqsubseteq \mathbf{U} \wedge \mathbf{V} = \mathbf{W}$. So, $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}} = \mathcal{H}_{\mathcal{A}}$, so that intersection has positive measure, and $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$ by the definition of the wedge, and every wall in $\mathcal{A}$ has colour nested in $\mathbf{W}$.

Similarly, the clean containers property implies Definition 3.1.(IV). Indeed, let $\mathbf{U}, \mathbf{V}$ be colours and let $\mathcal{A}, \mathcal{B}$ be inseparable sets with all elements of the former crossing all elements of the latter. Assume $Col(\mathcal{A}) \sqsubseteq \mathbf{U}, Col(\mathcal{B}) \sqsubseteq \mathbf{V}$. For each $\hat{a} \in \mathcal{A}$, we have that $Col(\hat{a}) \sqsubseteq Col(\hat{b})^\perp$ for all $\hat{b} \in \mathcal{B}$, so we take $\mathbf{U}' = \bigwedge_{\hat{b}\in\mathcal{B}} Col(\hat{b})^\perp \wedge \mathbf{U}$. Also, let $\mathbf{V}' = \bigwedge_{\hat{b}\in\mathcal{B}} Col(\hat{b})$.

Then $Col(\mathcal{A}) \sqsubseteq \mathbf{U}', Col(\mathcal{B}) \sqsubseteq \mathbf{V}'$, as required by Definition 3.1.(IV).

Moreover, $\mathbf{U}' \perp \mathbf{V}'$. Indeed, for any $\hat{b} \in \mathcal{B}$, we have $Col(\hat{b}) \perp \mathbf{U}'$ since $\mathbf{U}' \sqsubseteq Col(\hat{b})^\perp$. But $\mathbf{V}' \sqsubseteq Col(\hat{b})$, so $\mathbf{U}' \perp \mathbf{V}'$. So, every wall whose colour is nested in $\mathbf{U}'$ and every wall whose colour is nested in $\mathbf{V}'$ have orthogonal colours, and hence the walls cross. Thus Col is indeed a poset-colouring. □

Before proceeding to construct orthogonal poset-colourings, we illustrate one of their purposes with Theorem 3.13: an orthogonal poset-colouring has the property that finite depth implies the tangible filter condition. This will not be needed for the application to asymptotic cones of hierarchically hyperbolic spaces.

Example 3.11 is an example where finite depth fails.

Example 3.11 (Staircase)**.** Modify the median space $\mathbf{X}$ from Example 3.6 so that it consists of the union of the squares lying between $y = x$ and $y = 0$ in the standard tiling of $\mathbb{R}^2$ by 2–cubes (an infinite staircase). This is again a median metric space — and also a CAT(0) square complex — and each wall is parallel to exactly one hyperplane. Notice that the infinite staircase is a median subalgebra of $\mathbb{R}^2$ but not a median convex subspace.

Colour the walls according to the hyperplanes (i.e. the colour of a wall is the unique hyperplane to which it is parallel), again with a trivial $\sqsubseteq$–relation (one can insert an artificial $\sqsubseteq$–maximal element). This is now a finite-depth poset-colouring, and it satisfies the tangible filter condition.

Declare walls to be orthogonal if they cross. This is not an orthogonal poset-colouring, because it fails the clean containers condition.

On the other hand, in this example, the orthogonal poset-colouring constructed in Section 3.3 does not have finite depth but does satisfy the tangible filter condition. □

Example 3.12. The CAT(0) cube complex illustrated in Figure 6 has an orthogonal poset-colouring of the walls, described in Section 3.3 below. The reader can check that this orthogonal poset-colouring has infinite depth, and does not have the tangible filter property. Indeed, the colour of the i^{th} horizontal wall is the set of the first i horizontal walls, and the join of these colours (i.e. the $\sqsubseteq$–minimal colour $\sqsubseteq$–larger than all of them, which is just the set of all horizontal walls) witnesses failure of tangibility since there is no convex subcomplex crossing exactly the horizontal walls.

The difference with Example 3.11 is the following. In that example, the join of the colours associated to "horizontal" walls is just the unique maximal colour, whose associated filter is tangible. In this example, the join of the horizontal (blue) walls is non-maximal, since it is orthogonal to the colour of each purple wall. And its associated filter is not tangible. □

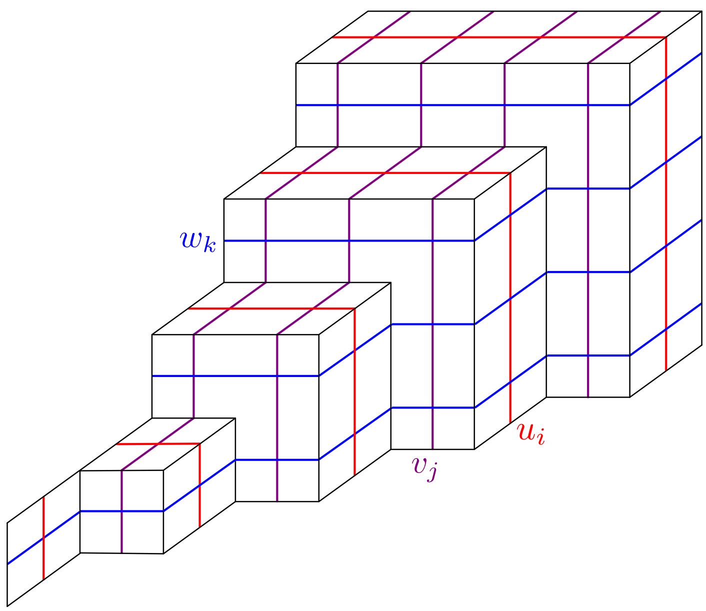

FIGURE 6. CAT(0) cube complex of infinite depth, whose orthogonal poset-colouring fails to satisfy the tangible filter condition.

Theorem 3.13 (Finite depth implies tangible for orthogonal poset-colourings)**.** *Let* $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ *be a complete, finite rank median metric space and let* $\mathcal{W}$ *be the set of walls. If there exists an orthogonal set* $(\mathfrak{F}^\bullet, \sqsubseteq, \perp)$ *and an orthogonal poset-colouring* $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ *of finite depth, then* Col *satisfies the tangible filter condition from* Definition 3.5.

Proof. Let $\mathbf{U} \in \mathfrak{F}$. Let $\mathcal{W}_\mathbf{U}$ be the set of walls with colour nested in $\mathbf{U}$ and $\sigma_\mathbf{U}$ be the associated filter as in Definition 3.3. Let $\mathbf{x}_0 \in \mathbf{X}$ be a fixed basepoint. Recall that we need to show that $\mathrm{fio}(\sigma_\mathbf{U} - \sigma_{\mathbf{x}_0}) < \infty$.

First suppose that $\mathbf{U}$ is the unique $\sqsubseteq$–maximal element of $\mathfrak{F}$. Then $\mathcal{W}_\mathbf{U} = \mathcal{W}$, the set of all walls. So $\sigma_\mathbf{U}$ is the empty set of halfspaces, and we are done.

So, assume that $\mathbf{U}$ is not $\sqsubseteq$–maximal. Consider the set $\sigma_\mathbf{U} - \sigma_{\mathbf{x}_0}$. This is the set of halfspaces u such that

- the wall $\hat{u}$ associated to u is not in $\mathcal{W}_{\mathbf{U}}$;
- some wall $\hat{h} \in \mathcal{W}_{\mathbf{U}}$ satisfies $h \subset u$ or $h^* \subset u$;
- $\mathbf{x}_0 \notin u$, i.e. $\mathbf{x}_0 \in u^*$.

The set $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$ is partially ordered by inclusion. The following claim about this partial order is the first place where we need the finite rank assumption.

Claim 1. Let $u, v \in \sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$, and let $\hat{u}, \hat{v}$ respectively denote the associated walls. Then one of the following holds:

- $u \subset v$;
- $v \subset u$;
- $\hat{u}$ and $\hat{v}$ cross.

Hence $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$ can be written as a disjoint union of at most N chains in the partial order $\subseteq$, where N is the rank of $(\mathbf{X}, \boldsymbol{\mu})$.

Proof of Claim 1. Suppose that $\hat{u}, \hat{v}$ do not cross.

First suppose that $u \cap v \neq \varnothing$ but neither u nor v contains the other. Then $u^* \cap v^* = \varnothing$, contradicting that $\mathbf{x}_0 \in u^* \cap v^*$. So, either $u \subset v$ or $v \subset u$, in which case we are done, or $u \cap v = \varnothing$. Assume the latter. Then $\mathbf{x}_0 \in u^* \cap v^*$, while we have walls $\hat{h}, \hat{w}$ such that, up to relabelling halfspaces, $h \subset u, w \subset v$, and $\hat{h}, \hat{w} \in \mathcal{W}_{\mathbf{U}}$. This contradicts inseparability of $\mathcal{W}_{\mathbf{U}}$ since $\hat{u}, \hat{v}$ separate $\hat{h}, \hat{w}$. Hence u, v are $\subseteq$–comparable.

Hence $\subseteq$–antichains have cardinality at most N, by [Bow13, Proposition 6.2]. So the "hence" part of the claim follows from Dilworth's theorem [Dil50]. □

By the Claim, there exists $k \leqslant N$ and $\subseteq$–chains $\mathcal{C}_1, \ldots, \mathcal{C}_k$ whose disjoint union is $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$. For each $i \leqslant k$, let $\overline{\mathcal{C}}_i$ be the *inseparable closure* of $\mathcal{C}_i$, defined in [Fio20, Section 2.1]: $\overline{\mathcal{C}}_i$ is the set of all halfspaces b such that for some $a, c \in \mathcal{C}_i$, we have $a \leqslant b \leqslant c$. Note that $\overline{\mathcal{C}}_i$ is contained in $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$. By construction, $\overline{\mathcal{C}}_i$ is inseparable and hence fio–measurable by [Fio20, Lemma 3.9].

Suppose that $\mathrm{fio}(\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}) = \infty$. Then, without loss of generality, $\mathrm{fio}(\overline{\mathcal{C}}_1) = \infty$, since $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$ is the union of the finitely many measurable sets $\overline{\mathcal{C}}_i$. (Recall that $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0} = \cup_i \mathcal{C}_i$, and for each i we have $\mathcal{C}_i \subset \overline{\mathcal{C}_i} \subset \sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$.)

Claim 2. We can choose a sequence of halfspaces $\{h_n\}$ in $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$ such that

- $h_{n+1} \subsetneq h_n$ for all $n \in \mathbb{N}$, and
- the inseparable set $\mathcal{H}_n$ of halfspaces h such that $\mathbf{x}_0 \in h_1^* \subset h^*$ and $h_n \subset h$ satisfies $\mathrm{fio}(\mathcal{H}_n) > n$.

Proof of Claim 2. By [Fio20, Corollary 2.27] (which hypothesises completeness and finite rank), there is a countable chain $\{h_n\}_{n \geqslant 1}$ in $\mathcal{C}_1$ that is cofinal in $\mathcal{C}_1$.

For each $n \geqslant 1$, let $\mathcal{H}_n$ be the set of halfspaces h such that $\mathbf{x}_0 \in h_1^* \subset h^*$ and $h_n \subset h$. Then $\mathcal{H}_n$ is inseparable (hence fio–measurable) and $\mathcal{H}_n \subset \overline{\mathcal{C}}_1$.

Observe that $\mathrm{fio}(\mathcal{H}_n)$ is unbounded as $n \to \infty$, by cofinality of $\{h_n\}$ in $\mathcal{C}_1$, the definition of the inseparable closure, and the fact that $\mathrm{fio}(\overline{\mathcal{C}_1}) = \infty$. So, we can pass to a subsequence with the desired properties. □

By definition of $\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}$, for each n there exists $\hat{w}_n \in \mathcal{W}_{\mathbf{U}}$ such that $w_n \subset h_n$ (up to relabelling the $\hat{w}_n$ halfspaces). Hence $w_n \subset h_i$ for all $i \leqslant n$. Let $\mathcal{W}_n$ be the set of walls crossing h_n.

Claim 3. For $m \leqslant n$, we have $\mathcal{W}_m \cap \mathcal{W}_{\mathbf{U}} \subseteq \mathcal{W}_n \cap \mathcal{W}_{\mathbf{U}}$. Moreover, for all $m \in \mathbb{N}$ there exists $n(m) > m$ such that the preceding containment is proper whenever $n \geqslant n(m)$.

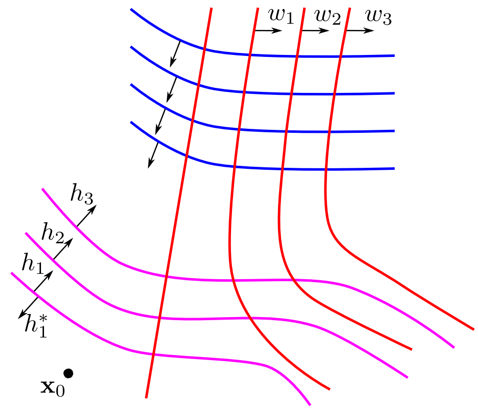

FIGURE 7. Proof of Theorem 3.13.

Proof of Claim 3. If $\hat{u}$ crosses $\hat{h}_m$ and does not cross $\hat{h}_n$, then $\hat{u}$ is separated from $\hat{w}_n$ by $\hat{h}_n$, so by inseparability of $\mathcal{W}_{\mathbf{U}}$, we have $\hat{u} \notin \mathcal{W}_{\mathbf{U}}$. This proves the first assertion, i.e. $\mathcal{W}_m \cap \mathcal{W}_{\mathbf{U}} \subseteq \mathcal{W}_n \cap \mathcal{W}_{\mathbf{U}}$ for $m \leqslant n$.

Suppose for some $m \in \mathbb{N}$ that

$$\mathcal{W}_m \cap \mathcal{W}_{\mathbf{U}} = \mathcal{W}_n \cap \mathcal{W}_{\mathbf{U}}$$

for all $n \geqslant m$. Then we can take $\hat{w}_n = \hat{w}_m$ for all $n \geqslant m$. Thus the set of halfspaces separating $\mathbf{x}_0$ from w_m includes h_n for all $n \geqslant m$, and hence includes all subsets of the form $\mathcal{H}_n, n \geqslant m$. Hence $\mathcal{H}(\mathbf{x}_0, w_m)$ has infinite measure, contradicting that every point in w_m lies at finite distance from $\mathbf{x}_0$. This proves the second assertion. □

In view of the previous claim, we can pass to a subsequence and assume that $\mathcal{W}_n \cap \mathcal{W}_{\mathbf{U}} \subsetneq \mathcal{W}_{n+1} \cap \mathcal{W}_{\mathbf{U}}$ for all n. (The subsequence is obtained by redefining h_2 to be $h_{n(1)}$, h_3 to be $h_{n(n(1))}$, etc.)

Note that if $\hat{v} \in \mathcal{W}_n$, then $\hat{v}$ crosses $\hat{h}_n$, by definition, so $Col(\hat{v}) \perp Col(\hat{h}_n)$ by Definition 3.7. Hence $\mathcal{W}_n \subset \mathcal{W}_{Col(\hat{h}_n)^{\perp}}$. (To see the latter containment, note that if $\hat{v} \in \mathcal{W}_n$, then $Col(\hat{v}) \perp Col(\hat{h}_n)$ implies $Col(\hat{v}) \sqsubseteq Col(\hat{h}_n)^{\perp}$, so $\hat{v} \in \mathcal{W}_{Col(\hat{h}_n)^{\perp}}$.)

On the other hand, if $\hat{v}$ is a wall with $Col(\hat{v}) \sqsubseteq Col(\hat{h}_n)^{\perp}$, i.e. $Col(\hat{v}) \perp Col(\hat{h}_n)$, then by the same definition, $\hat{v}$ and $\hat{h}_n$ cross. Hence $\hat{v} \in \mathcal{W}_{Col(\hat{h}_n)^{\perp}} = \mathcal{W}_n$.

The claim and the preceding discussion show that

$$\mathcal{W}_{Col(\hat{h}_n)^{\perp}} \cap \mathcal{W}_{\mathbf{U}} \subsetneq \mathcal{W}_{Col(\hat{h}_{n+1})^{\perp}} \cap \mathcal{W}_{\mathbf{U}}.$$

By the Lemma 3.14 below, we thus have:

$$Col(\hat{h}_n)^{\perp} \wedge \mathbf{U} \subsetneq Col(\hat{h}_{n+1})^{\perp} \wedge \mathbf{U}$$

for all n.

This contradicts finite depth, so $\sigma_{\mathbf{U}}$ must be tangible. □

Lemma 3.14. *Let $(\mathfrak{F}^{\bullet}, \sqsubseteq, \perp)$ be an orthogonal set and $Col : \mathcal{W} \to \mathfrak{F}^{\bullet}$ an orthogonal poset-colouring. Let $\mathcal{U}, \mathcal{V} \in \mathfrak{F}^{\bullet}$. Then*

- $\mathcal{W}_{\mathbf{U}} \subsetneq \mathcal{W}_{\mathbf{V}}$ *implies* $\mathbf{U} \subsetneq \mathbf{V}$.
- *If* $\mathbf{U} \wedge \mathbf{V}$ *is defined, i.e. if there exists* $\mathbf{W}$ *that is nested in both* $\mathbf{U}$ *and* $\mathbf{V}$*, then*

$$\mathcal{W}_{\mathbf{U} \wedge \mathbf{V}} = \mathcal{W}_{\mathbf{U}} \cap \mathcal{W}_{\mathbf{V}}.$$

Proof. If $\mathbf{V}$ is the unique $\sqsubseteq$–maximal element, then the first assertion holds automatically, so suppose not. Then by Definition 3.7, $\mathbf{V}^{\perp}$ is defined. By the same definition, every wall in

$\mathcal{W}_{\mathbf{V}}$ crosses every wall in $\mathcal{W}_{\mathbf{V}^\perp}$, so if $\mathcal{W}_{\mathbf{U}} \subsetneq \mathcal{W}_{\mathbf{V}}$, another application of the definition shows that $\mathbf{U} \perp \mathbf{V}^\perp$, i.e. $\mathbf{V}^\perp \sqsubseteq \mathbf{U}^\perp$. Yet another application of the definition then shows $\mathbf{U} \sqsubseteq \mathbf{V}$. Now, if $\mathbf{U} = \mathbf{V}$, then $\mathcal{W}_{\mathbf{U}} = \mathcal{W}_{\mathbf{V}}$, so if the containment in the statement is proper, so is the nesting. This proves the first assertion.

Now suppose that $\mathbf{U} \wedge \mathbf{V}$ is defined (so as to prove the second assertion). Suppose that $\hat{u}$ is a wall with $Col(\hat{u}) \sqsubseteq \mathbf{U} \wedge \mathbf{V}$. Then $Col(\hat{u}) \sqsubseteq \mathbf{U}$, so by definition, $\hat{u} \in \mathcal{W}_{\mathbf{U}} \cap \mathcal{W}_{\mathbf{V}}$. Hence $\mathcal{W}_{\mathbf{U} \wedge \mathbf{V}} \subset \mathcal{W}_{\mathbf{U}} \cap \mathcal{W}_{\mathbf{V}}$. On the other hand, if $Col(\hat{u}) \sqsubseteq \mathbf{U}$ and $Col(\hat{u}) \sqsubseteq \mathbf{V}$, then $Col(\hat{u}) \sqsubseteq \mathbf{U} \wedge \mathbf{V}$, which implies the other containment and hence concludes the proof of the second assertion. □

3.3. ☺ **Existence of orthogonal poset-colouring on median spaces.** We now construct an orthogonal poset-colouring for median metric spaces, called the *canonical orthogonal poset-colouring*. For our applications, we have in mind the case of a complete, connected, finite-rank median space, but the reader is also encouraged to have in mind the example of the median metric space consisting of the 0–skeleton of a CAT(0) cube complex. The reader familiar with the *factor system* construction in [BHS17b, Section 8] might notice a resemblance between that construction and the one in this section in the case of cube complexes.

Fix a wall $\hat{u}$. Recall that $\mathcal{W}(\hat{u})$ is the set of walls crossing $\hat{u}$. (Recall that $\hat{u}, \hat{v}$ *cross* if every halfspace associated to one intersects every halfspace associated to the other.)

Definition 3.15 (The set $\mathfrak{F}_0^\bullet$)**.** Let $\mathfrak{F}_0^\bullet$ be the set of all nonempty sets of walls of the form $\bigcap_{i\in I} \mathcal{W}(\hat{u}_i)$, where $\{\hat{u}_i\}_{i\in I}$ is a collection of walls. We adopt the convention that $\mathcal{W} \in \mathfrak{F}_0^\bullet$ — it is the intersection of sets $\mathcal{W}(\hat{u})$, taken over an empty set of walls $\hat{u}$. □

If $\mathcal{U} \in \mathfrak{F}_0^\bullet$, then $\mathcal{U}$ is inseparable, and therefore fio–measurable by [Fio20, Lemma 3.9]. Indeed, we have:

Lemma 3.16. *Let $\hat{u}, \hat{v}$ be walls that both cross a wall $\hat{w}$. Suppose that $\hat{h}$ is a wall separating $\hat{u}, \hat{v}$. Then $\hat{h}$ crosses $\hat{w}$. Hence each element of $\mathfrak{F}_0^\bullet$ is inseparable.*

Proof. We can label halfspaces so that $v \subset h \subset u$. Choose $x, y \in u^* \cap w, u^* \cap w^*$ and $z, t \in v \cap w, v \cap w^*$. Then $x \in h^* \cap w, y \in h^* \cap w^*, z \in h \cap w, t \in h \cap w^*$, so $\hat{h}, \hat{w}$ cross.

This shows that $\mathcal{W}(\hat{w})$ is inseparable. Since arbitrary intersections of inseparable sets are inseparable, every set in $\mathfrak{F}_0^\bullet$ is inseparable. □

We will use $\mathfrak{F}_0^\bullet$ to build an orthogonal set, starting with the following orthogonality and nesting relations on $\mathfrak{F}_0^\bullet$:

Definition 3.17 (Orthogonality in $\mathfrak{F}_0^\bullet$)**.** Let $\mathcal{U}, \mathcal{V} \in \mathfrak{F}_0^\bullet$. We write $\mathcal{U} \perp \mathcal{V}$ to mean that every wall in $\mathcal{U}$ crosses every wall in $\mathcal{V}$. □

Definition 3.18 (Nesting in $\mathfrak{F}_0^\bullet$)**.** We say that $\mathcal{U} \sqsubseteq \mathcal{V}$ if $\mathcal{U} \subset \mathcal{V}$. □

Immediately from the definitions, we have the following for all $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathfrak{F}_0^\bullet$:

- $\sqsubseteq$ is a partial order on $\mathfrak{F}_0^\bullet$ and $\mathcal{W}$ is the unique maximal element.
- $\sqsubseteq$ and $\perp$ are mutually exclusive, since no wall crosses itself.
- $\perp$ is anti-reflexive (since no wall crosses itself) and symmetric.
- If $\mathcal{Y} \sqsubseteq \mathcal{U}$ and $\mathcal{U} \perp \mathcal{V}$, then $\mathcal{Y} \perp \mathcal{V}$.

Remark 3.19 (Wedge property in $\mathfrak{F}_0^\bullet$)**.** Let $\mathcal{U}, \mathcal{V} \in \mathfrak{F}_0^\bullet$. Suppose that there exists $\mathcal{Y} \in \mathfrak{F}_0^\bullet$ with $\mathcal{Y} \sqsubseteq \mathcal{U}$ and $\mathcal{Y} \sqsubseteq \mathcal{V}$.

Write

$$\mathcal{U} = \bigcap_{i\in I} \mathcal{W}(\hat{u}_i)$$

and

$$\mathcal{V} = \bigcap_{i\in J} \mathcal{W}(\hat{v}_j).$$

Then

$$\mathcal{U} \cap \mathcal{V} = \bigcap_{i\in I, j\in J} \mathcal{W}(\hat{u}_i) \cap \mathcal{W}(\hat{v}_j),$$

so $\mathcal{U} \cap \mathcal{V} \in \mathfrak{F}_0^\bullet$. Since $\mathcal{Y}$ is contained in $\mathcal{U} \cap \mathcal{V}$, we have $\mathcal{Y} \sqsubseteq \mathcal{U} \cap \mathcal{V} \in \mathfrak{F}_0^\bullet$. So, we can define a wedge operator on $\mathfrak{F}_0^\bullet$ as in Definition 3.7 by $\mathcal{U} \wedge \mathcal{V} = \mathcal{U} \cap \mathcal{V}$. □

Remark 3.20. We will not need the following, but we observe that we can take wedges of arbitrary collections of subsets in $\mathfrak{F}_0^\bullet$, not just pairs, just by taking intersections. □

Lemma 3.21 (Clean containers)**.** *Let $\mathcal{U} \in \mathfrak{F}_0^\bullet$ be such that there exists $\mathcal{V} \in \mathfrak{F}_0^\bullet$ with $\mathcal{U} \bot \mathcal{V}$. Then there exists $\mathcal{U}^\perp \in \mathfrak{F}_0^\bullet$ such that $\mathcal{U}^\perp \bot \mathcal{U}$, and $\mathcal{Y} \in \mathfrak{F}_0^\bullet$ satisfies $\mathcal{Y} \bot \mathcal{U}$ if and only if $\mathcal{Y} \sqsubseteq \mathcal{U}^\perp$. Moreover, for all $\mathcal{U}, \mathcal{V}$, we have that $\mathcal{U} \sqsubseteq \mathcal{V}$ implies $\mathcal{V}^\perp \sqsubseteq \mathcal{U}^\perp$, and in particular*

$$\mathcal{U}^\perp = ((\mathcal{U}^\perp)^\perp)^\perp.$$

Proof. Write

$$\mathcal{U} = \bigcap_{i\in I} \mathcal{W}(\hat{u}_i),$$

where $\{\hat{u}_i\}_{i\in I}$ is a set of walls. Let

$$\mathcal{Y} = \bigcap_{\hat{w}\in\mathcal{U}} \mathcal{W}(\hat{w}).$$

In other words, $\mathcal{Y}$ is the set of walls $\hat{y}$ such that $\hat{y}$ crosses every wall in $\mathcal{U}$. By definition, $\mathcal{Y} \in \mathfrak{F}_0^\bullet$.

By assumption, $\mathcal{V} \bot \mathcal{U}$, so every wall in $\mathcal{U}$ crosses every wall in $\mathcal{V}$. Hence, $\mathcal{V} \subset \mathcal{Y}$, i.e. $\mathcal{V} \sqsubseteq \mathcal{Y}$. Since $\mathcal{V}$ is arbitrary, this will show that every element of $\mathfrak{F}_0^\bullet$ orthogonal to $\mathcal{U}$ is nested in $\mathcal{Y}$. In particular, $\mathcal{Y} \bot \mathcal{U}$. Let $\mathcal{U}^\perp = \mathcal{Y}$.

Now we prove the "moreover" statement. Suppose that $\mathcal{U}, \mathcal{V}$ satisfy $\mathcal{U} \sqsubseteq \mathcal{V}$, so that $\mathcal{U} \subset \mathcal{V}$. So, from the definition, we immediately get $\mathcal{V}^\perp \sqsubseteq \mathcal{U}^\perp$: any wall that crosses all walls in $\mathcal{V}$ must cross all walls in $\mathcal{U}$.

In particular, since $\mathcal{U}^\perp \bot \mathcal{U}$, we have $\mathcal{U} \sqsubseteq (\mathcal{U}^\perp)^\perp$, so $((\mathcal{U}^\perp)^\perp)^\perp \sqsubseteq \mathcal{U}^\perp$. On the other hand, letting $\mathcal{V} = \mathcal{U}^\perp$, we have $\mathcal{V}^\perp \bot \mathcal{V}$, so $\mathcal{V} \sqsubseteq (\mathcal{V}^\perp)^\perp$, i.e. $\mathcal{U}^\perp \sqsubseteq ((\mathcal{U}^\perp)^\perp)^\perp$. So $((\mathcal{U}^\perp)^\perp)^\perp = \mathcal{U}^\perp$, since $\sqsubseteq$ is a partial order. □

Now let $\mathfrak{F}_1^\bullet \subset \mathfrak{F}_0^\bullet$ be the set of $\mathcal{U} \in \mathfrak{F}_0^\bullet$ such that either $\mathcal{U} = \mathcal{W}$, or there exists $\mathcal{V} \in \mathfrak{F}_0^\bullet$ with $\mathcal{U} \bot \mathcal{V}$.

Lemma 3.22. *Suppose that $\mathcal{U}, \mathcal{V} \in \mathfrak{F}_1^\bullet$. Then*

- *if $\mathcal{U} \wedge \mathcal{V}$ is defined (i.e. $\mathcal{U} \cap \mathcal{V} \neq \emptyset$) then $\mathcal{U} \wedge \mathcal{V} \in \mathfrak{F}_1^\bullet$;*
- *$\mathcal{U}^\perp \in \mathfrak{F}_1^\bullet$ provided $\mathcal{U} \neq \mathcal{W}$.*

Proof. The second assertion follows from the definition of $\mathfrak{F}_1^\bullet$ and the fact that $\mathcal{U}^\perp \bot \mathcal{U}$. For the first assertion, note that if $\mathcal{Y} \bot \mathcal{U}$, then $\mathcal{Y} \bot (\mathcal{U} \wedge \mathcal{V})$. □

We would like to use $(\mathfrak{F}_1^\bullet, \sqsubseteq, \bot)$ as our orthogonal set, but it does not necessarily satisfy the "nesting is determined by orthogonality" condition from Definition 3.7.

To remedy this, we introduce the following equivalence relation on $\mathfrak{F}_1^\bullet$: we write $\mathcal{U} \sim \mathcal{V}$ to mean $\mathcal{U}^\perp = \mathcal{V}^\perp$. If $\mathcal{W}$ is the unique $\sqsubseteq$–maximal element of $\mathfrak{F}_1^\bullet$, then we declare $\mathcal{W}$ to be unique in its $\sim$–class. We let $\mathbf{U}$ denote the $\sim$–class of $\mathcal{U}$, and $\mathbf{V}$ the $\sim$–class of $\mathcal{V}$, etc.

Let $\mathfrak{F}^\bullet_\perp$ be the set of $\sim$–class representatives in $\mathfrak{F}^\bullet_1$. The following lemma defines a section of the quotient map $\mathfrak{F}^\bullet_1 \to \mathfrak{F}^\bullet_\perp$, and after proving it, we will therefore think of $\mathfrak{F}^\bullet_\perp$ as a subset of $\mathfrak{F}^\bullet_1$.

Lemma 3.23 ($\sqsubseteq$–maximal $\sim$–class representatives)**.** *Let $\mathbf{U} \in \mathfrak{F}^\bullet_\perp$. Then there is a unique $\sqsubseteq$–maximal $\mathcal{Y} \in \mathbf{U}$. If $\mathbf{U}$ is not the class of $\mathcal{W}$, then $\mathcal{Y}$ has the property that $\mathcal{Y} = (\mathcal{U}^\perp)^\perp$ for all $\mathcal{U} \in \mathbf{U}$. Conversely, any element of the form $(\mathcal{V}^\perp)^\perp$, with $\mathcal{V} \in \mathfrak{F}^\bullet_1$, is $\sqsubseteq$–maximal in its $\sim$–class.*

Proof. Let $\mathcal{U} \in \mathbf{U}$. If $\mathcal{U} = \mathcal{W}$, we are done, so suppose not. Then $\mathcal{U}^\perp$ is defined, and we have $\mathcal{U} \sqsubseteq (\mathcal{U}^\perp)^\perp$. If $\mathcal{V} \sim \mathcal{U}$, then $(\mathcal{U}^\perp)^\perp = (\mathcal{V}^\perp)^\perp$, proving the first assertion.

Now let $\mathcal{V} \in \mathfrak{F}^\bullet_1$. Let $\mathcal{Y} = (\mathcal{V}^\perp)^\perp$. Then $\mathcal{Y} \sim \mathcal{V}$ by Lemma 3.21. Also, $\mathcal{V} \sqsubseteq \mathcal{Y}$, and for any $\mathcal{V}' \sim \mathcal{V}$, applying the orthogonal complement operation twice to $\mathcal{V}'$ gives $\mathcal{Y}$. So $\mathcal{Y}$ is $\sqsubseteq$–maximal in the $\sim$–class. □

Thus we can view $\mathfrak{F}^\bullet_\perp$ as the subset of $\mathfrak{F}^\bullet_1$ containing exactly those elements that are $\sqsubseteq$–maximal in their $\sim$–class. So, $\mathfrak{F}^\bullet_\perp$ contains $\mathcal{W}$ and inherits the $\sqsubseteq$ and $\perp$ relations from $\mathfrak{F}^\bullet_1$.

By the preceding lemma, if $\mathcal{U} \in \mathfrak{F}^\bullet_\perp$, then we can write $\mathcal{U} = (\mathcal{V}^\perp)^\perp$ for some $\mathcal{V}$. Hence $\mathcal{U}^\perp = ((\mathcal{V}^\perp)^\perp)^\perp$, so by Lemma 3.23, $\mathfrak{F}^\bullet_\perp$ inherits the clean containers property.

Observe that $(\mathcal{U}^\perp)^\perp \cap (\mathcal{V}^\perp)^\perp = (((\mathcal{U}^\perp)^\perp \cap (\mathcal{V}^\perp)^\perp)^\perp)^\perp$, by Lemma 3.21, so the wedge property persists in $\mathfrak{F}^\bullet_\perp$ in view of Lemma 3.23. We now check that we have found an orthogonal set:

Lemma 3.24. *The triple $(\mathfrak{F}^\bullet_\perp, \sqsubseteq, \perp)$ is an orthogonal set.*

Proof. We have already seen that $\perp$ is a symmetric, anti-reflexive relation and $\sqsubseteq$ is a partial order with a unique maximal element. Moreover, $\sqsubseteq$ and $\perp$ are mutually exclusive and $\mathcal{U} \sqsubseteq \mathcal{V} \perp \mathcal{W}$ implies $\mathcal{U} \perp \mathcal{W}$. The wedge and clean containers properties from Definition 3.7 have already been verified, so it just remains to check that nesting is determined by orthogonality, i.e. if $\mathcal{U}, \mathcal{V} \in \mathfrak{F}^\bullet_\perp$, then $\mathcal{V}^\perp \sqsubseteq \mathcal{U}^\perp$ implies $\mathcal{U} \sqsubseteq \mathcal{V}$ (the other direction was checked in Lemma 3.21).

Suppose that $\mathcal{U} = (\mathcal{A}^\perp)^\perp$ and $\mathcal{V} = (\mathcal{B}^\perp)^\perp$ and $\mathcal{V}^\perp \sqsubseteq \mathcal{U}^\perp$. Then by Lemma 3.21, $\mathcal{B}^\perp \sqsubseteq \mathcal{A}^\perp$, so by another application of the same lemma, $\mathcal{U} \sqsubseteq \mathcal{V}$, as required.

Now we take care of the proper nesting part of the "nesting is determined by orthogonality" condition. Suppose that $\mathcal{V}^\perp \sqsubsetneq \mathcal{U}^\perp$. Then $\mathcal{U} \sqsubseteq \mathcal{V}$, by the preceding discussion, and $\mathcal{U} \neq \mathcal{V}$ since they have different orthogonal complements.

Conversely, suppose that $\mathcal{U} \sqsubsetneq \mathcal{V}$. We have by the preceding discussion that $\mathcal{V}^\perp \sqsubseteq \mathcal{U}^\perp$. We need to show that $\mathcal{V}^\perp \neq \mathcal{U}^\perp$. Writing $\mathcal{U} = (\mathcal{A}^\perp)^\perp$ and $\mathcal{V} = (\mathcal{B}^\perp)^\perp$, we have by Lemma 3.21 that $\mathcal{V}^\perp = \mathcal{U}^\perp$ implies $\mathcal{A}^\perp = \mathcal{B}^\perp$, from which, taking orthogonal complements once more, we get $\mathcal{U} = \mathcal{V}$, a contradiction. □

Now we can prove the main result:

Proposition 3.25 (Canonical orthogonal poset-colouring)**.** *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a median metric space and denote by $\mathcal{W}$ the set of walls. Then there is an orthogonal poset-colouring $Col_\perp : \mathcal{W} \to (\mathfrak{F}^\bullet_\perp, \sqsubseteq, \perp)$.*

We call the orthogonal poset-colouring described in this section, the *canonical orthogonal poset-colouring of a median space.*

We emphasise that in the proposition, the poset-colouring may have infinite (even uncountable) depth.

We need one more lemma:

Lemma 3.26 (Crossing pair has an abstract square)**.** *Let $\hat{h}, \hat{v}$ be crossing walls. Then there exist $\mathbf{U}, \mathbf{V} \in \mathfrak{F}_{\perp}^{\bullet}$ such that the following hold:*

- $\mathbf{U} \perp \mathbf{V}$.
- *There exist $\mathcal{U} \in \mathbf{U}, \mathcal{V} \in \mathbf{V}$ respectively containing $\hat{h}, \hat{v}$.*

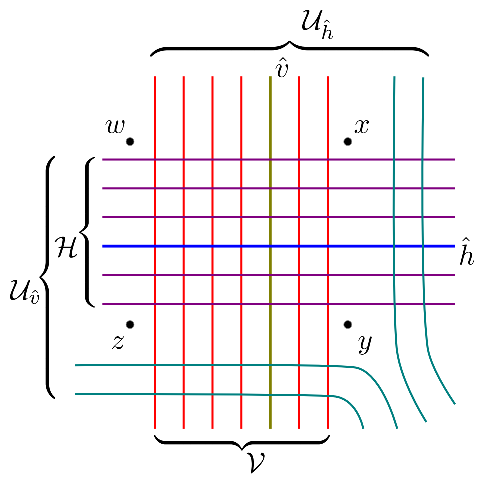

FIGURE 8. Setup of the proof of Lemma 3.26.

Proof. Choose w, x, y, z respectively lying in $h \cap v, h \cap v^*, h^* \cap v^*, h^* \cap v$. By [CDH10, Lemma 2.26], we can assume that w, x, y, z form a *rectangle*, so in particular $\hat{h}$ separates w, x from y, z, and $\hat{v}$ separates w, z from x, y. See Figure 8.

Let $\mathcal{V} = \mathcal{W}(w, z \mid x, y)$ be the set of walls separating w, z from x, y and let $\mathcal{R} = \mathcal{W}(w, x \mid y, z)$ be defined analogously. Each wall in $\mathcal{R}$ crosses each wall in $\mathcal{V}$, since w, x, y, z form a rectangle. Note that $\hat{h} \in \mathcal{R}$ and $\hat{v} \in \mathcal{V}$.

Let $\mathcal{U}_{\hat{h}} = \bigcap_{\hat{w} \in \mathcal{R}} \mathcal{W}(\hat{w})$ and let $\mathcal{U}_{\hat{v}} = \bigcap_{\hat{w} \in \mathcal{V}} \mathcal{W}(\hat{w})$. Then $\mathcal{U}_{\hat{h}}, \mathcal{U}_{\hat{v}} \in \mathfrak{F}_0^{\bullet}$, by definition.

Now, $\hat{v} \in \mathcal{U}_{\hat{h}}$. Indeed, $\hat{v}$ crosses every wall in $\mathcal{R}$.

Let $\mathcal{Y} = \bigcap_{\hat{u} \in \mathcal{U}_{\hat{h}}} \mathcal{W}(\hat{u})$. Then since $\hat{h} \in \mathcal{R}$, we have that $\hat{h}$ crosses every element of $\mathcal{U}_{\hat{h}}$, so $\hat{h} \in \mathcal{Y}$. So $\mathcal{Y}$ is nonempty, and therefore by definition lies in $\mathfrak{F}_0^{\bullet}$. By construction, $\mathcal{Y} \perp \mathcal{U}_{\hat{h}}$, so both $\mathcal{Y}, \mathcal{U}_{\hat{h}} \in \mathfrak{F}_1^{\bullet}$.

Let $\mathbf{V}$ be the $\sim$–class of $\mathcal{U}_{\hat{h}}$ and let $\mathbf{U}$ be the $\sim$–class of $\mathcal{Y}$. Then $\mathbf{U} \perp \mathbf{V}$, by the definition of $\mathcal{Y}$. Moreover, $\hat{h} \in \mathcal{Y}$ and $\hat{v} \in \mathcal{U}_{\hat{h}}$, as required. □

Now we can prove the existence of orthogonal poset-colourings.

Proof of Proposition 3.25. If $\mathbf{X}$ is a single point, then the proposition holds vacuously since there are no nontrivial walls, so we can assume the set of walls is nonempty.

Define a map $Col_{\perp} : \mathcal{W} \to \mathfrak{F}_{\perp}^{\bullet}$ as follows.

Let $\hat{w} = \{w, w^*\}$ be a wall. Let $I(\hat{w})$ be the set of walls $\hat{u}$ such that $\hat{w}$ crosses $\hat{u}$. So,

$$\mathcal{U}_{\hat{w}} = \bigcap_{\hat{u} \in I(\hat{w})} \mathcal{W}(\hat{u})$$

lies in $\mathfrak{F}_0^{\bullet}$ and contains $\hat{w}$. If $I(\hat{w}) = \varnothing$, then by our convention on intersections over the empty index set, $\mathcal{U}_{\hat{w}} = \mathcal{W}$. To see that $\mathcal{U}_{\hat{w}} \in \mathfrak{F}_0^{\bullet}$, refer to Definition 3.15 and observe that $\mathcal{U}_{\hat{w}}$ is nonempty since it contains $\hat{w}$.

If $I(\hat{w}) \neq \varnothing$, then let $\mathcal{Y} = \bigcap_{\hat{u} \in \mathcal{U}_{\hat{w}}} \mathcal{W}(\hat{u})$. Then $\mathcal{Y} \perp \mathcal{U}_{\hat{w}}$, and $\mathcal{Y}$ is nonempty since it contains $I(\hat{w})$. Thus $\mathcal{U}_{\hat{w}} \in \mathfrak{F}_1^{\bullet}$.

Therefore, we can sensibly define $Col_\perp(\hat{w}) \in \mathfrak{F}^\bullet_\perp$ to be the $\sim$–class of $\mathcal{U}_{\hat{w}}$. In particular, if $I(\hat{w}) = \varnothing$, i.e. $\hat{w}$ does not cross any walls, then $Col_\perp(\hat{w})$ is the $\sim$–class of $\mathcal{W}$.

We have already checked that $(\mathfrak{F}^\bullet_\perp, \sqsubseteq, \perp)$ satisfies the parts of Definition 3.7 not involving the map $Col_\perp$. It remains to complete the proof that $Col_\perp$ is a poset-colouring, i.e. to check the the four conditions in Definition 3.1, and to check that, if $Col_\perp(\hat{w})\perp Col_\perp(\hat{v})$, then $\hat{w}$ and $\hat{v}$ cross, and conversely, if $\hat{h}$ and $\hat{v}$ are crossing walls, then their colours are orthogonal.

Suppose that there exists a set $\mathcal{A} \sqsubseteq \mathbf{U}, \mathbf{V}$ as in Definition 3.1.(III). Then it suffices to take as a family the wedge $\mathbf{W} = \mathbf{U} \wedge \mathbf{V}$. Indeed, by definition, $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$; by Remark 3.19, we have that $Col(\mathcal{A}) \sqsubseteq \mathbf{W}$. Furthermore, since the set of halfspaces associated to $\mathcal{A}$ has positive fio–measure and since the halfspaces associated to $\mathbf{W}$ contain those associated to $\mathcal{A}$, it follows that they also have positive fio–measure. This verifies Definition 3.1.(III).

Suppose that $\hat{v}$ and $\hat{h}$ cross. By Lemma 3.26, we have $\mathcal{U}, \mathcal{V} \in \mathfrak{F}^\bullet_1$ such that $\hat{h} \in \mathcal{U}$ and $\hat{v} \in \mathcal{V}$, and $\mathcal{U}\perp\mathcal{V}$. We can write

$$\mathcal{U} = \bigcap_{i \in I} \mathcal{W}(\hat{u}_i),$$

so since $\hat{h} \in \mathcal{U}$, we have $I \subset I(\bar{h})$, and hence $Col_\perp(\hat{h}) \sqsubseteq \mathbf{U}$, where $\mathbf{U}$ is the $\sim$–class of $\mathcal{U}$. Similarly, $Col_\perp(\hat{v}) \sqsubseteq \mathbf{V}$. So $Col_\perp(\hat{h})\perp Col_\perp(\hat{v})$, and in particular $Col_\perp(\hat{h}), Col_\perp(\hat{v})$ are not $\sqsubseteq$–related.

Next, suppose that $\hat{h}, \hat{v}, \hat{u}$ are walls such that $Col_\perp(\hat{h}), Col_\perp(\hat{v}) \sqsubseteq \mathbf{V}$ and $\hat{u}$ separates $\hat{h}$ from $\hat{v}$. Let $\mathcal{V} \in \mathfrak{F}^\bullet_1$ be the unique $\sqsubseteq$–maximal representative of its $\sim$–class (given by Lemma 3.23). Writing

$$\mathcal{V} = \bigcap_{i \in I} \mathcal{W}(\hat{y}_i),$$

we have that $\hat{v}$ and $\hat{h}$ both cross every $\hat{y}_i$. Since $\hat{u}$ separates $\hat{v}, \hat{h}$, it follows that $\hat{u}$ crosses $\hat{y}_i$. Hence $Col_\perp(\hat{u}) \sqsubseteq \mathbf{V}$.

Next, suppose that $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet_\perp$. Suppose there exist sets $\mathcal{A}, \mathcal{B}$ of walls with $Col_\perp(\mathcal{A}) \sqsubseteq \mathbf{U}$ and $Col_\perp(\mathcal{B}) \sqsubseteq \mathbf{V}$. Suppose that all walls in $\mathcal{A}$ cross all walls in $\mathcal{B}$, and $\mathcal{A}, \mathcal{B}$ are nonempty.

Let $\mathcal{U}_A \in \mathfrak{F}^\bullet_1$ be the set of walls crossing all walls in $\mathcal{B}$. Note that $\mathcal{A} \subset \mathcal{U}_A$. Also every wall in $\mathcal{B}$ crosses every wall in $\mathcal{U}_A$, by definition, so $\mathcal{B} \subset \mathcal{U}_A^\perp$.

So $\mathcal{U}_A, \mathcal{U}_A^\perp$ respectively contain $\mathcal{A}, \mathcal{B}$ and are orthogonal in $\mathfrak{F}^\bullet_1$. Let $\mathbf{U}''$ and $\mathbf{V}''$ be their $\sim$–classes, so $\mathbf{U}''\perp\mathbf{V}''$. To conclude, just let $\mathbf{U}' = \mathbf{U}'' \wedge \mathbf{U}$ and let $\mathbf{V}' = \mathbf{V}'' \wedge \mathbf{V}$. This verifies Definition 3.1.(IV).

Finally, if $Col_\perp(\hat{v})\perp Col_\perp(\hat{h})$, then since $Col_\perp(\hat{v})$ is represented by a set of walls containing $\hat{v}$ and vice versa, the definition of orthogonality implies that $\hat{v}$ and $\hat{h}$ cross. Conversely, if $\hat{v}$ and $\hat{h}$ are crossing walls, then we saw above that $Col_\perp(\hat{h})\perp Col_\perp(\hat{v})$.

This completes the proof that $Col_\perp$ is an orthogonal poset-colouring. □

We next show that the canonical orthogonal poset-colouring is minimal and essentially unique in the following sense.

Proposition 3.27. *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a median metric space and denote by $\mathcal{W}$ the set of walls. Let $Col_\perp : \mathcal{W} \to (\mathfrak{F}^\bullet_\perp, \sqsubseteq, \perp)$ be the canonical orthogonal poset-colouring as described in Proposition 3.25 and let $Col'_\perp : \mathcal{W} \to ({\mathfrak{F}^\bullet_\perp}', \sqsubseteq', \perp')$ be an arbitrary orthogonal poset-colouring.*

Let ${\mathfrak{F}^\bullet_{\perp,p}}'$ be the subset of elements $\mathbf{V} \in {\mathfrak{F}^\bullet_\perp}'$ such that $\mathcal{W}_\mathbf{V} \neq \varnothing$ and consider $({\mathfrak{F}^\bullet_{\perp,p}}', \sqsubseteq, \perp)$ with the induced relations.

Then, there is a bijective map $f : ({\mathfrak{F}^\bullet_{\perp,p}}', \sqsubseteq, \perp') \to (\mathfrak{F}^\bullet_\perp, \sqsubseteq, \perp)$ that respects the relations, i.e. $\mathbf{U} \perp$ (resp. $\sqsubseteq$)$\mathbf{V}$ if and only if $f(\mathbf{U}) \perp$ (resp. $\sqsubseteq$)$f(\mathbf{V})$.

Proof. Let $\mathbf{V}' \in {\mathfrak{F}^\bullet_{\perp,p}}'$.

Suppose that $\mathbf{V}'$ is not the $\sqsubseteq$–maximal element. Then, since orthogonality determines nesting, it follows that there exists $\mathbf{W}' \in {\mathfrak{F}^\bullet_{\perp,p}}'$ such that $\mathbf{W}' \perp \mathbf{V}'$.

Since ${\mathfrak{F}^\bullet_{\perp}}'$ is an orthogonal poset-colouring, there exists ${\mathbf{V}'}^{\perp} \in {\mathfrak{F}^\bullet_{\perp,p}}'$. Since ${\mathbf{V}'}^{\perp} \perp \mathbf{V}'$, from the definition of orthogonal poset-colouring we have that each wall in $\mathcal{W}_{\mathbf{V}'^{\perp}}$ crosses each wall in $\mathcal{W}_{\mathbf{V}'}$. Conversely, if a wall $\hat{w}$ crosses each wall in $\mathcal{W}_{\mathbf{V}'}$, then from the definition of orthogonal poset-colouring, we have that $Col(\hat{w}) \perp \mathbf{V}'$. Since ${\mathbf{V}'}^{\perp}$ has nested all $\mathbf{W}' \in {\mathfrak{F}^\bullet_{\perp,p}}'$ such that $\mathbf{W}' \perp \mathbf{V}'$, we deduce that $\mathcal{W}_{\mathbf{V}'^{\perp}}$ is precisely the set of walls that cross each wall in $\mathcal{W}_{\mathbf{V}'}$.

Now, from the construction of the canonical orthogonal poset-colouring, see Definition 3.15, the set of walls that cross a given set of walls is an element of $\mathfrak{F}^\bullet_{\perp}$. Therefore, we deduce that there exists $\mathbf{V}^{\perp} \in \mathfrak{F}^\bullet_{\perp}$ such that $\mathcal{W}_{\mathbf{V}^{\perp}} = \mathcal{W}_{\mathbf{V}'^{\perp}} \subset \mathcal{W}$. Furthermore, since $\mathbf{V}^{\perp} \in \mathfrak{F}^\bullet_{\perp}$, we have $\mathbf{V}^{\perp\perp} \in \mathfrak{F}^\bullet_{\perp}$. From Lemma 3.8, we have that $\mathbf{V}^{\perp\perp} = \mathbf{V}$ and so $\mathbf{V} \in \mathfrak{F}^\bullet_{\perp}$ and since $\mathcal{W}_{\mathbf{V}}$ are the walls that cross $\mathcal{W}_{\mathbf{V}^{\perp}} = \mathcal{W}_{\mathbf{V}'^{\perp}}$, we have that $\mathcal{W}_{\mathbf{V}'} = \mathcal{W}_{\mathbf{V}}$.

We set that $f : {\mathfrak{F}^\bullet_{\perp,p}}' \to \mathfrak{F}^\bullet_{\perp}$ sends the $\sqsubseteq$-maximal element of ${\mathfrak{F}^\bullet_{\perp,p}}'$ to the $\sqsubseteq$-maximal element of $\mathfrak{F}^\bullet_{\perp}$ and, in the notation above, $\mathbf{V}'$ to $\mathbf{V}$. From the discussion above, we have that $\mathcal{W}_{\mathbf{V}'} = \mathcal{W}_{f(\mathbf{V}')}$ for all $\mathbf{V} \in {\mathfrak{F}^\bullet_{\perp,p}}'$. It follows that the map f is injective as if $f(\mathbf{V}') = f(\mathbf{W}')$, we have that $\mathcal{W}_{\mathbf{V}'} = \mathcal{W}_{f(\mathbf{V}')} = \mathcal{W}_{f(\mathbf{W}')} = \mathcal{W}_{\mathbf{W}'}$ and since nesting is determined by inclusion of the sets of walls, we have that $\mathbf{V}' = \mathbf{W}'$.

Finally, since

$$\mathcal{W} = \bigcup_{\mathbf{V}' \in {\mathfrak{F}^\bullet_{\perp,p}}'} \mathcal{W}_{\mathbf{V}'} = \bigcup_{\mathbf{V}' \in f({\mathfrak{F}^\bullet_{\perp,p}}')} \mathcal{W}_{f(\mathbf{V}')}$$

and by construction $\mathcal{W}_{\mathbf{V}} \neq \varnothing$ for all $\mathbf{V} \in \mathfrak{F}^\bullet_{\perp}$, it follows that $f({\mathfrak{F}^\bullet_{\perp,p}}') = \mathfrak{F}^\bullet_{\perp}$ and so f is surjective.

It follows that $f : {\mathfrak{F}^\bullet_{\perp,p}}' \to \mathfrak{F}^\bullet_{\perp}$ is a bijection. Since orthogonality of elements in $\mathfrak{F}^\bullet_{\perp}$ and ${\mathfrak{F}^\bullet_{\perp}}'$ is determined by the crossing of the walls with colours nested in those elements, we have that $\mathbf{U}' \perp \mathbf{V}'$ if and only if $f(\mathbf{U}') \perp f(\mathbf{V}')$; furthermore, since orthogonality determines nesting, we have that $\mathbf{U}' \sqsubseteq \mathbf{V}'$ if and only if $f(\mathbf{U}') \sqsubseteq f(\mathbf{V}')$. □

We also record the following fact about orthogonal poset-colourings.

Proposition 3.28. *Let $\mathbf{X}$ be a median metric space with more than one point, and let $Col : \mathcal{W} \to (\mathfrak{F}^\bullet, \sqsubseteq, \perp)$ be an orthogonal poset-colouring. For $\mathbf{U} \in \mathfrak{F}^\bullet$, recall that $\mathcal{W}_{\mathbf{U}}$ is the set of walls $\hat{h}$ with $Col(\hat{h}) \sqsubseteq \mathbf{U}$ and $\mathcal{H}_{\mathbf{U}}$ is the set of halfspaces associated to walls in $\mathcal{W}_{\mathbf{U}}$.*

Then for all walls $\hat{h}$,

$$\max\{\mathrm{fio}(\mathcal{H}_{Col(\hat{h})}), \mathrm{fio}(\mathcal{H}_{Col(\hat{h})^{\perp}}\} > 0;$$

the above quantity may be infinite. When $Col(\hat{h})$ is $\sqsubseteq$–maximal, we take the above to mean $\mathrm{fio}(\mathcal{H}_{Col(\hat{h})}) > 0$.

Proof. Let $\mathcal{T} = Col(\hat{h})$, let $\mathbf{T}$ be its $\sim$–class.

First suppose that no wall crosses $\hat{h}$. Then by Definition 3.7, $\mathbf{T}$ is $\sqsubseteq$–maximal, so $\mathcal{H}_{\mathbf{T}} = \mathcal{H}$, which has positive measure since $\mathbf{X}$ is not a single point.

Otherwise, some wall crosses $\hat{h}$. So, by the definition of an orthogonal poset-colouring, there exists $\mathbf{U} \in \mathfrak{F}^\bullet$, and some $\hat{v}$ with $Col(\hat{v}) = \mathbf{U}$, such that $\mathbf{U} \perp \mathbf{T}$ (and $\hat{h}$ and $\hat{v}$ cross). Let $w, x, y, z \in \mathbf{X}$ and $\mathcal{R}, \mathcal{V}$ be as in the proof of Lemma 3.26. Then the sets of halfspaces associated to $\mathcal{R}$ and $\mathcal{V}$ both have positive measure, since $\mathsf{d}_1(w, x) > 0$ and $\mathsf{d}_1(w, z) > 0$. As in the proof of Lemma 3.26, we have $\hat{h} \in \mathcal{R}$ and $\hat{v} \in \mathcal{V}$. Every wall in $\mathcal{V}$ crosses $\hat{h}$, so if some wall $\hat{u}$ crosses every wall crossing $\hat{h}$, then $\hat{u}$ crosses every wall in $\mathcal{V}$. Hence $\mathcal{V} \subset \mathcal{W}_{\mathbf{T}^{\perp}}$, and therefore $\mathcal{H}_{\mathbf{T}^{\perp}}$ has positive measure. □

4. Real cubings

We now introduce real cubings.

4.1. **Definition of a real cubing.** In this section, we introduce ℝ–cubings. First we need some notation:

Notation 4.1 (The ℓ_1–space associated to a set of based ℝ–trees)**.** Let $\mathfrak{F}^\bullet$ be a set, and, for each $\mathbf{F} \in \mathfrak{F}^\bullet$, let $(\mathcal{T}^\bullet\mathbf{F}, \mathsf{d}_\mathbf{F})$ be an ℝ–tree (i.e. a 0–hyperbolic geodesic metric space) with a basepoint $1_\mathbf{F}$. We denote by $\ell_1(\mathfrak{F}^\bullet)$ the subspace of $\prod_{\mathbf{F}\in\mathfrak{F}^\bullet} \mathcal{T}^\bullet\mathbf{F}$ consisting of those tuples $(\mathbf{x_F})_{\mathbf{F}\in\mathfrak{F}^\bullet}$ such that

$$\sum_{\mathbf{F}\in\mathfrak{F}^\bullet} \mathsf{d}_\mathbf{F}(\mathbf{x_F}, 1_\mathbf{F}) < \infty.$$

We equip $\ell_1(\mathfrak{F}^\bullet)$ with the ℓ_1 metric d_1, i.e.

$$\mathsf{d}_1((\mathbf{x_F}), (\mathbf{y_F})) = \sum_{\mathbf{F}\in\mathfrak{F}^\bullet} \mathsf{d}_\mathbf{F}(\mathbf{x_F}, \mathbf{y_F}).$$

Note that we do not impose any cardinality constraint on $\mathfrak{F}^\bullet$. But, in the above definition, and later, we use the following convention on sums: we are summing over the $\mathbf{F} \in \mathfrak{F}^\bullet$ for which the corresponding term is nonzero (hence positive). If there are uncountably many such terms, $(\mathbf{x_F})_\mathbf{F} \notin \ell_1(\mathfrak{F}^\bullet)$. So we are really only interested in points for which the set of such $\mathbf{F}$ is countable, and then the infinite sum can be interpreted in the usual way since the property of convergence is independent of the order in which we sum the terms, in view of positivity of the nonzero terms. □

Definition 4.2 (ℝ–cubing)**.** The nonempty set $\boxed{\mathfrak{F}^\bullet}$ together with the collection $\{\boxed{\mathcal{T}^\bullet\mathbf{W}} : \mathbf{W} \in \mathfrak{F}^\bullet\}$ of ℝ–trees $(\mathcal{T}^\bullet\mathbf{W}, \mathsf{d}_\mathbf{W})$, each with a fixed basepoint $1_\mathbf{W} \in \mathcal{T}^\bullet\mathbf{W}$ is a *real cubing index set* whenever it satisfies the following four properties:

(1) **(Nesting.)** $\mathfrak{F}^\bullet$ is equipped with a partial order $\boxed{\sqsubseteq}$, called *nesting*. For $\mathbf{V}, \mathbf{W} \in \mathfrak{F}^\bullet$, we say $\mathbf{V}$ is *nested* in $\mathbf{W}$ when $\mathbf{V} \sqsubseteq \mathbf{W}$. (We emphasise that $\mathbf{W} \sqsubseteq \mathbf{W}$.)

For each $\mathbf{W} \in \mathfrak{F}^\bullet$, we denote by $\mathfrak{F}^\bullet_\mathbf{W}$ the set of $\mathbf{V} \in \mathfrak{F}^\bullet$ such that $\mathbf{V} \sqsubseteq \mathbf{W}$. Moreover, for all $\mathbf{V}, \mathbf{W} \in \mathfrak{F}^\bullet$ with $\mathbf{V} \sqsubsetneq \mathbf{W}$, there is an associated point $\boxed{\rho^\mathbf{V}_\mathbf{W}} \in \mathcal{T}^\bullet\mathbf{W}$. There is also a map $\boxed{\rho^\mathbf{W}_\mathbf{V} : \mathcal{T}^\bullet\mathbf{W} \to \mathcal{T}^\bullet\mathbf{V}}$ whenever $\mathbf{V} \sqsubsetneq \mathbf{W}$.

(2) **(Orthogonality.)** $\mathfrak{F}^\bullet$ has a symmetric and anti-reflexive relation called *orthogonality*: we write $\mathbf{V}\boxed{\perp}\mathbf{W}$ when $\mathbf{V}, \mathbf{W}$ are orthogonal. Also, whenever $\mathbf{V} \sqsubseteq \mathbf{W}$ and $\mathbf{W}\perp\mathbf{U}$, we require that $\mathbf{V}\perp\mathbf{U}$. If $\mathbf{V}\perp\mathbf{W}$, then $\mathbf{V}, \mathbf{W}$ are not $\sqsubseteq$–comparable.

(3) **(Transversality.)** If $\mathbf{V}, \mathbf{W} \in \mathfrak{F}^\bullet$ are not orthogonal and neither is nested in the other, then we say $\mathbf{V}, \mathbf{W}$ are *transverse*, denoted $\mathbf{V}\boxed{\pitchfork}\mathbf{W}$. If $\mathbf{V}\pitchfork\mathbf{W}$, then there are points $\boxed{\rho^\mathbf{V}_\mathbf{W}} \in \mathcal{T}^\bullet\mathbf{W}$ and $\rho^\mathbf{W}_\mathbf{V} \in \mathcal{T}^\bullet\mathbf{V}$ so that the following holds. If $\mathbf{U} \sqsubseteq \mathbf{V}$ or $\mathbf{U}\perp\mathbf{V}$, then $\rho^\mathbf{U}_\mathbf{W} = \rho^\mathbf{V}_\mathbf{W}$ whenever $\mathbf{W} \in \mathfrak{F}^\bullet$ satisfies either $\mathbf{V} \sqsubsetneq \mathbf{W}$ or $\mathbf{V}\pitchfork\mathbf{W}$ and $\mathbf{W} \not\perp \mathbf{U}$.

(4) **(Finite complexity.)** There exists $\boxed{\chi^\bullet} \in \mathbb{N} \cup \{0\}$, the *complexity* of $(\mathbf{X}, \mathfrak{F}^\bullet)$, so that any $\sqsubseteq$–chain in $\mathfrak{F}^\bullet$ has cardinality at most $\chi^\bullet$. Similarly, any subset of $\mathfrak{F}^\bullet$ whose elements are pairwise orthogonal has cardinality at most $\chi^\bullet$.

The nonempty path-connected complete metric space $(\boxed{\mathbf{X}}, \boxed{\mathsf{d}_\mathbf{X}})$ is a *real cubing* (or ℝ–cubing) if there exists a real cubing index set $\boxed{\mathfrak{F}^\bullet}$ for which the following hold:

(5) **(Consistency, realisation, isometric embedding)** The space $\mathbf{X}$ is a subspace of $\ell_1(\mathfrak{F}^\bullet)$, and $\mathsf{d}_\mathbf{X}$ is the restriction to $\mathbf{X}$ of the ℓ_1 metric.

Moreover, $\mathbf{X}$ has the following property, where $\boxed{\pi_\mathbf{W}} : \ell_1(\mathfrak{F}^\bullet) \to \mathcal{T}^\bullet\mathbf{W}$ is the natural projection for each $\mathbf{W} \in \mathfrak{F}^\bullet$. A point $\mathbf{x} \in \ell_1(\mathfrak{F}^\bullet)$ belongs to $\mathbf{X}$ if and only if

- $\pi_\mathbf{W}(\mathbf{x}) \in \pi_\mathbf{W}(\mathbf{X})$ for all $\mathbf{W} \in \mathfrak{F}^\bullet$; and

- if $\mathbf{V} \pitchfork \mathbf{W}$, then
$$\min\left\{\mathsf{d}_{\mathbf{W}}(\pi_{\mathbf{W}}(\mathbf{x}), \rho^{\mathbf{V}}_{\mathbf{W}}), \mathsf{d}_{\mathbf{V}}(\pi_{\mathbf{V}}(\mathbf{x}), \rho^{\mathbf{W}}_{\mathbf{V}})\right\} = 0,$$
and
- if $\mathbf{V} \subsetneq \mathbf{W}$, then
$$\min\left\{\mathsf{d}_{\mathbf{W}}(\pi_{\mathbf{W}}(\mathbf{x}), \rho^{\mathbf{V}}_{\mathbf{W}}), \mathsf{d}_{\mathbf{V}}(\pi_{\mathbf{V}}(\mathbf{x}), \rho^{\mathbf{W}}_{\mathbf{V}}(\pi_{\mathbf{W}}(\mathbf{x})))\right\} = 0.$$

The above equalities are called *consistency* conditions on tuples in $\ell_1(\mathfrak{F}^\bullet)$.

We require that for all $\mathbf{V}, \mathbf{W} \in \mathfrak{F}^\bullet$, we have that $\rho^{\mathbf{V}}_{\mathbf{W}} \in \pi_{\mathbf{W}}(\mathbf{X})$ whenever $\mathbf{V} \sqsubseteq \mathbf{W}$ or $\mathbf{V} \pitchfork \mathbf{W}$.

(6) **(Bounded geodesic image.)** If $\mathbf{V} \subsetneq \mathbf{W}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, then $\pi_{\mathbf{V}}(\mathbf{x}) \neq \pi_{\mathbf{V}}(\mathbf{y})$ only if the geodesic in $\mathcal{T}^\bullet\mathbf{W}$ from $\pi_{\mathbf{W}}(\mathbf{x})$ to $\pi_{\mathbf{W}}(\mathbf{y})$ contains $\rho^{\mathbf{V}}_{\mathbf{W}}$.

We also refer to the pair $(\mathbf{X}, \mathfrak{F}^\bullet)$ as a *real cubing.* This notation implicitly includes all of the data: ℝ–trees, $\rho^\bullet_\bullet$-points and maps, etc.

The space $\mathbf{X}$ is complete by hypothesis, but the associated ℝ–trees need not be, even if (as will be the case in practice), the maps $\pi_{\mathbf{U}} : \mathbf{X} \to \mathcal{T}^\bullet\mathbf{U}$ are surjective. □

Remark 4.3. Real trees were introduced by Alperin and Moss [AM85] as metric completions of simplicial trees and in their influential work, Morgan and Shalen [MS84] studied real trees and groups acting on them as degenerations of hyperbolic spaces, or in other words, asymptotic cones of hyperbolic spaces. In this first setting, real trees were defined as complete metric spaces. Later, the definition was generalised to be a 0-hyperbolic space and nowadays real trees do not need to be complete.

As in the original setting for trees, our main interest is in spaces that are complete for some other reason, e.g. asymptotic cones of hierarchically hyperbolic groups. For this reason we require real cubings to be complete metric spaces. This assumption is mainly for convenience and we believe that, with the corresponding adjustments, it may be dropped (but path-connectedness must be retained).

For instance, one of the implications of completeness that we use is the fact that real cubings are geodesics spaces: we show that they are complete, connected median spaces. However, a path-connected, not necessarily complete metric space $\mathbf{X}$ satisfying the rest of Definition 4.2 can be shown to be a geodesic space as follows. First, the proof of Lemma 4.7 below shows that $\mathbf{X}$ is a median metric space, without using completeness.

Then, the completion $\overline{\mathbf{X}}$ is again median, by [DK18, Proposition 6.42]. For each $\mathbf{W} \in \mathfrak{F}^\bullet$, the lipschitz projection $\pi_{\mathbf{W}} : \mathbf{X} \to \mathcal{T}^\bullet\mathbf{W}$ extends to a lipschitz projection $\pi_{\mathbf{W}} : \overline{\mathbf{X}} \to \overline{\mathcal{T}^\bullet\mathbf{W}}$ between completions, which continues to be a median-preserving map. As a complete, path-connected median space, $\overline{\mathbf{X}}$ is a geodesic space, and the extended $\pi_{\mathbf{W}}$ must send geodesics to geodesics since it is a median homomorphism. But for any pair of points $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, the geodesic γ from $\mathbf{x}$ to $\mathbf{y}$ in $\overline{\mathbf{X}}$ must remain inside $\mathbf{X}$, since its image in each $\overline{\mathcal{T}^\bullet\mathbf{W}}$ must remain in $\mathcal{T}^\bullet\mathbf{W}$ since it cannot contain any valence–1 points (except the endpoints), and the completion of any ℝ–tree is obtained by adding valence–1 points [AB87, MNO92].

But, for our purposes, we will just work with complete ℝ–cubings. □

Remark 4.4. Definition 4.2 is motivated by the definition of a hierarchically hyperbolic space (see Part 2), with a few important differences.

For example, we do not require that $\mathfrak{F}$ has a unique $\sqsubseteq$–maximal element, although this will often hold in practice.

Also, in contrast with the definition of a hierarchically hyperbolic space, the hyperbolic spaces associated to elements of the index set of an HHS structure have been replaced with ℝ–trees, and various coarse points have become points, coarse equalities have become equalities, etc. Most importantly, several statements that are axioms in the hierarchically hyperbolic space setting have analogues in this setting that are *consequences* of the other axioms; we now establish these. □

As we mentioned, real cubings do not require a unique maximal element. However, the next lemma allows us to assume without of generality when convenient, that the real cubing has a unique $\sqsubseteq$-maximal element.

Lemma 4.5. *Let $(\mathbf{X}, \mathfrak{F}^{\bullet})$ be a real cubing. Then we can modify the index set $\mathfrak{F}^{\bullet}$ by adding a unique $\sqsubseteq$-maximal element and obtain an index set $\mathfrak{F}^{\bullet}_m$ so that $(\mathbf{X}, \mathfrak{F}^{\bullet}_m)$ is a real cubing.*

Proof. If $\mathfrak{F}^{\bullet}$ does not have a $\sqsubseteq$-maximal element, we define the set $\mathfrak{F}^{\bullet}_m$ to be $\mathfrak{F}^{\bullet}$ together with a new element $\mathbf{S}$, The relations and real trees from $\mathfrak{F}^{\bullet}$ are preserved in $\mathfrak{F}^{\bullet}_m$. For all $\mathbf{U} \in \mathfrak{F}^{\bullet}$ we declare that $\mathbf{U} \subsetneq \mathbf{S}$ and define $\mathcal{T}\mathbf{S}$ to be a point p. The ρ maps are defined as follows: for all $\mathbf{U} \in \mathfrak{F}^{\bullet}$, $\rho^{\mathbf{U}}_{\mathbf{S}} = p$ and $\rho^{\mathbf{S}}_{\mathbf{U}}$ is an arbitrary point in $\mathcal{T}\mathbf{U}$.

Then it is routine to check that $\mathfrak{F}^{\bullet} \cup \{\mathbf{S}\}$ is a real cubing index set.

Note that adding elements to the index set whose associated real trees are points does not affect the set of consistent tuples. More precisely, the natural map

$$\prod_{\mathbf{U}\in\mathfrak{F}^{\bullet}} \mathcal{T}\mathbf{U} \to \{p\} \times \prod_{\mathbf{U}\in\mathfrak{F}^{\bullet}} \mathcal{T}\mathbf{U}$$

sends consistent, ℓ_1 points to consistent ℓ_1 points bijectively. Since $\mathcal{T}\mathbf{S}$ is a point, the geodesic bounded property is satisfied trivially. Therefore, $(\mathbf{X}, \mathfrak{F}^{\bullet}_m)$ is a real cubing.

Note that the clean containers property (see Definition 4.40) persists, and the wedge property persists (if it held in the first place). □

The next statement should be compared to the uniqueness axiom (Definition 10.1.(9)) in the definition of an HHS. While they are similar, the following statement is weaker, for example because it does not imply that $\{\mathcal{T}^{\bullet}\mathbf{W} : \mathbf{W} \in \mathfrak{F}^{\bullet}\}$ contains ℝ–trees of arbitrarily large diameter even when $\mathbf{X}$ is unbounded.

Lemma 4.6 (Weak uniqueness)**.** *Let $(\mathbf{X}, \mathfrak{F}^{\bullet})$ be an ℝ–cubing. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. If $\mathbf{x} \neq \mathbf{y}$, then there exists $\mathbf{W} \in \mathfrak{F}^{\bullet}$ such that $\pi_{\mathbf{W}}(\mathbf{x}) \neq \pi_{\mathbf{W}}(\mathbf{y})$.*

Proof. This is immediate from the definition. □

As an analogue of the fact that hierarchically hyperbolic spaces are coarse median spaces (see [BHS19, Section 7]), ℝ–cubings are median metric spaces:

Lemma 4.7 (ℝ–cubings are median)**.** *Let $(\mathbf{X}, \mathfrak{F}^{\bullet})$ be an ℝ–cubing. Fix $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$. Let $\mathbf{W} \in \mathfrak{F}^{\bullet}$ and let $\mu_{\mathbf{W}}$ be the median of $\pi_{\mathbf{W}}(\mathbf{x}), \pi_{\mathbf{W}}(\mathbf{y}), \pi_{\mathbf{W}}(\mathbf{z})$ in the ℝ–tree $\mathcal{T}^{\bullet}\mathbf{W}$. Then there exists a unique $\mu = \mu(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbf{X}$ such that $\pi_{\mathbf{W}}(\mu) = \mu_{\mathbf{W}}$ for all $\mathbf{W} \in \mathfrak{F}^{\bullet}$.*

In particular, $(\mathbf{X}, \mathsf{d}_{\mathbf{X}})$ is a complete, connected, median, geodesic metric space and $\mu : \mathbf{X}^3 \to \mathbf{X}$ is the median operator.

Before giving a detailed proof, we sketch a more conceptually revealing proof. First, equip $\prod_{\mathbf{U}\in\mathfrak{F}^{\bullet}} \mathcal{T}^{\bullet}\mathbf{U}$ with the product median coming from the median metrics on the real trees $\mathcal{T}^{\bullet}\mathbf{U}$. Note that $\ell_1(\mathfrak{F}^{\bullet})$ is a median subalgebra, and in fact a median metric space. So to prove that $\mathbf{X}$ is a connected median space whose median has the property given in the statement, we just need to show that $\mathbf{X}$ is a median subalgebra of $\ell_1(\mathfrak{F}^{\bullet})$. But each of the consistency conditions in Definition 4.2 defines a median subalgebra of $\mathcal{T}^{\bullet}\mathbf{U} \times \mathcal{T}^{\bullet}\mathbf{V}$ for some $\mathbf{U}, \mathbf{V}$, and by definition $\mathbf{X}$ is the intersection of the preimages of these subalgebras under the natural projection $\ell_1(\mathfrak{F}^{\bullet}) \to \mathcal{T}^{\bullet}\mathbf{U} \times \mathcal{T}^{\bullet}\mathbf{V}$, and hence a median subalgebra. This viewpoint is explained more fully in Section 7.

Proof of Lemma 4.7. Fix $\mathbf{W} \in \mathfrak{F}^\bullet$. For each distinct $a, b \in \{\pi_\mathbf{W}(\mathbf{x}), \pi_\mathbf{W}(\mathbf{y}), \pi_\mathbf{W}(\mathbf{z})\}$, let $[a,b]$ be the unique geodesic in $\mathcal{T}^\bullet\mathbf{W}$ joining a to b (and hence passing through $\mu_\mathbf{W}$).

Since $\mathbf{X}$ is path-connected, and $\pi_\mathbf{W}$ is continuous, $[a,b] \subset \pi_\mathbf{W}(\mathbf{X})$. Indeed, any $\mathbf{X}$–path from, say, $\mathbf{x}$ to $\mathbf{y}$ is sent by $\pi_\mathbf{W}$ to a path in $\mathcal{T}^\bullet\mathbf{W}$ joining $\pi_\mathbf{W}(\mathbf{x}), \pi_\mathbf{W}(\mathbf{y})$ and hence containing $[\pi_\mathbf{W}(\mathbf{x}), \pi_\mathbf{W}(\mathbf{y})]$ in its image.

Suppose that $\mathbf{V} \pitchfork \mathbf{W}$. Then by Definition 4.2.(5), up to switching $\mathbf{W}, \mathbf{V}$, we have that $\rho^\mathbf{W}_\mathbf{V}$ coincides with at least two of the points $\pi_\mathbf{V}(\mathbf{x}), \pi_\mathbf{V}(\mathbf{y}), \pi_\mathbf{V}(\mathbf{z})$. Hence $\mu_\mathbf{V} = \rho^\mathbf{W}_\mathbf{V}$, or the same holds with $\mathbf{V}$ and $\mathbf{W}$ reversed.

Similarly, if $\mathbf{V} \subsetneq \mathbf{W}$ and $\rho^\mathbf{V}_\mathbf{W}$ coincides with at least two of $\pi_\mathbf{W}(\mathbf{x}), \pi_\mathbf{W}(\mathbf{y}), \pi_\mathbf{W}(\mathbf{z})$, then $\mu_\mathbf{W} = \rho^\mathbf{V}_\mathbf{W}$.

So, suppose that $\mu_\mathbf{W} \neq \rho^\mathbf{V}_\mathbf{W}$. Then $\rho^\mathbf{V}_\mathbf{W}$ does not lie on the geodesic between, say, $\pi_\mathbf{W}(\mathbf{x})$ and $\pi_\mathbf{W}(\mathbf{y})$, while $\mu_\mathbf{W}$ does.

By Definition 4.2, $\pi_\mathbf{V}(\mathbf{x}) = \pi_\mathbf{V}(\mathbf{y}) = \mu_\mathbf{V}$, and each of these points coincides with $\rho^\mathbf{W}_\mathbf{V}(\pi_\mathbf{W}(\mathbf{x}))$ and $\rho^\mathbf{W}_\mathbf{V}(\pi_\mathbf{W}(\mathbf{y}))$. Now, $\rho^\mathbf{W}_\mathbf{V}$ is constant on each component of $\pi_\mathbf{W}(\mathbf{X}) - \{\rho^\mathbf{V}_\mathbf{W}\}$, and $\mu_\mathbf{W}$ lies on a geodesic from $\pi_\mathbf{W}(\mathbf{x})$ to $\pi_\mathbf{W}(\mathbf{y})$ and is not equal to $\rho^\mathbf{V}_\mathbf{W}$. So $\rho^\mathbf{W}_\mathbf{V}(\mu_\mathbf{W}) = \mu_\mathbf{V}$.

We have verified that $(\mu_\mathbf{W})_{\mathbf{W} \in \mathfrak{F}^\bullet}$ satisfies the conditions from Definition 4.2.(5), whence there exists $\mu \in \mathbf{X}$ such that $\pi_\mathbf{W}(\mu) = \mu_\mathbf{W}$ for all $\mathbf{W}$. The point μ is unique by Lemma 4.6.

μ **is a median:** Given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, let $\mu = \mu(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Define

$$S = S(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{1}{2}(\mathsf{d}_\mathbf{X}(\mathbf{x}, \mathbf{y}) + \mathsf{d}_\mathbf{X}(\mathbf{z}, \mathbf{y}) + \mathsf{d}_\mathbf{X}(\mathbf{x}, \mathbf{z})).$$

Define $T = T(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mu) = \sum_{a \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}} \mathsf{d}_\mathbf{X}(a, \mu)$.

For each $\mathbf{W} \in \mathfrak{F}^\bullet$, define

$$S_\mathbf{W} = S(\pi_\mathbf{W}(\mathbf{x}), \pi_\mathbf{W}(\mathbf{y}), \pi_\mathbf{W}(\mathbf{z}))$$

and

$$T_\mathbf{W} = T(\pi_\mathbf{W}(\mathbf{x}), \pi_\mathbf{W}(\mathbf{y}), \pi_\mathbf{W}(\mathbf{z}), \pi_\mathbf{W}(\boldsymbol{\mu}))$$

analogously. Since $\pi_\mathbf{W}(\mu)$ is the median of $\pi_\mathbf{W}(\mathbf{x}), \pi_\mathbf{W}(\mathbf{y}), \pi_\mathbf{W}(\mathbf{z})$ in $\mathcal{T}^\bullet\mathbf{W}$, we have $S_\mathbf{W} = T_\mathbf{W}$ for all $\mathbf{W}$, by characterisation (C2) of a median from [Bow16b, p. 6].

From the characterisation of $\mathsf{d}_\mathbf{X}$ as the restriction to $\mathbf{X}$ of the ℓ_1 metric, it follows immediately from the above that $S = T$. Another application of characterisation (C2) from [Bow16b] now shows that $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto \mu$ defines a median on $\mathbf{X}$.

Conclusion: Thus far, we know that $(\mathbf{X}, \mathsf{d}_\mathbf{X})$ is a median metric space that is path-connected and complete by hypothesis. So, by Lemma 4.6 of [Bow16b], $\mathbf{X}$ is a geodesic space. □

4.2. **ρ–consistency and nonempty product regions.** Another difference between the definition of a real cubing and a hierarchically hyperbolic space has to do with the fact that for real cubings, we have assumed the analogue of the HHS realisation theorem (Theorem 10.5) but omitted an analogue of the *partial realisation axiom* (Definition 10.1.(8)). Nonetheless, a weaker version of the omitted axiom holds:

Proposition 4.8. *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be a real cubing. Let $\mathbf{U} \in \mathfrak{F}^\bullet$. Let $\mathbf{P}_\mathbf{U}$ be the set of $\mathbf{x} \in \mathbf{X}$ such that $\pi_\mathbf{V}(\mathbf{x}) = \rho^\mathbf{U}_\mathbf{V}$ whenever $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \subsetneq \mathbf{V}$. Then $\mathbf{P}_\mathbf{U} = \varnothing$ only if $\pi_\mathbf{U}(\mathbf{X})$ is a single point.*

The converse of the proposition does not hold. Very typically, we will encounter real cubings where some $\pi_\mathbf{U}(\mathbf{X})$ is a single point and $\mathbf{P}_\mathbf{U} \neq \varnothing$. These $\mathbf{U}$ are totally innocuous. This is why, in Proposition 4.10, when we discard certain $\mathbf{U}$, without affecting $\mathbf{X}$, to get a new real cubing structure, we discard *only* those $\mathbf{U}$ with $\mathbf{P}_\mathbf{U} = \varnothing$. Other $\mathbf{U}$ stay, even if $\pi_\mathbf{U}(\mathbf{X})$ is a single point. This flexibility is useful in our applications to asymptotic cones.

Proof of Proposition 4.8. Suppose $\mathbf{U} \in \mathfrak{F}^\bullet$ satisfies $\pi_{\mathbf{U}}(\mathbf{x}) \neq \pi_{\mathbf{U}}(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, i.e. $\pi_{\mathbf{U}}(\mathbf{X})$ is not a single point.

Let γ be a geodesic in $\mathbf{X}$ from $\mathbf{x}$ to $\mathbf{y}$, which exists by Lemma 4.7. Since $\pi_{\mathbf{U}}$ is a lipschitz (hence continuous) median homomorphism (the former because of Definition 4.2.(5) and the latter by the characterisation of the median in Lemma 4.7), the composition $\pi_{\mathbf{U}} \circ \gamma$ is a (continuous) path and in fact an unparameterised geodesic in $\mathcal{T}^\bullet\mathbf{U}$ from $\pi_{\mathbf{U}}(\mathbf{x})$ to $\pi_{\mathbf{U}}(\mathbf{y})$. Hence we can choose $\mathbf{t} \in \gamma$ such that $\pi_{\mathbf{U}}(\mathbf{t})$ is an interior point of $\pi_{\mathbf{U}} \circ \gamma$, i.e. $\pi_{\mathbf{U}}(\mathbf{t}) \notin \{\pi_{\mathbf{U}}(\mathbf{x}), \pi_{\mathbf{U}}(\mathbf{y})\}$.

As above, we have for any other $\mathbf{V} \in \mathfrak{F}^\bullet$ that $\pi_{\mathbf{V}} \circ \gamma$ is (after reparameterising) the geodesic in the real tree $\mathcal{T}^\bullet\mathbf{V}$ from $\pi_{\mathbf{V}}(\mathbf{x})$ to $\pi_{\mathbf{V}}(\mathbf{y})$. In particular, $\pi_{\mathbf{V}}(\mathbf{t})$ lies on the geodesic in $\mathcal{T}^\bullet\mathbf{V}$ from $\pi_{\mathbf{V}}(\mathbf{x})$ to $\pi_{\mathbf{V}}(\mathbf{y})$.

We now argue that $\mathbf{t} \in \mathbf{P}_{\mathbf{U}}$, which will complete the proof.

Suppose that $\mathbf{V} \in \mathfrak{F}^\bullet$ satisfies $\mathbf{V} \pitchfork \mathbf{U}$. Then by Definition 4.2.(5) (the consistency condition), one of the following holds:

- $\pi_{\mathbf{V}}(\mathbf{x}) = \pi_{\mathbf{V}}(\mathbf{y}) = \rho^{\mathbf{U}}_{\mathbf{V}}$, in which case, since $\pi_{\mathbf{V}}(\mathbf{t})$ lies on the geodesic $\pi_{\mathbf{V}} \circ \gamma$, we get $\pi_{\mathbf{V}}(\mathbf{t}) = \rho^{\mathbf{U}}_{\mathbf{V}}$.
- $\pi_{\mathbf{V}}(\mathbf{x}) \neq \pi_{\mathbf{V}}(\mathbf{y})$ so that, up to interchanging the roles of $\mathbf{x}, \mathbf{y}$, we have $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ and $\pi_{\mathbf{U}}(\mathbf{y}) = \rho^{\mathbf{V}}_{\mathbf{U}}$. Since $\pi_{\mathbf{U}}(\mathbf{t}) \neq \pi_{\mathbf{U}}(\mathbf{y})$, consistency demands that $\pi_{\mathbf{V}}(\mathbf{t}) = \rho^{\mathbf{U}}_{\mathbf{V}}$.

Now suppose that $\mathbf{U} \sqsubsetneq \mathbf{V}$. The consistency and bounded geodesic image (Definition 4.2.(6)) imply that the geodesic $\pi_{\mathbf{V}} \circ \gamma$ passes through $\rho^{\mathbf{U}}_{\mathbf{V}}$. The point $\pi_{\mathbf{V}}(\mathbf{t})$ also lies on this geodesic. So if $\rho^{\mathbf{U}}_{\mathbf{V}} \neq \pi_{\mathbf{V}}(\mathbf{t})$, then consistency and bounded geodesic image imply that $\pi_{\mathbf{U}}(\mathbf{t}) \in \{\pi_{\mathbf{U}}(\mathbf{x}), \pi_{\mathbf{U}}(\mathbf{y})\}$, violating our choice of $\mathbf{t}$. Hence $\mathbf{t} \in \mathbf{P}_{\mathbf{U}}$, and we are done. □

The sets $\mathbf{P}_{\mathbf{U}}$ — the *standard product regions* in $\mathbf{X}$ — will be examined in more detail in Section 4.10, in which the name "product region" is justified. For now, we are only concerned with whether and when these subsets are nonempty.

Definition 4.9 (Nonempty products)**.** The ℝ–cubing $(\mathbf{X}, \mathfrak{F}^\bullet)$ has *nonempty products* if $\mathbf{P}_{\mathbf{U}} \neq \varnothing$ for all $\mathbf{U} \in \mathfrak{F}^\bullet$. □

Proposition 4.10. *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be a real cubing. Let $\mathfrak{F}^\bullet_1$ be the set of $\mathbf{U} \in \mathfrak{F}^\bullet$ with $\mathbf{P}_{\mathbf{U}} \neq \varnothing$. Then $(\mathbf{X}, \mathfrak{F}^\bullet_1)$ is a real cubing with nonempty products, where the relations $\sqsubseteq, \perp, \pitchfork$, the points/maps $\rho^\bullet_\bullet$, and the real trees are inherited from $(\mathbf{X}, \mathfrak{F}^\bullet)$.*

Proof. Let $\mathbf{X}'$ be the set of points in $\ell_1(\mathfrak{F}^\bullet_1)$ satisfying the consistency conditions in Definition 4.2.(5). By definition, $(\mathbf{X}', \mathfrak{F}^\bullet_1)$ is a real cubing provided $\mathbf{X}'$, with the ℓ_1–metric, is complete and path-connected.

The natural projection $\prod_{\mathbf{V} \in \mathfrak{F}^\bullet} \mathcal{T}^\bullet\mathbf{V} \to \prod_{\mathbf{U} \in \mathfrak{F}^\bullet_1} \mathcal{T}^\bullet\mathbf{U}$ induced by forgetting factors in $\mathfrak{F}^\bullet - \mathfrak{F}^\bullet_1$ induces a 1–lipschitz map $\mathbf{X} \to \mathbf{X}'$, since consistent points in the larger product are sent to consistent points in the small product, and the property of being finite distance from the basepoint is preserved. Since $\pi_{\mathbf{U}}(\mathbf{X})$ is a single point for $\mathbf{U} \in \mathfrak{F}^\bullet - \mathfrak{F}^\bullet_1$, by Proposition 4.8, the map $\mathbf{X} \to \mathbf{X}'$ is in fact an isometric embedding.

By another application of Proposition 4.8, every element of $\mathbf{X}'$ uniquely extends to an element of $\prod_{\mathbf{V} \in \mathfrak{F}^\bullet} \mathcal{T}^\bullet\mathbf{V}$ whose $\mathbf{V}$–coordinate is the point $\pi_{\mathbf{V}}(\mathbf{X})$ for $\mathbf{V} \notin \mathfrak{F}^\bullet_1$. The consistency condition is automatic for this extended tuple, so the map $\mathbf{X} \to \mathbf{X}'$ is surjective. Hence $\mathbf{X}'$ is isometric to $\mathbf{X}$, and thus complete and path-connected. Hence $(\mathbf{X}, \mathfrak{F}^\bullet_1)$ is a real cubing, and it has nonempty products by construction. □

Example 4.11. The reader interested in how empty products can arise should consider the example where $\mathbf{X} = \mathbb{R}$, and $\mathfrak{F}^\bullet$ consists of pairwise transverse elements $\mathbf{V}_n$, with $\mathcal{T}^\bullet\mathbf{V}_n = [n, n+1]$ for $n \in \mathbb{Z}$, and an additional $\mathbf{U}$, transverse to all $\mathbf{V}_n$, with $\mathcal{T}^\bullet\mathbf{U}$ a single point. Define $\rho^{\mathbf{V}_n}_{\mathbf{V}_{n-1}} = n$ and $\rho^{\mathbf{U}}_{\mathbf{V}_n} = n + 1$ for all n. Then $\mathbf{P}_{\mathbf{U}} = \varnothing$. □

The preceding proposition says that if we are only interested in $\mathbf{X}$ itself, we can assume that it has nonempty products. In practice, and in particular in our applications to asymptotic cones, we will be working with real cubings that have nonempty products by construction, and we will not need to apply the above proposition.

Finally, nonempty products yields the following consequence of the consistency condition in Definition 4.2, which will be useful in Section 6.

Lemma 4.12 (ρ–consistency)**.** *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be a real cubing with nonempty products. Let $\mathbf{U}, \mathbf{V}, \mathbf{W} \in \mathfrak{F}^\bullet$ satisfy*

- $\mathbf{U} \subsetneq \mathbf{V}$ *or* $\mathbf{U} \pitchfork \mathbf{V}$*, and*
- $\mathbf{U} \subsetneq \mathbf{W}$ *or* $\mathbf{U} \pitchfork \mathbf{W}$*.*

Then, if $\mathbf{V} \pitchfork \mathbf{W}$*, we have*

$$\min\{\mathsf{d}_{\mathbf{W}}(\rho^{\mathbf{U}}_{\mathbf{W}}, \rho^{\mathbf{V}}_{\mathbf{W}}), \mathsf{d}_{\mathbf{V}}(\rho^{\mathbf{U}}_{\mathbf{V}}, \rho^{\mathbf{W}}_{\mathbf{V}})\} = 0,$$

and if $\mathbf{V} \subsetneq \mathbf{W}$*, we have*

$$\min\{\mathsf{d}_{\mathbf{W}}(\rho^{\mathbf{U}}_{\mathbf{W}}, \rho^{\mathbf{V}}_{\mathbf{W}}), \mathsf{d}_{\mathbf{V}}(\rho^{\mathbf{U}}_{\mathbf{V}}, \rho^{\mathbf{W}}_{\mathbf{V}}(\rho^{\mathbf{U}}_{\mathbf{W}}))\} = 0.$$

More generally, the above conclusion holds without the nonempty products assumption whenever $\mathbf{U}, \mathbf{V}, \mathbf{W}$ are as above and, additionally, $\mathbf{P}_{\mathbf{U}} \neq \varnothing$.

Proof. It suffices to prove the "more generally" statement. Assume $\mathbf{P}_{\mathbf{U}} \neq \varnothing$ and choose $\mathbf{x} \in \mathbf{P}_{\mathbf{U}}$. Then $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ and $\pi_{\mathbf{W}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{W}}$. The lemma now follows by applying Definition 4.2.(5) to $\mathbf{x}$. ☐

4.3. **Real cubing rank and contractibility.** Recall that the *rank* of $\mathbf{X}$ as a median space is the maximal $n \in \mathbb{N}$ such that there is a median preserving embedding of an n–cube $\{0,1\}^n$ in $\mathbf{X}$.

Lemma 4.13. *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be an $\mathbb{R}$–cubing. Then $\mathbf{X}$ is a finite-rank median space.*

Remark 4.14 (Finite rank in practice)**.** The $\mathbb{R}$–cubings arising later in the paper will often be (spaces bilipschitz equivalent to) asymptotic cones of bounded-rank coarse median spaces with uniform coarse median constants. They are thus finite-rank topological median algebras, by [Bow13, Theorem 2.3], and, by modifying the metric in its bilipschitz class (but not modifying the median), they are finite-rank median metric spaces [Bow18b, Theorem 6.9]. But the preceding lemma says finite rank can be checked directly from the definition of an $\mathbb{R}$–cubing. ☐

Proof of Lemma 4.13. Fix $n \geqslant 0$ and let C be a median algebra isomorphic to $\{0,1\}^n$. Let $\phi : C \to \mathbf{X}$ be an injective, median preserving map. Let $\mathbf{0}$ be the ϕ–image of $(0,0,\ldots,0)$, let $e_i \in C$ be the vector whose i^{th} coordinate is 1 and all other coordinates are 0, and let $\mathbf{x}_i = \phi(e_i)$.

Let $i \neq j$. Then there exist $\mathbf{U}_i, \mathbf{U}_j \in \mathfrak{F}^\bullet$ such that $\pi_{\mathbf{U}_i}(\mathbf{0}) \neq \pi_{\mathbf{U}_i}(\mathbf{x}_i)$, and the same holds with i replaced by j. Let e_{ij} be the vector in C with 1 in the i and j coordinates and 0 elsewhere. Let $\mathbf{x}_{ij} = \phi(e_{ij})$. Note that $\mu(\mathbf{0}, \mathbf{x}_i, \mathbf{x}_{ij}) = \mathbf{x}_i$ and $\mu(\mathbf{0}, \mathbf{x}_j, \mathbf{x}_{ij}) = \mathbf{x}_j$, and $\mu(\mathbf{x}_{ij}, \mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_{ij}$. Also, $\mu(\mathbf{0}, \mathbf{x}_i, \mathbf{x}_j) = \mathbf{0}$.

Hence $\pi_{\mathbf{U}_i}(\mathbf{x}_i)$ lies on the unique geodesic in $\mathcal{T}^\bullet\mathbf{U}_i$ joining the projections of $\mathbf{0}$ and $\mathbf{x}_{ij}$. The same holds in $\mathbf{U}_j$, with $\pi_{\mathbf{U}_j}(\mathbf{x}_i)$ replacing $\pi_{\mathbf{U}_i}(\mathbf{x}_i)$. On the other hand, the projection of $\mathbf{0}$ to $\mathcal{T}^\bullet\mathbf{U}_i$ lies on the geodesic joining the projections of $\mathbf{x}_i, \mathbf{x}_j$, and the same holds with i and j reversing roles. Finally, the projection of $\mathbf{x}_{ij}$ lies on the geodesic joining the projections of $\mathbf{x}_i, \mathbf{x}_j$.

We claim that $\mathbf{U}_i \perp \mathbf{U}_j$. If not, then, up to relabelling, one of the following holds:

- $\mathbf{U}_i = \mathbf{U}_j$. Then each of $\pi_{\mathbf{U}_i}(\mathbf{x}_i), \pi_{\mathbf{U}_i}(\mathbf{x}_j)$ lies on the geodesic joining the other to $\pi_{\mathbf{U}_i}(\mathbf{0})$. This is impossible since $\pi_{\mathbf{U}_i}(\mathbf{x}_i), \pi_{\mathbf{U}_i}(\mathbf{x}_j)$ both differ from $\pi_{\mathbf{U}_j}(\mathbf{0})$. Hence $\mathbf{U}_i \neq \mathbf{U}_j$.
- $\mathbf{U}_i \pitchfork \mathbf{U}_j$. By consistency, up to reversing the roles of i and j, we have $\rho^{\mathbf{U}_j}_{\mathbf{U}_i} = \pi_{\mathbf{U}_i}(\mathbf{x}_{ij}) \neq \pi_{\mathbf{U}_i}(\mathbf{0})$, since $\pi_{\mathbf{U}_i}(\mathbf{x}_i)$ lies on the geodesic from $\pi_{\mathbf{U}_i}(\mathbf{0})$ to $\pi_{\mathbf{U}_i}(\mathbf{x}_{ij})$.

  By consistency, $\pi_{\mathbf{U}_j}(\mathbf{0}) = \rho^{\mathbf{U}_i}_{\mathbf{U}_j}$. But then by consistency, $\pi_{\mathbf{U}_i}(\mathbf{x}_j) = \pi_{\mathbf{U}_i}(\mathbf{x}_{ij})$, contradicting that the projection of $\mathbf{0}$ lies on the geodesic from that of $\mathbf{x}_i$ to that of $\mathbf{x}_j$. Hence $\mathbf{U}_i$ is not transverse to $\mathbf{U}_j$.
- $\mathbf{U}_i \subsetneq \mathbf{U}_j$ or vice versa. Assume the former. Then $\rho^{\mathbf{U}_i}_{\mathbf{U}_j}$ lies on the $\mathcal{T}^\bullet U_j$–geodesic joining the projections of $\mathbf{0}$ and $\mathbf{x}_{ij}$, by consistency and bounded geodesic image axioms. So, by the same axioms, either $\pi_{\mathbf{U}_i}(\mathbf{x}_j)$ coincides with the $\mathbf{U}_i$–projection of $\mathbf{0}$ or $\mathbf{x}_{ij}$, contradicting either that $\pi_{\mathbf{U}_i}(\mathbf{x}_i)$ lies on the geodesic joining the projections of $\mathbf{0}$ and $\mathbf{x}_j$, or that $\pi_{\mathbf{U}_i}(\mathbf{x}_{ij})$ lies on the geodesic joining the projections of $\mathbf{x}_i$ and $\mathbf{x}_i$.

Hence we have found n pairwise orthogonal elements $\mathbf{U}_1, \ldots, \mathbf{U}_n$ of $\mathfrak{F}^\bullet$, whence $n \leqslant \chi^\bullet$ by the finite complexity axiom. So, the rank of $\mathbf{X}$ as a median space is at most $\chi^\bullet < \infty$. □

From the preceding, and [Bow16b, Theorem 1.1], we get:

Corollary 4.15 (Contractibility)**.** *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be an $\mathbb{R}$–cubing. Then $\mathbf{X}$ is bilipschitz homeomorphic to a CAT(0) space, and is in particular contractible.*

4.4. ☺ **Discrete real cubings.** We next show that discrete real cubings are precisely CAT(0) cube complexes. We say that a real cubing $(\mathbf{X}, \mathfrak{F}^\bullet)$ is *discrete* if each $\mathcal{T}^\bullet\mathbf{W}, \mathbf{W} \in \mathfrak{F}^\bullet$ is a simplicial tree, and $\rho^{\mathbf{V}}_{\mathbf{W}}$ is a vertex of $\mathcal{T}^\bullet\mathbf{W}$ whenever it is defined and a single point.

Theorem 4.16. *Every discrete real cubing is median-preservingly, ℓ_1–isometric to a finite-dimensional CAT(0) cube complex. Conversely, every finite-dimensional CAT(0) cube complex is median-preservingly ℓ_1–isometric to a discrete real cubing.*

Proof. In Example 4.25 we will show that CAT(0) cube complexes are discrete real cubings (the associated trees are single edges and the ρ–points are vertices), so it suffices to show the converse.

If $(\mathbf{X}, \mathfrak{F}^\bullet)$ is a discrete real cubing, then it is a finite-rank connected median subalgebra in a (possibly infinite) product of simplicial trees, by Lemma 4.7 and Lemma 4.13. Hence $\mathbf{X}$ is a finite-dimensional CAT(0) cube complex. □

The preceding theorem is not much more informative than the fact that any CAT(0) cube complex is an isometrically embedded median subspace of an (infinite) cube. Later we show that one gets a more useful real cubing structure on $\mathbf{X}$ when the orthogonal poset-colouring of the hyperplanes in $\mathbf{X}$ has finite depth, and we relate this to conditions in [BHS17b] ensuring that $\mathbf{X}$ is a hierarchically hyperbolic space.

4.5. **Convex subspaces and gates.** We now study convex subspaces of $\mathbb{R}$–cubings.

Definition 4.17 (Convexity)**.** Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be a $\mathbb{R}$–cubing. We say that $\mathbf{Y} \subset \mathbf{X}$ is *convex* if the following hold:

- $\pi_{\mathbf{W}}(\mathbf{Y})$ is a subtree of $\pi_{\mathbf{W}}(\mathbf{X})$ for all $\mathbf{W} \in \mathfrak{F}^\bullet$;
- if $\mathbf{x} \in \mathbf{X}$ satisfies $\pi_{\mathbf{W}}(\mathbf{x}) \in \pi_{\mathbf{W}}(\mathbf{Y})$ for all $\mathbf{W} \in \mathfrak{F}^\bullet$, then $\mathbf{x} \in \mathbf{Y}$.

We will almost always be interested in convex sets that are also closed. □

Lemma 4.18. *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be an $\mathbb{R}$–cubing and let μ be the median on $\mathbf{X}$ from Lemma 4.7. Then $\mathbf{Y} \subseteq \mathbf{X}$ is convex only if $\mathbf{Y}$ is median-convex, i.e. $\mu(\mathbf{y}, \mathbf{y}', \mathbf{x}) \in \mathbf{Y}$ whenever $\mathbf{y}, \mathbf{y}' \in \mathbf{Y}$ and $\mathbf{x} \in \mathbf{X}$.*

Proof. Suppose that $\mathbf{Y}$ is convex. Fix $\mathbf{y}, \mathbf{y}', \mathbf{x}$ as in the statement and let μ be their median. For each $\mathbf{W} \in \mathfrak{F}^\bullet$, the point $\mu_{\mathbf{W}}$ lies on the geodesic from $\pi_{\mathbf{W}}(\mathbf{y})$ to $\pi_{\mathbf{W}}(\mathbf{y}')$ and hence in $\pi_{\mathbf{W}}(\mathbf{Y})$, by the first condition in Definition 4.17. Hence, by the second condition, $\mu \in \mathbf{Y}$. $\square$

The next lemma reflects a general fact about median metric spaces:

Lemma 4.19 (Gates in $\mathbb{R}$–cubings)**.** *Let* $(\mathbf{X}, \mathfrak{F}^\bullet)$ *be a* $\mathbb{R}$*–cubing and let* $\mathbf{Y} \subset \mathbf{X}$ *be closed and convex. Then there is a* 1*–lipschitz retraction* $\mathfrak{g}_{\mathbf{Y}} = \mathfrak{g} : \mathbf{X} \to \mathbf{Y}$*. Moreover, for all* $\mathbf{x} \in \mathbf{X}$ *and* $\mathbf{U} \in \mathfrak{F}^\bullet$*, the point* $\pi_{\mathbf{U}}(\mathfrak{g}_{\mathbf{Y}}(\mathbf{x}))$ *is the closest point in the subtree* $\pi_{\mathbf{U}}(\mathbf{Y}) \subset \mathcal{T}^\bullet\mathbf{U}$ *to* $\pi_{\mathbf{U}}(\mathbf{x})$*.*

Proof. By the preceding lemma, $\mathbf{Y}$ is median convex in $\mathbf{X}$, and hence there is a unique map $\mathfrak{g}_{\mathbf{Y}} : \mathbf{X} \to \mathbf{Y}$ such that $\mathsf{d}_{\mathbf{X}}(\mathbf{x}, \mathfrak{g}_{\mathbf{Y}}(\mathbf{x})) = \mathsf{d}_{\mathbf{X}}(\mathbf{x}, \mathbf{Y})$ for all $\mathbf{x} \in \mathbf{X}$. The gate map $\mathfrak{g}_{\mathbf{Y}}$ is a retraction by definition and is always 1–lipschitz [CDH10, Lemma 2.13].

Let $\mathbf{x} \in \mathbf{X}$. For any $\mathbf{y} \in \mathbf{Y}$, we have a geodesic $\gamma_{\mathbf{y}}$ from $\mathbf{x}$ to $\mathbf{y}$ passing through $\mathfrak{g}_{\mathbf{Y}}(\mathbf{x})$.

Fix $\mathbf{W} \in \mathfrak{F}^\bullet$. Then for any $\mathbf{y} \in \mathbf{Y}$, we have that $\pi_{\mathbf{W}} \circ \gamma_{\mathbf{y}}$ is (after reparametrising) a geodesic of $\mathcal{C}\mathbf{W}$, since $\pi_{\mathbf{W}}$ is a median-preserving map. So, for any point $p \in \pi_{\mathbf{W}}(\mathbf{Y})$, we have that $\mu_{\mathbf{W}}(\pi_{\mathbf{W}}(\mathbf{x}), \pi_{\mathbf{W}}(\mathfrak{g}_{\mathbf{Y}}(\mathbf{x})), p) = \pi_{\mathbf{W}}(\mathfrak{g}_{\mathbf{Y}}(\mathbf{x}))$. Hence $\pi_{\mathbf{W}}(\mathfrak{g}_{\mathbf{Y}}(\mathbf{x}))$ is the closest point of $\pi_{\mathbf{W}}(\mathbf{Y})$ to $\pi_{\mathbf{W}}(\mathbf{x})$. $\square$

We note the following converse:

Lemma 4.20 (Gated implies convex)**.** *Let* $(\mathbf{X}, \mathfrak{F}^\bullet)$ *be an* $\mathbb{R}$*–cubing and let* $\mathbf{Y} \subset \mathbf{X}$ *have the property that there is a* 1*–lipschitz retraction* $\mathfrak{g} : \mathbf{X} \to \mathbf{Y}$ *satisfying the conclusion of Lemma 4.19. Then* $\mathbf{Y}$ *is convex.*

Proof. Let $\mathbf{U} \in \mathfrak{F}^\bullet$. Since $\mathfrak{g}$ is continuous and $\mathbf{X}$ is connected, $\mathbf{Y}$ is connected. Continuity of $\pi_{\mathbf{U}}$ implies that $\pi_{\mathbf{U}}(\mathbf{Y})$ is connected, i.e. a subtree of $\mathcal{T}^\bullet\mathbf{U}$. Suppose that $\mathbf{x} \in \mathbf{X}$ satisfies $\pi_{\mathbf{V}}(\mathbf{x}) \in \pi_{\mathbf{V}}(\mathbf{Y})$ for all $\mathbf{V}$. Then by hypothesis, $\mathfrak{g}(\mathbf{x})$ has the same image in $\mathcal{T}^\bullet\mathbf{V}$ as $\mathbf{x}$, for all $\mathbf{V}$, so Lemma 4.6 implies that $\mathfrak{g}(\mathbf{x}) = \mathbf{x}$. Hence $\mathbf{x} \in \mathbf{Y}$. Thus $\mathbf{Y}$ is convex, by Definition 4.17. $\square$

One can also define convex hulls from the $\mathbb{R}$–cubing viewpoint, and relate this to the median-convex hull.

Definition 4.21 (Convex hulls)**.** Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be an $\mathbb{R}$–cubing, and let $A \subset \mathbf{X}$. Then *convex hull* $\mathrm{Hull}^\bullet(A)$ of A is the intersection of all closed convex subsets of $\mathbf{X}$ that contain A. Let $\mathrm{Hull}^\bullet_1(A)$ be the intersection of all closed, median-convex subsets containing A. $\square$

Lemma 4.22. *Let* $A \subset \mathbf{X}$*. Then* $\mathrm{Hull}^\bullet(A) = \mathrm{Hull}^\bullet_1(A)$*.*

In particular, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$*, the subspace* $\mathrm{Hull}^\bullet(\{\mathbf{x}, \mathbf{y}\})$ *and* $\mathrm{Hull}^\bullet_1(\{\mathbf{x}, \mathbf{y}\})$ *coincide with the* median interval *between* $\mathbf{x}$ *and* $\mathbf{y}$*, i.e. the set of all* $\mathbf{z}$ *with* $\mu(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{z}$*.*

Proof. Let M be a closed convex set containing A. By Lemma 4.18, M is median-convex and closed. So $\mathrm{Hull}^\bullet_1(A) \subset \mathrm{Hull}^\bullet(A)$.

Conversely, suppose that $M \supset A$ is median-convex and closed. Then M admits a gate map $\mathfrak{g} : \mathbf{X} \to M$. Since $\mathfrak{g}$ is continuous and surjective and $\pi_{\mathbf{W}}$ is continuous for all $\mathbf{W} \in \mathfrak{F}^\bullet$, $\pi_{\mathbf{W}}(M)$ is connected, and hence a subtree. Moreover, since $\pi_{\mathbf{W}}$ takes the median μ to the median $\mu_{\mathbf{W}}$, and $\mathfrak{g}$ is characterised by the fact that $\mu(\mathbf{y}, \mathbf{x}, \mathfrak{g}(\mathbf{x})) = \mathfrak{g}(\mathbf{x})$ whenever $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in M$ (see e.g. [Bow20, Section 4]), we see that $\pi_{\mathbf{W}}(\mathfrak{g}(\mathbf{x}))$ is the closest point of $\pi_{\mathbf{W}}(M)$ to $\pi_{\mathbf{W}}(\mathbf{x})$. Hence, by Lemma 4.20, M is convex. Hence $\mathrm{Hull}^\bullet_1(A) \subset \mathrm{Hull}^\bullet(A)$.

The statement about median intervals follows immediately from the first part of the lemma, once we recall that the median interval between $\mathbf{x}, \mathbf{y}$ is exactly $\mathrm{Hull}^\bullet_1(\{\mathbf{x}, \mathbf{y}\})$ (e.g. [CDH10, Corollary 2.15] combined with the face that any closed median-convex set containing $\mathbf{x}, \mathbf{y}$ contains the median interval). $\square$

The final property of convexity established here is:

Lemma 4.23. *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be an $\mathbb{R}$–cubing. Let $\mathbf{Y} \subset \mathbf{X}$ be closed and convex. Then $\mathbf{Y}$ is an $\mathbb{R}$–cubing.*

Proof. For each $\mathbf{U} \in \mathfrak{F}^\bullet$, let $\mathcal{T}_1^\bullet\mathbf{U}$ be the image of $\mathcal{T}^\bullet\mathbf{U}$ under closest-point projection to closure of the subtree $\pi_{\mathbf{U}}(\mathbf{Y})$. The indexing set in our new $\mathbb{R}$–cubing structure will be $\mathfrak{F}^\bullet$, and $\mathcal{T}_1^\bullet\mathbf{U}$ will be the $\mathbb{R}$–tree associated to $\mathbf{U}$. Given $\mathbf{U}, \mathbf{V}$ transverse, let $(\rho')^{\mathbf{U}}_{\mathbf{V}}$ be the image of $\rho^{\mathbf{U}}_{\mathbf{V}}$ under the gate map $\mathcal{T}^\bullet\mathbf{V} \to \mathcal{T}_1^\bullet\mathbf{V}$, and define $(\rho')^{\mathbf{U}}_{\mathbf{V}}$ likewise when $\mathbf{U} \subsetneq \mathbf{V}$. In this case, let $(\rho')^{\mathbf{V}}_{\mathbf{U}}$ be obtained by composing the above closest-point projection to $\pi_{\mathbf{U}}(\mathbf{Y})$ with $\rho^{\mathbf{V}}_{\mathbf{U}}$. It is now easily verified that this data determines a $\mathbb{R}$–cubing structure on $\mathbf{Y}$. ☐

The preceding properties mirror statements about *hierarchically quasiconvex* subsets of hierarchically hyperbolic spaces, in [BHS19, Section 5].

4.6. ☺ **Motivating examples.** We now discuss some basic examples of real cubings.

Example 4.24 (Finite products of $\mathbb{R}$–trees)**.** Let $\mathfrak{F}^\bullet$ be a finite set, and for each $\mathbf{U} \in \mathfrak{F}^\bullet$, let $\mathcal{T}^\bullet\mathbf{U}$ be a (based) $\mathbb{R}$–tree. Let $\pitchfork$ be the empty relation on $\mathfrak{F}^\bullet$, let $\sqsubseteq$ be the trivial reflexive relation, and declare $\mathbf{U} \perp \mathbf{V}$ for all distinct $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet$. Then $(\prod_{\mathbf{U} \in \mathfrak{F}^\bullet} \mathcal{T}^\bullet\mathbf{U}, \mathfrak{F}^\bullet)$ is an $\mathbb{R}$–cubing. In particular, if the $\mathbb{R}$–trees are lines, we see that $(\mathbb{R}^n, \mathfrak{F}^\bullet)$ is an $\mathbb{R}$–cubing for any $n \geqslant 0$, where $\mathbb{R}^n$ is given the ℓ_1 metric. ☐

Example 4.25 (Cubings are $\mathbb{R}$–cubings)**.** Let $\mathbf{X}$ be a finite-dimensional CAT(0) cube complex with base 0–cube x_0. Let $\mathfrak{F}^\bullet$ be the set of hyperplanes. For each $\mathbf{H} \in \mathfrak{F}^\bullet$, let $\mathcal{T}^\bullet\mathbf{H}$ be a 1–cube dual to $\mathbf{H}$, which we identify with $[-\frac{1}{2}, \frac{1}{2}]$ in the following way. Let $\mathbf{H}^{\pm}$ be the two halfspaces of $\mathbf{X}$ associated to $\mathbf{H}$, and identify $\pm\frac{1}{2}$ with the 0–cube of $\mathcal{T}^\bullet\mathbf{H}$ lying in $\mathbf{H}^{\pm}$. We base $\mathcal{T}^\bullet\mathbf{H}$ at $\pm\frac{1}{2}$ according to whether $x_0 \in \mathbf{H}^{\pm}$.

Declare $\mathbf{H} \perp \mathbf{H}'$ if $\mathbf{H}, \mathbf{H}'$ cross (i.e. they are distinct and have nonempty intersection). Otherwise, if $\mathbf{H} \neq \mathbf{H}'$, declare $\mathbf{H} \pitchfork \mathbf{H}'$.

Given transverse (i.e. disjoint) $\mathbf{H}, \mathbf{H}'$, let $\rho^{\mathbf{H}}_{\mathbf{H}'}$ be $\pm\frac{1}{2}$ if $\mathbf{H} \subset (\mathbf{H}')^{\pm}$.

Let $\mathbf{Y} \subset \ell_1(\mathfrak{F}^\bullet)$ be the subspace consisting of all consistent tuples $(\mathbf{y_H})_{\mathbf{H} \in \mathfrak{F}^\bullet}$. So, $(\mathbf{Y}, \mathfrak{F}^\bullet)$ is a $\mathbb{R}$–cubing (with trivial $\sqsubseteq$ relation).

We can now define an isometric embedding $(\mathbf{X})^{(0)} \to \mathbf{Y}$ as follows, where $(\mathbf{X})^{(0)}$ is equipped with the graph-metric from $(\mathbf{X})^{(1)}$ (recall that this means that the distance between $x, y \in (\mathbf{X})^{(0)}$ is the number of hyperplanes separating x, y).

Given a 0–cube x, and a hyperplane $\mathbf{H}$, let $x_{\mathbf{H}} \in \mathcal{T}^\bullet\mathbf{H}$ be $\pm\frac{1}{2}$ if $x \in \mathbf{H}^{\pm}$.

If $\mathbf{H}, \mathbf{H}'$ are transverse, then either $\mathbf{H}$ separates x from $\mathbf{H}'$, or $\mathbf{H}', x$ lie in the same halfspace of $\mathbf{X}$ associated to $\mathbf{H}$. In the later case, $x_{\mathbf{H}} = \rho^{\mathbf{H}'}_{\mathbf{H}}$, and in the former case, $\mathbf{H}'$ does not separate $\mathbf{H}, x$, so $\rho^{\mathbf{H}}_{\mathbf{H}'} = x_{\mathbf{H}'}$. Hence the tuple $(x_{\mathbf{H}})$ is consistent, and thus defines a unique point $\mathbf{y}(x) \in \mathbf{Y}$.

By definition, if $\mathbf{H} \in \mathfrak{F}^\bullet$ and $x, x' \in (\mathbf{X})^{(0)}$, then $\mathsf{d}_{\mathbf{H}}(\pi_{\mathbf{H}}(\mathbf{y}(x)), \pi_{\mathbf{H}}(\mathbf{y}(x'))) \neq 0$ if and only if $\mathbf{H}$ separates x, x', so $x \mapsto \mathbf{y}(x)$ defines an isometric embedding. This map can be extended to the open cubes of $\mathbf{X}$ to produce an ℓ_1–isometry $\mathbf{X} \to \mathbf{Y}$, when $\mathbf{X}$ is given the piecewise–ℓ_1 metric from, e.g. [Mie14]. In other words, finite-dimensional CAT(0) cube complexes are $\mathbb{R}$–cubings. (Note that we need finite dimension to ensure that the finite complexity axiom is satisfied.)

Because of the lack of nesting and the way the $\mathbb{R}$–trees and projections were defined, this example does not exhibit an important phenomenon that one can see in more general $\mathbb{R}$–cubings: the set $\mathbf{Y}$ of consistent tuples is in this case contained in the set of points $(x_{\mathbf{H}})$ such that $x_{\mathbf{H}} \neq (x_0)_{\mathbf{H}}$ for only finitely many values of $\mathbf{H}$. Indeed, let $\mathbf{H} \pitchfork \mathbf{H}'$. Without loss of generality, $\rho^{\mathbf{H}'}_{\mathbf{H}} = \frac{1}{2}$ and $\rho^{\mathbf{H}}_{\mathbf{H}'} = -\frac{1}{2}$. If $x_{\mathbf{H}} \in (-\frac{1}{2}, \frac{1}{2})$, then by consistency, $x_{\mathbf{H}'} = -\frac{1}{2}$. So, the set of $\mathbf{H}$ where $x_{\mathbf{H}}$ is not at the endpoint of the associated interval is pairwise-orthogonal and thus finite. The set of $\mathbf{H}$ where $x_{\mathbf{H}}$ is at an endpoint of the associated interval different from $(x_0)_{\mathbf{H}}$ is finite since we only consider points in $\ell_1(\mathfrak{F}^\bullet)$. So, there are finitely many $\mathbf{H}$ where the x and x_0–coordinates differ. (This reflects surjectivity of $\mathbf{X} \to \mathbf{Y}$.)

In more complicated examples, like asymptotic cones of hierarchically hyperbolic spaces, the basepoint x_0 need not have the property that $\mathsf{d}_{\mathbf{H}}(x_0, \rho^{\mathbf{H}'}_{\mathbf{H}})$ is either 0 or 1, and one finds consistent points that differ from the basepoint on infinitely many coordinates. □

Remark 4.26. The cubical example illustrates a key difference between the notion of a $\mathbb{R}$–cubing and a hierarchically hyperbolic space: the uniqueness axiom for hierarchically hyperbolic spaces (Definition 10.1.(9)) can not hold for the above construction, since the $\mathcal{T}^\bullet\mathbf{H}$ have uniformly bounded diameter. So, a $\mathbb{R}$–cubing is not simply a hierarchically hyperbolic space with each associated hyperbolic space being an $\mathbb{R}$–tree and all coarse equalities in the definition replaced with equalities. Indeed, there are simple examples of CAT(0) cube complexes of finite dimension that do not admit hierarchically hyperbolic structures that are compatible with the cubical/median structure [BHS17b, HS20]. □

Example 4.27 (Trees of flats)**.** Let $G = \langle a, b, c \mid [a, b] = 1\rangle \cong \mathbb{Z}^2 * \mathbb{Z}$. Let X be the presentation complex (a nonpositively-curved square complex) and let $\widetilde{X}$ be its universal cover. As a CAT(0) cube complex, $\widetilde{X}$ has a unique metric making each cube convex and isometric to a Euclidean unit cube with the ℓ_1 metric [Mie14]; let d_1 denote this metric.

Consider the following index set $\mathfrak{F}^\bullet$, consisting of all cosets in G of canonical cyclic subgroups:

- $g\mathbf{A} = g\langle a\rangle$.
- $g\mathbf{B} = g\langle b\rangle$.
- $g\mathbf{C} = g\langle c\rangle$.

We declare $g\mathbf{A} \perp g\mathbf{B}$ for each g, and all other pairs are transverse. For each $\mathbf{U} \in \mathfrak{F}^\bullet$, let $\mathcal{T}^\bullet\mathbf{U}$ be an isometric copy of $\mathbb{R}$.

We leave the following as exercises for the reader:

- Define projections in such a way that the above data makes $(\widetilde{X}, \mathsf{d}_1)$ a real cubing with index set $\mathfrak{F}^\bullet$.
- Show that the real cubing metric defined in $\widetilde{X}$ (as a subspace of $\ell_1(\mathfrak{F}^\bullet)$) coincides with d_1.
- Note that $(\widetilde{X}, \mathsf{d}_1, \boldsymbol{\mu})$ is a complete connected median space of finite rank. Describe the orthogonal poset-colouring for $(\widetilde{X}, \mathsf{d}_1, \boldsymbol{\mu})$ and show that it satisfies the tangible filter condition.
- Modify X to a square complex X' by blowing up the vertex to an edge, with one endpoint contained in a torus and one contained in a circle. Find a real cubing structure on the universal cover $\widetilde{X'}$ whose index set contains a unique $\sqsubseteq$–maximal element whose associated $\mathbb{R}$–tree is the Bass-Serre tree of the above splitting of G. Show that this can be done in such a way that the $\mathbb{R}$–cubing structure is simultaneously an HHS structure, see Definition 10.1.
- What is the relations between the two metrics?
- Note that $(\widetilde{X'}, \mathsf{d}_1', \boldsymbol{\mu}')$ is a complete connected median space of finite rank. Describe the orthogonal poset-colouring for $(\widetilde{X'}, \mathsf{d}_1', \boldsymbol{\mu}')$ and show that it satisfies the tangible filter condition.

Together with Example 4.25, this shows that a space can admit multiple distinct real cubing structures. □

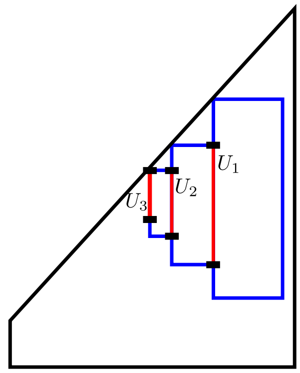

FIGURE 9. First, produce a rectangle satisfying the hypotheses of Claim 4, and then apply the claim to produce the red interval, which embeds in some $\mathcal{T}^\bullet U_n$. Then build a new rectangle with top-left corner on L, and its right vertical side contained in the red interval. This rectangle yields $\mathbf{U}_{n+1}$ by another application of the claim, and $\mathbf{U}_{n+1} \subsetneq \mathbf{U}_n$.

4.7. ☺ **Motivating non-example and some discussion of trapezoids.** Not every complete, connected, finite-rank median metric space is isometric to a real cubing. In this section, we given an example.

Let $\mathbf{X}$ be the (closed) trapezoid in $\mathbb{R}^2$ determined by the points $(0,0),(0,1),(1,0),(1,2)$. Let d_1 be the metric on $\mathbf{X}$ obtained by restricting the ℓ_1 metric on $\mathbb{R}^2$ to $\mathbf{X}$. Let $\boldsymbol{\mu}$ be the usual product median on $\mathbb{R}^2$.

Observe that $\mathbf{X}$ is a median subalgebra of $\mathbb{R}^2$. So, $(\mathbf{X},\mathsf{d}_1,\boldsymbol{\mu})$ is a complete, connected median metric space of rank 2.

Proposition 4.28. *There does not exist an $\mathbb{R}$–cubing into which $\mathbf{X}$ isometrically embeds, preserving the median, as a median-convex subspace. In particular, $\mathbf{X}$ is not an $\mathbb{R}$–cubing.*

Remark 4.29. By construction, $\mathbf{X}$ is an isometrically embedded median subalgebra of an $\mathbb{R}$–cubing, namely $\mathbb{R}^2$ with the ℓ_1 metric and median $\boldsymbol{\mu}$, but it is not median-convex since, for example, $\boldsymbol{\mu}((0,0),(0,2),(1,2)) = (0,2) \notin \mathbf{X}$ but $(0,0),(1,2) \in \mathbf{X}$. □

Sketch of Proposition 4.28. Suppose that $(\mathbf{Y},\mathfrak{F}^\bullet)$ is an $\mathbb{R}$–cubing and $\mathbf{X} \to \mathbf{Y}$ is a median-preserving isometric embedding with convex image. Then by Lemma 4.23, $(\mathbf{X},\mathfrak{F}^\bullet)$ is an $\mathbb{R}$–cubing.

So it suffices to derive a contradiction from the assumption that there is a real cubing structure $(\mathbf{X},\mathfrak{F}^\bullet)$ giving rise to the metric d_1 and the median $\boldsymbol{\mu}$.

The key fact is the following consequence of the definition of an $\mathbb{R}$–cubing, which we leave as an exercise[5]

Claim 4. Let L be the Euclidean line segment in $\mathbf{X}$ joining $(0,1)$ to $(1,2)$. Let $p \in L - \{(0,1),(1,2)\}$. Let $[a,b]\times[c,d]$ be a nontrivial axis-parallel rectangle in $\mathbf{X}$ (necessarily median-convex) with $(a,d)=p$. Then there exist $e<e' \in (c,d)$ such that there is a $\mathbf{U} \in \mathfrak{F}^\bullet$ on which the projections of (a,e) and (a,e') differ and on the segment from (a,e) to (a,e'), the projection $\pi_{\mathbf{U}}:\mathbf{X}\to\mathcal{T}^\bullet\mathbf{U}$ is an isometric embedding.

Applying the preceding claim iteratively, as shown in Figure 9, one finds $\subsetneq$–chains of arbitrary length, contradicting Definition 4.2.(4).

The remaining things to check are:

[5]Apologies; we will not leave things as exercises in this paper where a full proof is needed for the main results, but we have sometimes done so in explaining examples.

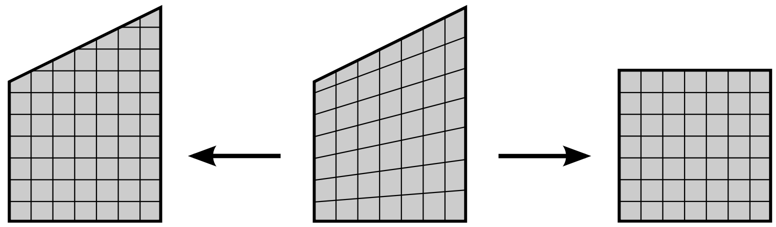

FIGURE 10. At left is the median metric space $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$. The lines indicate some of the walls determined by the median $\boldsymbol{\mu}$. Since this is just the subspace metric and median inherited from the plane, these walls are vertical or horizontal. The middle space is $\mathbf{X}$, equipped with the same metric d_1, but with a different median, indicated again by some of the walls. *This is not a median metric space.* For example, the right and left vertical segments cross exactly the same walls but have different lengths. At right is $[0,1]^2$, with the product median and ℓ_1 metric; it is a median metric space. The right arrow is a median isomorphism, but not an isometry (it is bilipschitz). The left arrow is an isometry, but not a median isomorphism.

- one must produce the sequence of rectangles as in Figure 9;
- one must check that any two parallel segments in one of the rectangles are "supported" on the same ℝ–trees;
- one must check that we do not have $\mathbf{U}_n = \mathbf{U}_{n+1}$, which is essentially because of Definition 4.2.(5).

We leave the details to the reader, since this is an illustrative example. $\square$

We note that $\mathbf{X}$ also supports a different median, $\boldsymbol{\mu}'$, obtained by pulling back the product median on $[0,1]^2$ under the homeomorphism $\mathbf{X} \to [0,1]^2$ that scales each vertical segment down so that it has length 1.

The triple $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu}')$ is not a median metric space. However, the above homeomorphism shows that we can change the metric in its bilipschitz class, *preserving the median*, to obtain a median metric space, namely $[0,1]^2$. This is an instance of an important result of Bowditch that we shall use later. See Figure 10 for a summary.

4.8. **Automorphisms of real cubings.** We will later be interested not only in ℝ–cubings, but in group actions on them. We first define the notion of a morphism of ℝ–cubings, which is a lipschitz map at the level of the space, and which preserves the ℝ–cubing structure.

Definition 4.30 (Morphism of ℝ–cubings)**.** Let $(\mathbf{X}, \mathfrak{F}^{\bullet}), (\mathbf{Y}, \mathfrak{H}^{\bullet})$ be ℝ–cubings and $K \geqslant 1$. A *K–morphism* is a triple $(f, I, \{f_{\mathbf{U}} : \mathbf{U} \in \mathfrak{F}^{\bullet}\})$ where

- $f : \mathbf{X} \to \mathbf{Y}$ is a K–lipschitz map;
- $I : \mathfrak{F}^{\bullet} \to \mathfrak{H}^{\bullet}$ is a map preserving $\sqsubseteq, \perp, \pitchfork$;
- for each $\mathbf{U} \in \mathfrak{F}^{\bullet}$, the map $f_{\mathbf{U}} : \mathcal{T}^{\bullet}\mathbf{U} \to \mathcal{T}^{\bullet} I(\mathbf{U})$ is a K–lipschitz injective map,

and all of the following hold:

- for all $\mathbf{x} \in \mathbf{X}$ and $\mathbf{U} \in \mathfrak{F}^{\bullet}$, we have $f_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x})) = \pi_{I(\mathbf{U})}(f(\mathbf{x}))$;
- if $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$ satisfy $\mathbf{U} \pitchfork \mathbf{V}$, then $f_{\mathbf{V}}(\rho^{\mathbf{U}}_{\mathbf{V}}) = \rho^{I(\mathbf{U})}_{I(\mathbf{V})}$;
- if $\mathbf{U} \sqsubsetneq \mathbf{V}$, then $f_{\mathbf{V}}(\rho^{\mathbf{U}}_{\mathbf{V}}) = \rho^{I(\mathbf{U})}_{I(\mathbf{V})}$.

$\square$

Remark 4.31 ("Downward" $\rho^{\bullet}_{\bullet}$ maps and morphisms)**.** Let $f, I, \{f_{\mathbf{U}}\}$ be as in Definition 4.30. Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$ satisfy $\mathbf{U} \subsetneq \mathbf{V}$. Then for any $\mathbf{x} \in \mathbf{X}$, the following holds. If $\pi_{\mathbf{V}}(\mathbf{x}) \neq \rho^{\mathbf{U}}_{\mathbf{V}}$, then by consistency, we have $\rho^{\mathbf{V}}_{\mathbf{U}}(\pi_{\mathbf{V}}(\mathbf{x})) = \pi_{\mathbf{U}}(\mathbf{x})$.

On the other hand, $f_{\mathbf{V}}(\rho^{\mathbf{V}}_{\mathbf{U}}) = \rho^{I(\mathbf{U})}_{I(\mathbf{V})}$ so, since $f_{\mathbf{V}}$ is injective and $\pi_{\mathbf{V}}(\mathbf{x}) \neq \rho^{\mathbf{U}}_{\mathbf{V}}$, we have $\pi_{I(\mathbf{V})}(f(\mathbf{x})) = f_{\mathbf{V}}(\pi_{\mathbf{V}}(\mathbf{x})) \neq \rho^{I(\mathbf{U})}_{I(\mathbf{V})}$. Hence, by consistency, $\rho^{I(\mathbf{V})}_{I(\mathbf{U})}(f_{\mathbf{V}}(\pi_{\mathbf{V}}(\mathbf{x})) = \pi_{I(\mathbf{U})}(f(\mathbf{x})) = f_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x}))$. In summary, the equality

$$\rho^{I(\mathbf{V})}_{I(\mathbf{U})} \circ f_{\mathbf{V}} = f_{\mathbf{U}} \circ \rho^{\mathbf{V}}_{\mathbf{U}}$$

holds in $\pi_{\mathbf{V}}(\mathbf{X})$ *away from* the point $\rho^{\mathbf{V}}_{\mathbf{U}}$. □

Lemma 4.32. *Let $(f, I, \{f_{\mathbf{U}}\})$ be as in Definition 4.30. Suppose that:*

- *I is injective;*
- *if $\mathbf{U} \notin \mathrm{im}(I)$ then $\pi_{\mathbf{U}} \circ f$ is constant;*
- *each $f_{\mathbf{U}}$ is an isometric embedding.*

Then f is an isometric embedding.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Let $\mathbf{U} \in \mathfrak{F}^{\bullet}$. Then $\mathsf{d}_{I(\mathbf{U})}(f(\mathbf{x}), f(\mathbf{y})) = \mathsf{d}_{\mathbf{U}}(f(\mathbf{x}), f(\mathbf{y}))$ by hypothesis, so $\mathsf{d}_{\mathbf{X}}(\mathbf{x}, \mathbf{y}) \leqslant \mathsf{d}_{\mathbf{Y}}(f(\mathbf{x}), f(\mathbf{y}))$. Next, suppose that $\mathbf{U} \in \mathfrak{H}^{\bullet}$. If $\mathbf{U} \notin \mathrm{im}(I)$, then the second hypothesis implies that $\pi_{\mathbf{U}}(f(\mathbf{x})) = \pi_{\mathbf{U}}(f(\mathbf{y}))$. The two preceding facts immediately show that f is an isometric embedding. □

Remark 4.33 (Recognising isometries)**.** Suppose that $(\mathbf{X}, \mathfrak{F}^{\bullet})$ and $(\mathbf{Y}, \mathfrak{H}^{\bullet})$ are ℝ–cubings with the property that each map $\pi_{\mathbf{U}} : \mathbf{X} \to \mathcal{T}^{\bullet}\mathbf{U}$ and $\pi_{\mathbf{V}} : \mathbf{Y} \to \mathcal{T}^{\bullet}\mathbf{V}$ is surjective (which can always be arranged by replacing various ℝ–trees with subtrees). Let $(f, I, \{f_{\mathbf{U}}\})$ be a morphism from $(\mathbf{X}, \mathfrak{F}^{\bullet})$ to $(\mathbf{Y}, \mathfrak{H}^{\bullet})$. Suppose that I is bijective and each $f_{\mathbf{U}}$ is an isometry. Then f is an isometric embedding by Lemma 4.32, and, moreover, it follows from Definition 4.2.(5) that f is an isometry. □

If $\mathbf{X} = \mathbf{Y}$, and $\mathfrak{F}^{\bullet} = \mathfrak{H}^{\bullet}$, and I is bijective, and f and each $f_{\mathbf{U}}$ is an isometry, then $(f, I, \{f_{\mathbf{U}}\})$ is an *automorphism* of $(\mathbf{X}, \mathfrak{F}^{\bullet})$. If, instead, there exists K such that f and each $f_{\mathbf{U}}$ is a K–bilipschitz map, and I is bijective, then $(f, I, \{f_{\mathbf{U}}\})$ is a *K–automorphism*.

Let $(f, I, \{f_{\mathbf{U}}\})$ and $(g, J, \{g_{\mathbf{U}}\})$ be automorphisms of $(\mathbf{X}, \mathfrak{F}^{\bullet})$. Then $f \circ g$ is an isometry, $I \circ J$ is a bijection preserving $\sqsubseteq, \perp, \pitchfork$, and for each $\mathbf{U}$, $f_{J(\mathbf{U})} \circ g_{\mathbf{U}}$ is an isometry, so $(f \circ g, I \circ J, \{f_{J(\mathbf{U})} \circ g_{\mathbf{U}}\})$ is an automorphism. There is an identity automorphism defined in the obvious way. The inverse of $(f, I, \{f_{\mathbf{U}}\})$ is $(f^{-1}, I^{-1}, \{f^{-1}_{\mathbf{U}}\})$.

Hence the set of 1–automorphisms forms a group, denoted $\mathrm{Aut}(\mathfrak{F}^{\bullet})$, of isometries of $\mathbf{X}$. An action of a group Γ on $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is a homomorphism $\Gamma \to \mathrm{Aut}(\mathfrak{F}^{\bullet})$.

Such an action in particular determines an action of Γ on the set $\mathfrak{F}^{\bullet}$, preserving the relations, and an isometric action of Γ on $\mathbf{X}$.

4.9. **Local real cubings and groves.** Let $(\mathbf{X}, \mathfrak{F}^{\bullet})$ be a ℝ–cubing. We say that $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is a *local* ℝ–cubing if there exists $\mathbf{x}_0 \in \mathbf{X}$ such that the following hold for all $\mathbf{V}, \mathbf{W} \in \mathfrak{F}^{\bullet}$:

- if $\mathbf{W} \pitchfork \mathbf{V}$, then $\pi_{\mathbf{V}}(\mathbf{x}_0) = \rho^{\mathbf{W}}_{\mathbf{V}}$;
- if $\mathbf{V} \subsetneq \mathbf{W}$, then $\pi_{\mathbf{W}}(\mathbf{x}_0) = \rho^{\mathbf{V}}_{\mathbf{W}}$.

If $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is a local ℝ–cubing, then we refer to $\mathfrak{F}^{\bullet}$ (together with the relations $\sqsubseteq, \pitchfork, \perp$, the ℝ–trees $\mathcal{T}^{\bullet}\mathbf{U}, \mathbf{U} \in \mathfrak{F}^{\bullet}$, and the points/maps $\rho^{\mathbf{W}}_{\mathbf{V}}$) as a *grove* (i.e. a collection of trees, all in one place).

Remark 4.34. Observe that if $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is a local real cubing, and $\mathbf{x}_0 \in \mathbf{X}$ is as above, then for any $\mathbf{x} \in \mathbf{X}$, the set of $\mathbf{U} \in \mathfrak{F}^{\bullet}$ with $\pi_{\mathbf{U}}(\mathbf{x}) \neq \pi_{\mathbf{U}}(\mathbf{x}_0)$ is a collection of pairwise orthogonal elements and hence finite. This follows from consistency and the definition of a local real cubing. □

Remark 4.35 (The "sheaf" of groves associated to an ℝ–cubing)**.** Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be an ℝ–cubing.

Recall that each $\mathbf{U} \in \mathfrak{F}^\bullet$ can be associated to a subspace $\mathbf{P_U} \subset \mathbf{X}$ as follows: let $\mathbf{P_U}$ be the set of $\mathbf{x} \in \mathbf{X}$ such that $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ whenever $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$.

Given $\mathbf{x}, \mathbf{y} \in \mathbf{P_U}$, observe that $\mathsf{d}_{\mathbf{W}}(\pi_{\mathbf{W}}(\mathbf{x}), \pi_{\mathbf{W}}(\mathbf{y})) > 0$ only if $\mathbf{W} \sqsubseteq \mathbf{U}$ or $\mathbf{W} \perp \mathbf{U}$.

For each $\mathbf{x} \in \mathbf{X}$, let $\mathfrak{F}^\bullet_{\mathbf{x}}$ be the subset of $\mathfrak{F}^\bullet$ consisting of those $\mathbf{U} \in \mathfrak{F}^\bullet$ such that $\mathbf{x} \in \mathbf{P_U}$. (In other words, $\mathbf{U} \in \mathfrak{F}^\bullet_{\mathbf{x}}$ if and only if, for all $\mathbf{V}$ with $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$, we have $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{x})$.)

Let $\mathbf{X_x}$ be the set of $\mathbf{y} \in \mathbf{X}$ such that $\pi_{\mathbf{V}}(\mathbf{y}) \neq \pi_{\mathbf{V}}(\mathbf{x})$ only if $\mathbf{V} \in \mathfrak{F}^\bullet_{\mathbf{x}}$. Then $(\mathbf{X_x}, \mathfrak{F}^\bullet_{\mathbf{x}})$ is a local ℝ–cubing and $\mathfrak{F}^\bullet_{\mathbf{x}}$ is a grove. This continues to hold if we replace each ℝ–tree $\mathcal{T}^\bullet\mathbf{V}$ by the subtree $\pi_{\mathbf{V}}(\mathbf{X_x})$.

There are other variants on this. Define $\mathfrak{F}^\bullet_{\mathbf{x}}$ as above. Let $\mathbf{X}'_{\mathbf{x}}$ be the set of all $\mathbf{y} \in \mathbf{X}$ such that the set of $\mathbf{V}$ with $\pi_{\mathbf{V}}(\mathbf{x}) \neq \pi_{\mathbf{V}}(\mathbf{y})$ is a set of pairwise-orthogonal elements of $\mathfrak{F}^\bullet_{\mathbf{x}}$. Then replace each $\mathcal{T}^\bullet\mathbf{V}, \mathbf{V} \in \mathfrak{F}^\bullet_{\mathbf{x}}$ by the ℝ–tree $\pi_{\mathbf{V}}(\mathbf{X}'_{\mathbf{x}})$. This again yields a local ℝ–cubing. Similarly, we can consider the real cubing $(\mathbf{X}'', \mathfrak{F}^\bullet)$ consisting of that $\mathbf{y}$ whose projection differs from that of $\mathbf{x}$ on at most one element of $\mathfrak{F}^\bullet_{\mathbf{x}}$. □

Remark 4.36 (Translating the local structure)**.** We will typically be interested in the following situation. Let Γ act transitively on $\mathbf{X}$ by K–automorphisms (for fixed $K > 0$). Then for each $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, choose $(g, I_g, \{g_{\mathbf{W}}\}) \in \Gamma$ such that $g\mathbf{x} = \mathbf{y}$. Then for all $\mathbf{U} \in \mathfrak{F}^\bullet_{\mathbf{x}}$, we have $I_g(\mathbf{U}) \in \mathbf{X_y}$, and so g induces an invertible K–morphism of ℝ–cubings $(\mathbf{X_x}, \mathfrak{F}^\bullet_{\mathbf{x}}) \to (\mathbf{X_y}, \mathfrak{F}^\bullet_{\mathbf{y}})$.

Indeed, suppose that $\mathbf{U} \in \mathfrak{F}^\bullet_{\mathbf{x}}$. Then for all $\mathbf{V}$ with $\mathbf{V} \pitchfork I_g(\mathbf{U})$ or $I_g(\mathbf{U}) \subsetneq \mathbf{V}$, we have that $\mathbf{U}$ is either properly nested in, or transverse to, $I_g^{-1}(\mathbf{V})$. Hence, since $\mathbf{U} \in \mathfrak{F}^\bullet_{\mathbf{x}}$, we have $\pi_{I_g^{-1}(\mathbf{V})}(\mathbf{x}) = \rho^{\mathbf{U}}_{I_g^{-1}(\mathbf{V})}$. Since $(g, I_g, \{g_{\mathbf{W}}\})$ is a morphism, we have $\pi_{\mathbf{V}}(\mathbf{y}) = \rho^{I_g(\mathbf{U})}_{\mathbf{V}}$, whence $I_g(\mathbf{U}) \in \mathfrak{F}^\bullet_{\mathbf{y}}$. So, $I_g : \mathfrak{F}^\bullet_{\mathbf{x}} \to \mathfrak{F}^\bullet_{\mathbf{y}}$ is injective, and considering $(g^{-1}, I_g^{-1}, \{g^{-1}_{\mathbf{W}}\})$ shows that it is surjective.

For each $\mathbf{W} \in \mathfrak{F}^\bullet_{\mathbf{x}}$, the map $g_{\mathbf{W}} : \mathcal{T}^\bullet\mathbf{W} \to \mathcal{T}^\bullet I_g(\mathbf{W})$ is K–bilipschitz by definition.

To conclude, it thus suffices to show that the K–bilipschitz map $g : \mathbf{X} \to \mathbf{Y}$ restricts to a map $g : \mathbf{X_x} \to \mathbf{Y_y}$. Suppose that $\mathbf{z} \in \mathbf{X_x}$. Then $\pi_{\mathbf{W}}(\mathbf{x}) = \pi_{\mathbf{W}}(\mathbf{z})$ for all $\mathbf{W} \in \mathfrak{F}^\bullet - \mathfrak{F}^\bullet_{\mathbf{x}}$. Now, if $\mathbf{W} \in \mathfrak{F}^\bullet - \mathfrak{F}^\bullet_{\mathbf{y}}$, then $I_g(\mathbf{W}) \in \mathfrak{F}^\bullet - \mathfrak{F}^\bullet_{\mathbf{x}}$, so $\mathbf{x}, \mathbf{z}$ have identical image in $\mathcal{T}^\bullet I_g^{-1}(\mathbf{W})$. Hence $\pi_{\mathbf{W}}(\mathbf{y}) = \pi_{\mathbf{W}}(g(\mathbf{z}))$, so $g(\mathbf{z}) \in \mathbf{X_y}$, as required.

The main case of interest is where $K = 1$, i.e. all of the maps above are isometries. The exact same conclusion holds if we had used the local ℝ–cubing $(\mathbf{X}'_{\mathbf{x}}, \mathfrak{F}^\bullet_{\mathbf{x}})$ (with the modified ℝ–trees) constructed in the previous remark. □

4.10. **Standard product regions, wedges, and clean containers in real cubings.** Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be an ℝ–cubing. Recall that for each $\mathbf{U} \in \mathfrak{F}^\bullet$, there is an associated subspace $\mathbf{P_U}$ consisting of exactly those points $\mathbf{x} \in \mathbf{X}$ such that

$$\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$$

whenever $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$. Recall from Proposition 4.10 that we can discard some elements of $\mathfrak{F}^\bullet$ and assume that each $\mathbf{P_U}$ is nonempty, but we don't require this in this section.

For each $p \in \mathbf{P_U}$, let $F^p_{\mathbf{U}}$ be the subset of $\mathbf{P_U}$ consisting of those $\mathbf{x}$ with $\pi_{\mathbf{V}}(\mathbf{x}) = \pi_{\mathbf{V}}(p)$ whenever $\mathbf{V} \perp \mathbf{U}$. Similarly, let $E^p_{\mathbf{U}}$ be the set of $\mathbf{x} \in \mathbf{P_U}$ be such that $\pi_{\mathbf{V}}(\mathbf{x}) = \pi_{\mathbf{V}}(p)$ whenever $\mathbf{V} \sqsubseteq \mathbf{U}$.

Proposition 4.37. *For all* $\mathbf{U} \in \mathfrak{F}^\bullet$ *and* $p \in \mathbf{P_U}$*, we have the following:*

(1) *The subspaces* $E^p_{\mathbf{U}}, F^p_{\mathbf{U}}, \mathbf{P_U}$ *are closed and median-convex.*

(2) *The inclusions* $E^p_{\mathbf{U}}, F^p_{\mathbf{U}} \to \mathbf{X}$ *extend to a median-preserving isometric embedding* $F^p_{\mathbf{U}} \times E^p_{\mathbf{U}} \to \mathbf{X}$ *with image* $\mathbf{P}_{\mathbf{U}}$*. In particular,* $F^p_{\mathbf{U}}$ *and* $F^q_{\mathbf{U}}$ *are parallel for all* $p, q \in \mathbf{P}_{\mathbf{U}}$*.*
(3) *The map* $\pi_{\mathbf{U}} : \mathbf{X} \to \mathcal{T}^{\bullet}\mathbf{U}$ *restricts on* $F_{\mathbf{U}}$ *to a surjection to* $\pi_{\mathbf{U}}(\mathbf{X})$*.*
(4) *Let* $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$*. Then* $\mathbf{V} \sqsubseteq \mathbf{U}$ *implies* $\mathfrak{g}_{F_{\mathbf{U}}} : F_{\mathbf{V}} \to F_{\mathbf{U}}$ *is an isometric embedding. Similarly,* $\mathbf{U} \perp \mathbf{V}$ *implies* $\mathfrak{g}_{E_{\mathbf{U}}} : F_{\mathbf{V}} \to E_{\mathbf{U}}$ *is an isometric embedding.*

Proof. Each real tree $\mathcal{T}^{\bullet}\mathbf{V}$ is Hausdorff, so singletons are closed. Each $\pi_{\mathbf{V}}$ is continuous, so since $\mathbf{P}_{\mathbf{U}}, F^p_{\mathbf{U}}, E^p_{\mathbf{U}}$ are all defined as intersections of preimages of points in various real trees, each of those sets is closed. Convexity follows in each case from the definition and Definition 4.17 and Definition 4.2.(5). This proves (1).

Given $f \in F^p_{\mathbf{U}}, e \in E^p_{\mathbf{U}}$, consider the point in $\ell^1(\mathfrak{F}^{\bullet})$ whose $\mathbf{V}$–coordinate is $\pi_{\mathbf{V}}(f)$ for $\mathbf{V} \sqsubseteq \mathbf{U}$, and $\pi_{\mathbf{V}}(e)$ for $\mathbf{V} \perp \mathbf{U}$, and $\rho^{\mathbf{U}}_{\mathbf{V}}$ otherwise. This tuple is consistent in view of Lemma 4.12 (ρ–consistency) and the fact that $\mathbf{P}_{\mathbf{U}} \neq \varnothing$.

This tuple thus determines a unique point in $\mathbf{X}$, by Definition 4.2.(5). This defines an isometric embedding $F^p_{\mathbf{U}} \times E^p_{\mathbf{U}} \to \mathbf{X}$ whose image is $\mathbf{P}_{\mathbf{U}}$. This proves assertion (2) once we observe that the parallelism claim follows since, in a product median space, any two sections of the natural projection to one of the factors are parallel.

Note that the image of $F^p_{\mathbf{U}}$ in $\mathcal{C}\mathbf{U}$ coincides with that of $\mathbf{P}_{\mathbf{U}}$, so to prove (3), it suffices to show that $\mathbf{P}_{\mathbf{U}}$ surjects to $\pi_{\mathbf{U}}(\mathbf{X})$. To see this, choose $\pi_{\mathbf{U}}(\mathbf{x})$, and note that $\mathfrak{g}_{\mathbf{P}_{\mathbf{U}}}(\mathbf{x}) \in \mathbf{P}_{\mathbf{U}}$ has the same image in $\mathcal{T}^{\bullet}\mathbf{U}$ as $\mathbf{x}$ does.

The final assertion follows from the definitions of the spaces involved. □

We often refer to any closed convex subspace parallel to $F^p_{\mathbf{U}}$ as $F_{\mathbf{U}}$, and similarly for $E_{\mathbf{U}}$, when the choice of parallel copy is not important (for example, when we are just interested in the set of walls crossing $F_{\mathbf{U}}$ — see Lemma 2.15).

Corollary 4.38. *Let* $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$ *satisfy* $\mathbf{U} \perp \mathbf{V}$*. Then* $P_{\mathbf{U}}$ *contains a closed convex subspace of the form* $F_{\mathbf{U}} \times F_{\mathbf{V}}$*. In particular, if* $\hat{h}, \hat{v}$ *are walls respectively crossing* $F_{\mathbf{U}}$ *and* $F_{\mathbf{V}}$*, then* $\hat{h}$ *and* $\hat{v}$ *cross.*

Proof. The first assertion follows from Proposition 4.37. The assertion about crossing walls then follows. Indeed, let $f \in h \cap F_{\mathbf{U}}$ and $f' \in h^* \cap F_{\mathbf{U}}$ and $e, e' \in v \cap F_{\mathbf{V}}, v^* \cap F_{\mathbf{V}}$. Then by Lemma 2.15, $(f, e) \in h \cap v, (f', e) \in h^* \cap v, (f', e') \in h^* \cap v^*, (f, e') \in h \cap v^*$, so $\hat{h}$ and $\hat{v}$ cross. □

The following two restrictions on the combinatorics of the index set are often useful:

Definition 4.39 (Wedges)**.** We say that $\mathfrak{F}^{\bullet}$ has *wedges* if, for all $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$ for which there exists $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$, there is a unique $\sqsubseteq$–maximal such $\mathbf{W}$, denoted $\mathbf{U} \wedge \mathbf{W}$. □

Definition 4.40 (Clean containers)**.** We say that $\mathfrak{F}^{\bullet}$ has *clean containers* if for all $\mathbf{U}$ such that there exists $\mathbf{W}$ with $\mathbf{W} \perp \mathbf{U}$, there exists $\mathbf{U}^{\perp} \in \mathfrak{F}^{\bullet}$ such that $\mathbf{U} \perp \mathbf{U}^{\perp}$ and $\mathbf{U} \perp \mathbf{V}$ if and only if $\mathbf{V} \sqsubseteq \mathbf{U}^{\perp}$. □

Remark 4.41. If $\mathfrak{F}^{\bullet}$ has wedges and clean containers, then the following holds for all $\mathbf{V}$. Suppose that $\mathbf{U} \sqsubseteq \mathbf{V}$. Then, if there exists $\mathbf{W} \sqsubseteq \mathbf{V}$ with $\mathbf{U} \perp \mathbf{W}$, we have a unique $\sqsubseteq$–maximal such $\mathbf{W}$, namely $\mathbf{V} \wedge \mathbf{U}^{\perp}$, and necessarily $\mathbf{U} \perp (\mathbf{V} \wedge \mathbf{U}^{\perp})$. □

The next lemma is due to Berlai and Robbio [BR20a], who were working with hierarchically hyperbolic spaces, rather than real cubings (but the arguments are formally identical):

Lemma 4.42 (Joins)**.** *Suppose* $\mathfrak{F}^{\bullet}$ *has wedges and a* $\sqsubseteq$*–maximal element. Let* $\{\mathbf{U}_i\}_{i \in I} \subset \mathfrak{F}^{\bullet}$*. Then there exists a unique element of* $\mathfrak{F}^{\bullet}$*, denoted*

$$\bigvee_{i \in I} \mathbf{U}_i,$$

that is $\sqsubseteq$–minimal with the property that each $\mathbf{U}_i$ is nested in it.

Proof. There exists $\mathbf{U}$ such that $\mathbf{U}_i \sqsubseteq \mathbf{U}$ for all i; at minimum we can take $\mathbf{U}$ to be the $\sqsubseteq$–maximal element. Since the length of $\sqsubseteq$–chains is bounded, we can assume that $\mathbf{U}$ is $\sqsubseteq$–minimal with the above property. If $\mathbf{V}$ is also $\sqsubseteq$–minimal with the property that $\mathbf{U}_i \sqsubseteq \mathbf{V}$ for all i, then $\mathbf{U} \wedge \mathbf{V}$ is defined and $\mathbf{U}_i \sqsubseteq \mathbf{U} \wedge \mathbf{V}$ for all i. Now, $\mathbf{U} \wedge \mathbf{V} \sqsubseteq \mathbf{U}$, so by $\sqsubseteq$–minimality, $\mathbf{U} \wedge \mathbf{V} = \mathbf{U}$. Hence $\mathbf{U} \sqsubseteq \mathbf{V}$, contradicting $\sqsubseteq$–minimality of $\mathbf{V}$ unless $\mathbf{U} = \mathbf{V}$. This shows that $\mathbf{U}$ is unique, establishing the existence of $\bigvee_{i \in I} \mathbf{U}_i$. ☐

Lemma 4.43 (Wedges of arbitrary subsets)**.** *Suppose that $\mathfrak{F}^\bullet$ has wedges. Let $\{\mathbf{U}_i\}_{i \in I} \subset \mathfrak{F}^\bullet$ and suppose that there exists $\mathbf{V} \in \mathfrak{F}^\bullet$ with $\mathbf{V} \sqsubseteq \mathbf{U}_i$ for all i. Then there exists a unique $\sqsubseteq$–maximal such $\mathbf{V}$, denoted $\bigwedge_i \mathbf{U}_i$.*

Proof. Let $\mathbf{V}$ be as in the statement. By finite complexity, there is a $\sqsubseteq$–maximal such $\mathbf{V}$. If $\mathbf{V}, \mathbf{V}'$ are both nested in $\mathbf{U}_i$ for all $\mathbf{U}_i$, then so is $\mathbf{V} \vee \mathbf{V}'$. This contradicts maximality of $\mathbf{V}$ unless $\mathbf{V} = \mathbf{V}'$, and we are done. ☐

5. Characterisation of real cubings among median spaces

5.1. **Finite-depth tangible poset-colourings give real cubings.** The main theorem of the section is:

Theorem 5.1 ($\mathbb{R}$–cubings from poset-colourings)**.** *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete, connected median metric space of rank $N < \infty$ and let $\mathbf{x}_0 \in \mathbf{X}$ be a basepoint. Suppose that there exists $D < \infty$ and a depth–D poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^\bullet_1$ of the walls of $\mathbf{X}$ that satisfies the tangible filter condition.*

For each $\mathbf{U} \in \mathfrak{F}^\bullet_1$, let $\mathbf{F_U} \subset \mathbf{X}$ be the closed convex subspace obtained by applying Theorem 2.18 and Corollary 2.19 to the tangible filter $\sigma_{\mathbf{U}}$. Let $\mathfrak{F}^\bullet \subset \mathfrak{F}^\bullet_1$ be the set of $\mathbf{U} \in \mathfrak{F}^\bullet$ such that $\mathsf{diam}(\mathbf{F_U}) > 0$.

Then to each $\mathbf{U} \in \mathfrak{F}^\bullet$ we can associate a based $\mathbb{R}$–tree $\mathcal{T}^\bullet\mathbf{U}$ and a 1–lipschitz median-preserving map $\pi_{\mathbf{U}} : \mathbf{X} \to \mathcal{T}^\bullet\mathbf{U}$ such that the product map $\mathbf{X} \to \ell_1(\mathfrak{F}^\bullet)$ is an isometric embedding making $(\mathbf{X}, \mathfrak{F}^\bullet)$ an $\mathbb{R}$–cubing in the sense of Definition 4.2. Moreover, this real cubing has nonempty products.

Remark 5.2. In Section 26, we will apply the above theorem to the asymptotic cone of a hierarchically hyperbolic space, after changing the metric in its bilipschitz equivalence class (using a theorem of Bowditch) to make it a median metric space. Since completeness and connectedness are automatic for asymptotic cones, and finite rank will be verified using the HHS structure, the main work will be to produce a finite-depth poset-colouring of the walls satisfying the tangible filter condition. ☐

The proof of Theorem 5.1 will occupy the rest of the subsection.

Fixing $\mathbf{U} \in \mathfrak{F}^\bullet_1$, we take the filter $\sigma_{\mathbf{U}}$ which, by hypothesis, satisfies

$$\mathrm{fio}(\sigma_{\mathbf{U}} - \sigma_{\mathbf{x}_0}) < \infty.$$

So, there is a (nonempty) closed, median-convex subspace $\mathbf{F_U}$ such that the set $\sigma_{\mathbf{F_U}}$ of halfspaces containing $\mathbf{F_U}$ is morally measurable and satisfies

$$\mathrm{fio}(\sigma_{\mathbf{U}} \triangle \sigma_{\mathbf{F_U}}) = 0.$$

We collect some properties of the subspaces $\mathbf{F_U}$. Since $\mathbf{F_U}$ is closed and convex, there is a gate map $\mathfrak{h}_{\mathbf{U}} : \mathbf{X} \to \mathbf{F_U}$. So far, we are not insisting on $\mathbf{U} \in \mathfrak{F}^\bullet$, i.e. we allow the case where $\mathbf{F_U}$ is a single point.

Lemma 5.3. *Let* $\mathbf{U} \in \mathfrak{F}_1^\bullet$ *be a colour. Let* $\mathcal{H}(\mathbf{F_U})$ *be the set of halfspaces* h *such that* h, h^* *both intersect* $\mathbf{F_U}$. *Let* $\mathcal{H}_\mathbf{U}$ *be the set of halfspaces associated to walls in* $\mathcal{W}_\mathbf{U}$.

Then both of these sets of halfspaces are morally measurable, and $\operatorname{fio}(\mathcal{H}_\mathbf{U} \triangle \mathcal{H}(\mathbf{F_U})) = 0$.

Proof. By Definition 3.1 and [Fio20, Lemma 3.9], $\mathcal{H}_\mathbf{U}$ is morally measurable ($\mathcal{H}_\mathbf{U}$ is inseparable). By [Fio20, Lemma 3.6], $\mathcal{H}(\mathbf{F_U})$ is morally measurable.

We have $\mathcal{H}(\mathbf{F_U}) = \mathcal{H} - (\sigma_{\mathbf{F_U}} \sqcup \sigma^*_{\mathbf{F_U}})$ where, for any set S of halfspaces, S^* is obtained from S by replacing each halfspace in S by its complement. Similarly, $\mathcal{H}_\mathbf{U}$ satisfies $\mathcal{H}_\mathbf{U} = \mathcal{H} - (\sigma_\mathbf{U} \cup \sigma^*_\mathbf{U})$.

A computation shows that

$$\mathcal{H}_\mathbf{U} \triangle \mathcal{H}(\mathbf{F_U}) \subset (\sigma_\mathbf{U} \triangle \sigma_{\mathbf{F_U}}) \cup \left(\sigma^*_\mathbf{U} \triangle \sigma^*_{\mathbf{F_U}}\right).$$

The defining property of $\mathbf{F_U}$ is that

$$\operatorname{fio}(\sigma_\mathbf{U} \triangle \sigma_{\mathbf{F_U}}) = 0.$$

So it suffices to show that

$$\operatorname{fio}(\sigma^*_\mathbf{U} \triangle \sigma^*_{\mathbf{F_U}}) = 0.$$

Recall from Lemma 2.20 that the involution $^* : \mathcal{H} \to \mathcal{H}$ is measure-preserving. Moreover, we have, for any sets A, B of halfspaces, that $(A - B)^* = A^* - B^*$, so the claim follows. □

Next, we eliminate ambiguity arising from the fact that $\sigma_\mathbf{U}$ depends on $\mathbf{x}_0$, to the extent that $\mathbf{x}_0$ influenced our choice of $\sigma_\mathbf{U}$.

Lemma 5.4 (Parallel copies of $\mathbf{F_U}$)**.** *Let* $\mathbf{U} \in \mathfrak{F}_1^\bullet$. *Let* $\mathbf{F}$ *be a closed, median-convex subspace of* $\mathbf{X}$ *such that* $\operatorname{fio}(\mathcal{H}_\mathbf{U} \triangle \mathcal{H}(\mathbf{F})) = 0$. *Then the gate map* $\mathfrak{h}_\mathbf{U} : \mathbf{F} \to \mathbf{F_U}$ *is a median-preserving isometry whose inverse is the restriction to* $\mathbf{F_U}$ *of the gate map* $\mathfrak{h} : \mathbf{X} \to \mathbf{F}$.

In particular, $\mathbf{F_U}$ *and* $\mathbf{F}$ *are crossed by* ***exactly*** *the same walls.*

Proof. This follows from Lemma 2.21. Indeed, by Lemma 5.3, $\operatorname{fio}(\mathcal{H}_\mathbf{U} \triangle \mathcal{H}(\mathbf{F_U}) = 0$, and by assumption, $\operatorname{fio}(\mathcal{H}_\mathbf{U} \triangle \mathcal{H}(\mathbf{F})) = 0$, so $\operatorname{fio}(\mathcal{H}(\mathbf{F_U}) \triangle \mathcal{H}(\mathbf{F})) = 0$, and Lemma 2.21 applies, showing that $\mathbf{F_U}$ and $\mathbf{F}$ are parallel and cross the same walls. □

Remark 5.5. The reader should think of any $\mathbf{F}$ in the previous lemma (including $\mathbf{F_U}$) as a "geometric realisation" of the colour $\mathbf{U}$. However, the reader is warned that although, by the lemma, any two such "geometric realisations" cross the same walls, the set of walls crossing a given "geometric realisation" is not exactly the set $\mathcal{W}_\mathbf{U}$ — the corresponding sets of halfspaces differ by a set of measure 0. □

Fixing $\mathbf{U} \in \mathfrak{F}_1^\bullet$, consider all of the closed convex subspaces $\mathbf{F}$ satisfying the hypotheses of Lemma 5.4. We call such an $\mathbf{F}$ a *parallel copy* of $\mathbf{F_U}$.

Next, we take care of the "combinatorial" part of the construction of an ℝ–cubing, i.e. the part not requiring the ℝ–trees, by equipping $\mathfrak{F}^\bullet \subset \mathfrak{F}_1^\bullet$ with the relations required by Definition 4.2.

Definition 5.6 (Nesting)**.** $\sqsubseteq$ is just the partial order on $\mathfrak{F}^\bullet$ already mentioned. So, chains have length bounded by $N < \infty$, and there is a unique $\sqsubseteq$–maximal $\mathbf{S}$. □

Definition 5.7 (Orthogonal colours)**.** The colours $\mathbf{U}, \mathbf{V} \subset \mathfrak{F}^\bullet$ are *orthogonal* if there exist parallel copies $\mathbf{F}'_\mathbf{U}, \mathbf{F}'_\mathbf{V}$ of $\mathbf{F_U}, \mathbf{F_V}$ respectively, such that the inclusions $\mathbf{F}'_\mathbf{U}, \mathbf{F}'_\mathbf{V} \to \mathbf{X}$ extend to a median-preserving isometric embedding $\mathbf{F}'_\mathbf{U} \times \mathbf{F}'_\mathbf{V} \to \mathbf{X}$ with convex image. (Here the product is given the ℓ_1 metric and product median, where the median and metric on the factors are those inherited from $\mathbf{X}$.) This is denoted $\mathbf{U} \perp \mathbf{V}$. We emphasise that orthogonality has only been defined on $\mathfrak{F}^\bullet$, so if $\mathbf{U} \perp \mathbf{V}$, then *both* factors of the associated product region are nontrivial. □

If $\mathbf{U}, \mathbf{V}$ are not $\sqsubseteq$–related or orthogonal, then they are *transverse*, denoted $\mathbf{U} \pitchfork \mathbf{V}$.

Next, we verify all of the properties of $\sqsubseteq, \perp, \pitchfork$ from Definition 4.2 not involving the $\mathbb{R}$–trees, which we have not yet constructed.

We first need a "geometric" description of nesting mirroring that for orthogonality.

Lemma 5.8. *We have $\mathbf{U} \sqsubseteq \mathbf{V}$ only if, up to replacing $\mathbf{F_U}$ by a parallel copy, we have $\mathbf{F_U} \subset \mathbf{F_V}$.*

Proof. Arguing almost exactly as in the proof of Lemma 5.4, using that $\mathbf{U} \sqsubseteq \mathbf{V}$, one shows that the gate map to $\mathbf{F_V}$ restricts on $\mathbf{F_U}$ to an isometric embedding. Hence, if $\mathbf{U} \sqsubseteq \mathbf{V}$, then $\mathbf{F_U} \subset \mathbf{F_V}$, up to changing $\mathbf{F_U}$ in its parallelism class (replace it by the image of the aforementioned gate map). □

From the preceding lemma, we get that $\sqsubseteq$ and $\perp$ are mutually exclusive as relations on $\mathfrak{F}^\bullet$. From the definition, $\perp$ is symmetric and anti-reflexive (note that anti-reflexivity uses that $\mathbf{F_U}$ is not a single point).

From Lemma 5.8, we have the following. Suppose that $\mathbf{U} \perp \mathbf{V}$ and $\mathbf{W} \sqsubseteq \mathbf{U}$. Then $\mathbf{W} \perp \mathbf{V}$. Indeed, we have a product subspace $\mathbf{F_U} \times \mathbf{F_V}$, and $\mathbf{F_W} \subset \mathbf{F_U}$ (after choosing an appropriate parallel copy of $\mathbf{F_W}$, holding the given parallel copy of $\mathbf{F_U}$ fixed). So we have a product $\mathbf{F_W} \times \mathbf{F_V}$. Thus $\mathbf{W} \perp \mathbf{V}$.

The part of the finite complexity axiom (Definition 4.2.(4)) involving $\sqsubseteq$ holds by our assumption that the depth is $D < \infty$, so we just need to check the part about orthogonality. This, together with anti-reflexivity, is why we have passed to $\mathfrak{F}^\bullet$.

Lemma 5.9. *Let $\mathbf{U}_1, \ldots, \mathbf{U}_k \in \mathfrak{F}^\bullet$ be pairwise orthogonal. Then $k \leqslant N$, where N is the rank of $(\mathbf{X}, \boldsymbol{\mu})$.*

Proof. For each $i \leqslant k$, choose distinct $\mathbf{x}_i, \mathbf{y}_i \in \mathbf{F}_{\mathbf{U}_i}$ (the choice of parallel copy does not matter). Let $\hat{w}_i$ be a wall separating $\mathbf{x}_i, \mathbf{y}_i$. Then $\hat{w}_i$ separates the gates of $\mathbf{x}_i, \mathbf{y}_i$ in any parallel copy of $\mathbf{F}_{\mathbf{U}_i}$ lying in the product region $\mathbf{F}_{\mathbf{U}_i} \times \mathbf{F}_{\mathbf{U}_j}$ which exists by the orthogonality assumption. From the product structure, it follows that $\hat{w}_i$ and $\hat{w}_j$ cross. Hence $\mathcal{W}$ contains a set of k distinct pairwise-crossing walls, so $k \leqslant N$ by [Bow13, Proposition 6.2]. □

Hence the finite complexity axiom is satisfied with $\chi^\bullet \geqslant \max\{N, D\}$. We now move on to the construction of the $\mathbb{R}$–trees, having verified the parts of Definition 4.2 not involving the real trees.

5.1.1. *Construction of $\mathcal{T}^\bullet\mathbf{U}$.* We now construct the real tree $\mathcal{T}^\bullet\mathbf{U}$ for $\mathbf{U} \in \mathfrak{F}^\bullet$ as a quotient of $\mathbf{F_U}$. The same exact construction would work for $\mathbf{U} \in \mathfrak{F}_1^\bullet - \mathfrak{F}^\bullet$, but for these colours, the associated $\mathbb{R}$–tree, as a quotient of $\mathbf{F_U}$, would be a point, so we ignore these. (It may also happen that the construction below gives a trivial $\mathbb{R}$–tree for certain $\mathbf{U} \in \mathfrak{F}^\bullet$.)

The goal is to define a pseudometric $\mathbf{D_U}$ on $\mathbf{X}$ so that the metric quotient is a connected, rank-1 median metric space, where the median $\boldsymbol{\mu}_\mathbf{U}$ is induced by $\boldsymbol{\mu}$. We will then deduce that the quotient is an $\mathbb{R}$–tree. This $\mathbb{R}$–tree need not be complete.

Fix $\mathbf{U} \in \mathfrak{F}^\bullet$. For any $\mathbf{x} \in \mathbf{X}$, we let $\bar{\mathbf{x}} = \mathfrak{h}_\mathbf{U}(\mathbf{x})$. So, we have implicitly fixed a parallel copy $\mathbf{F_U}$ onto which we are projecting. From the definition of parallelism, and Lemma 2.15, it will become clear during the construction that the choice of parallel copy has no effect on the outcome.

Definition 5.10 (t–distance, $\mathcal{A}_\mathbf{U}$)**.** Given $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, let

$$t_\mathbf{U}(\mathbf{x}, \mathbf{y}) = \mathsf{d}_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}).$$

Note that

$$t_\mathbf{U}(\mathbf{x}, \mathbf{y}) = \mathrm{fio}(\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_\mathbf{U}).$$

Indeed, the halfspaces separating $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ are exactly those that separate $\mathbf{x}, \mathbf{y}$ and lie in $\mathcal{H}(\mathbf{F_U})$. But by Lemma 5.3, $\mathcal{H}(\mathbf{F_U})$ coincides with $\mathcal{H}_\mathbf{U}$ up to a set of measure 0.

We will use the notation

$$\mathcal{A}_\mathbf{U} = \mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_\mathbf{U}$$

when the points $\mathbf{x}, \mathbf{y}$ are understood (in what come next, we hold $\mathbf{x}, \mathbf{y}$ fixed but vary $\mathbf{U}$, so will also use notation $\mathcal{A}_\mathbf{V}$ etc., defined in the same way with $\mathbf{V}$ replacing $\mathbf{U}$, but always with respect to the same $\mathbf{x}, \mathbf{y}$). □

In particular, recall that $\mathbf{S}$ is the unique $\sqsubseteq$–maximal element of $\mathfrak{F}^\bullet$ and $\mathbf{X} = \mathbf{F_S}$. (We can assume $\mathbf{S} \in \mathfrak{F}^\bullet$, for otherwise $\mathbf{F_S} = \mathbf{X}$ is a single point, and we are done.) Then $\bar{\mathbf{x}} = \mathbf{x}$ and $\bar{\mathbf{y}} = \mathbf{y}$. Also, $\mathcal{H}_\mathbf{S} = \mathcal{H}$. So, $\mathsf{d}_1(\mathbf{x}, \mathbf{y}) = t_\mathbf{S}(\mathbf{x}, \mathbf{y})$.

The next step is to define a quantity $s_\mathbf{U}(\mathbf{x}, \mathbf{y})$. The idea is that $s_\mathbf{U}(\mathbf{x}, \mathbf{y})$ will measure the contribution to $t_\mathbf{U}(\mathbf{x}, \mathbf{y})$ coming from positive-measure subsets $\mathcal{H}_\mathbf{V} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ with $\mathbf{V} \subsetneq \mathbf{U}$.

Definition 5.11 (s–distance for minimal $\mathbf{U}$)**.** If $\mathbf{U}$ is $\sqsubseteq$–minimal, then $\mathcal{H}_\mathbf{U}$ is the set of all halfspaces associated to walls whose colour is exactly $\mathbf{U}$, and we define

$$s_\mathbf{U}(\mathbf{x}, \mathbf{y}) = t_\mathbf{U}(\mathbf{x}, \mathbf{y}).$$

This will form the basis for an inductive definition of $s_\mathbf{U}$, where induction is on the *level* of $\mathbf{U}$, i.e. the maximum possible length of a $\sqsubseteq$–chain with highest element $\mathbf{U}$. So, in this induction, we will use finite depth. □

For $\mathbf{U}$ not $\sqsubseteq$–minimal, we define the quantity $s_\mathbf{U}(\mathbf{x}, \mathbf{y})$ as follows. First, recall that $\mathcal{A}_\mathbf{U} = \mathcal{H}_\mathbf{U} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ is fio–measurable, since it is the intersection of two inseparable sets (see [Fio20, Lemma 3.9]).

For each $\mathbf{V} \subsetneq \mathbf{U}$,

$$\mathrm{fio}(\mathcal{A}_\mathbf{V}) = t_\mathbf{V}(\mathbf{x}, \mathbf{y}).$$

We define a measurable set $\mathcal{P}_\mathbf{V}$ of halfspaces as follows. If $\mathbf{V}$ is $\sqsubseteq$–minimal, then $\mathcal{P}_\mathbf{V} = \mathcal{A}_\mathbf{V}$ (which is fio–measurable).

Proceed inductively on the $\sqsubseteq$–level of $\mathbf{V}$. Let $\mathcal{V}_\mathbf{V}$ be the set of $\mathbf{W} \subsetneq \mathbf{V}$ with $\mathrm{fio}(\mathcal{A}_\mathbf{W}) > 0$. Let

$$\mathcal{P}_\mathbf{V} = \mathcal{A}_\mathbf{V} - \bigcup_{\mathbf{W} \in \mathcal{V}_\mathbf{V}} \mathcal{P}_\mathbf{W}.$$

Assume by induction that each $\mathcal{P}_\mathbf{W} \subset \mathcal{A}_\mathbf{W}$, each $\mathcal{P}_\mathbf{W}$ is fio–measurable, and $\mathcal{P}_\mathbf{W} \cap \mathcal{P}_\mathbf{T}$ has measure 0 whenever $\mathbf{T} \neq \mathbf{W}$.

Now suppose that $\mathbf{V}' \subsetneq \mathbf{U}$ has level at most that of $\mathbf{V}$. We first claim that $\mathcal{P}_\mathbf{V} \cap \mathcal{P}_{\mathbf{V}'}$ has measure 0 whenever $\mathbf{V} \neq \mathbf{V}'$.

There are three cases:

- If $\mathbf{V} \perp \mathbf{V}'$, then $\mathcal{H}(\mathbf{F_V}) \cap \mathcal{H}(\mathbf{F}_{\mathbf{V}'}) = \varnothing$, so $\mathrm{fio}(\mathcal{H}_\mathbf{V} \cap \mathcal{H}_{\mathbf{V}'}) = 0$ by Lemma 5.3. So $\mathcal{A}_\mathbf{V} \cap \mathcal{A}_{\mathbf{V}'}$ has measure 0, so the same is true upon replacing A with $\mathcal{P}$.

  (The assertion $\mathcal{H}(\mathbf{F_V}) \cap \mathcal{H}(\mathbf{F}_{\mathbf{V}'}) = \varnothing$ follows from the fact that, up to parallelism (which does not affect the sets of halfspaces in question by Lemma 2.15), we have a convex product region $\mathbf{F_V} \times \mathbf{F}_{\mathbf{V}'}$. So, no wall can simultaneously separate points in $\mathbf{F_V}$ and $\mathbf{F}_{\mathbf{V}'}$ since the image of one under the gate map to the other is a single point.)
- If $\mathbf{V}' \subsetneq \mathbf{V}$, then $\mathcal{P}_\mathbf{V} \subset \mathcal{A}_\mathbf{V} - \mathcal{P}_{\mathbf{V}'}$, as required.
- If $\mathbf{V} \pitchfork \mathbf{V}'$, then let $\mathbf{F} = \mathfrak{h}_\mathbf{V}(\mathbf{F}_{\mathbf{V}'})$. So, up to a set of measure 0, the halfspaces in $\mathcal{A}_\mathbf{V} \cap \mathcal{A}_{\mathbf{V}'}$ are in $\mathcal{H}(\mathbf{F}) \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$. If this set has measure 0, we are done.

  Otherwise, note that $\mathcal{H}(\mathbf{F}) = \mathcal{H}(\mathbf{F_V}) \cap \mathcal{H}(\mathbf{F}_{\mathbf{V}'})$. So, by Lemma 5.3, the measure of

$$\mathcal{A} = \mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_\mathbf{V} \cap \mathcal{H}_{\mathbf{V}'}$$

is positive.

Moreover, letting $\mathcal{W}_{\mathcal{A}}$ be the set of walls whose associated halfspaces are in $\mathcal{A}$, we have $Col(\mathcal{W}_{\mathcal{A}}) \sqsubseteq \mathbf{V}, \mathbf{V}'$. Furthermore, since $\mathcal{H}_{\mathbf{V}}, \mathcal{H}_{\mathbf{V}'}$ and $\mathcal{H}(\mathbf{x}, \mathbf{y})$ are inseparable sets of walls, so is their intersection $\mathcal{A}$.

Thus, by Definition 3.1, we have colours $\{\mathbf{W}_i\}_{i\in I} \sqsubseteq \mathbf{V}', \mathbf{V}$ such that $\text{fio}(\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{A}) > 0$ for all $i \in I$ and such that $\mathcal{W}_{\mathcal{A}} \subset \bigcup_{i\in I} \mathcal{W}_{\mathbf{W}_i}$, up to replacing $\mathcal{A}$ by a subset differing from it on a null set.

Now, up to null sets, we have $\mathcal{A}_{\mathbf{V}} \cap \mathcal{A}_{\mathbf{V}'} = \mathcal{A} \subset \bigcup_{i\in I} \mathcal{A}_{\mathbf{W}_i}$. Moreover, since $\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{A} \subset \mathcal{H}_{\mathbf{W}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{y}) = \mathcal{A}_{\mathbf{W}_i}$ and by assumption $\text{fio}(\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{A}) > 0$, we have that $\text{fio}(\mathcal{A}_{\mathbf{W}_i}) > 0$ and so $\mathbf{W}_i \in \mathcal{V}_{\mathbf{V}} \cap \mathcal{V}_{\mathbf{V}'}$. Hence, $\mathcal{P}_{\mathbf{V}} \cap \mathcal{P}_{\mathbf{V}'}$ has measure 0, as required.

Next, we check that $\mathcal{P}_{\mathbf{V}}$ is fio–measurable. For each $n \geqslant 1$, let $\mathcal{V}^n_{\mathbf{V}}$ be the set of $\mathbf{W} \in \mathcal{V}_{\mathbf{V}}$ with $\text{fio}(\mathcal{P}_{\mathbf{W}}) > \frac{1}{n}$. For any finite $\mathcal{F} \subset \mathcal{V}^n_{\mathbf{V}}$, the preceding discussion and the inclusion-exclusion principle gives

$$\text{fio}\left(\bigcup_{\mathbf{W}\in\mathcal{F}} \mathcal{P}_{\mathbf{W}}\right) > \frac{|\mathcal{F}|}{n}.$$

Hence

$$|\mathcal{F}| \leqslant n \cdot \text{fio}(\mathcal{A}_{\mathbf{V}}) \leqslant n \cdot \mathsf{d}_1(\mathbf{x}, \mathbf{y}) < \infty,$$

so since this bound is independent of $\mathcal{F}$, we have that $\mathcal{V}^n_{\mathbf{V}}$ is finite. So $\mathcal{V}_{\mathbf{V}} = \cup_{n\geqslant 1} \mathcal{V}^n_{\mathbf{V}}$ is countable. Hence $\mathcal{P}_{\mathbf{V}}$ is the complement in a fio–measurable set of a countable union of measurable sets, so it is fio–measurable.

We thus have that $\mathcal{P}_{\mathbf{U}}$ is measurable, and we define

$$s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = \text{fio}(\mathcal{P}_{\mathbf{U}}).$$

Note that

$$s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = t_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) - \sum_{\mathbf{V}\sqsubsetneq\mathbf{U}} s_{\mathbf{V}}(\mathbf{x}, \mathbf{y}).$$

We have shown above that the sum on the right has countably many nonzero terms, all positive, so the sum is well-defined. More precisely, choosing an enumeration $\{\mathbf{V}_n\}_{n\geqslant 1}$ of the elements of the countable set $\mathcal{V}_{\mathbf{U}}$, we have

$$\begin{aligned} s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) &= \text{fio}\left(\mathcal{A}_{\mathbf{U}} - \bigcup_{\mathbf{V}\in\mathcal{V}_{\mathbf{U}}} \mathcal{P}_{\mathbf{V}}\right) \\ &= t_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) - \text{fio}\left(\bigcup_{n\geqslant 1} \mathcal{P}_{\mathbf{V}_n}\right). \end{aligned}$$

For any $n \geqslant 1$, we have $\text{fio}(\mathcal{P}_{\mathbf{V}_n} \cap \mathcal{P}_{\mathbf{V}_m}) = 0$ for all $m \neq n$, so we can write $\mathcal{P}_{\mathbf{V}_n} = \mathcal{P}'_{\mathbf{V}_n} \sqcup \mathcal{Q}_n$, where $\text{fio}(\mathcal{Q}_n) = 0$ and $\mathcal{P}'_{\mathbf{V}_n} \cap \mathcal{P}'_{\mathbf{V}_m} = \varnothing$ for $n \neq m$. So

$$\sum_n \text{fio}(\mathcal{P}'_n) \leqslant \text{fio}\left(\bigcup_n \mathcal{P}_n\right) \leqslant \sum_n \text{fio}(\mathcal{P}'_n) + \sum_m \text{fio}(\mathcal{Q}_m),$$

whence $\text{fio}\left(\bigcup_n \mathcal{P}_n\right) = \sum_{\mathbf{V}\sqsubsetneq\mathbf{U}} s_{\mathbf{U}}(\mathbf{x}, \mathbf{y})$, as required.

In particular, taking $\mathbf{U} = \mathbf{S}$, we have

$$\mathsf{d}_1(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{V}\in\mathfrak{F}^\bullet} s_{\mathbf{V}}(\mathbf{x}, \mathbf{y}).$$

This will be essential later when we verify Definition 4.2.(5).

Now we can construct the $\mathbb{R}$–tree $\mathcal{T}^\bullet\mathbf{U}$.

Proposition 5.12 (Building $\mathcal{T}^\bullet\mathbf{U}$). *Let $\mathbf{U} \in \mathfrak{F}^\bullet$. Then $s_{\mathbf{U}} : \mathbf{X}^2 \to [0,\infty)$ is a pseudometric.*

Let $(\mathcal{T}^\bullet\mathbf{U}, \mathbf{D_U})$ be the metric quotient of the pseudometric space $(\mathbf{X}, s_{\mathbf{U}})$. Let $\pi_{\mathbf{U}} : \mathbf{X} \to \mathcal{T}^\bullet\mathbf{U}$ be the quotient map. For each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, let

$$\boldsymbol{\mu}_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x}), \pi_{\mathbf{U}}(\mathbf{y}), \pi_{\mathbf{U}}(\mathbf{z})) = \pi_{\mathbf{U}}(\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z})).$$

Then $\boldsymbol{\mu}_{\mathbf{U}}$ is independent of the implicit choices of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $(\mathcal{T}^\bullet, \mathbf{D_U}, \boldsymbol{\mu}_{\mathbf{U}})$ is a connected median metric space, and in fact an $\mathbb{R}$–tree.

Moreover, $\pi_{\mathbf{U}} : (\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu}) \to (\mathcal{T}^\bullet\mathbf{U}, \mathbf{D_U}, \boldsymbol{\mu}_{\mathbf{U}})$ is a 1–lipschitz median homomorphism.

Finally, for all $\mathbf{x} \in \mathbf{X}$, we have $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathfrak{h}_{\mathbf{U}}(\mathbf{x}))$, and if $\mathbf{F}, \mathbf{F}'$ are parallel copies of $\mathbf{F_V}$ for $\mathbf{V} \sqsubsetneq \mathbf{U}$ or $\mathbf{V} \pitchfork \mathbf{U}$, then $\pi_{\mathbf{U}}(\mathbf{F}) = \pi_{\mathbf{U}}(\mathbf{F}')$ is a single point, which we denote $\rho^{\mathbf{V}}_{\mathbf{U}}$.

Later, where it is unlikely to cause confusion, for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, we will write $\mathbf{D_U}(\mathbf{x}, \mathbf{y})$ to mean $\mathbf{D_U}(\pi_{\mathbf{U}}(\mathbf{x}), \pi_{\mathbf{U}}(\mathbf{y}))$.

Proof of Proposition 5.12. First we check that $s_{\mathbf{U}}$ is a pseudometric; as part of this argument, we will establish an identity that will also be used to see that $\mathbf{D_U}$ is a median metric.

Symmetry of $s_{\mathbf{U}}$ is automatic, since $\mathcal{H}(\mathbf{x}, \mathbf{y}) = \mathcal{H}(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ and let $\mathbf{m} = \boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

We have

$$\mathcal{H}(\mathbf{x}, \mathbf{y}) = \mathcal{H}(\mathbf{x}, \mathbf{z}) \cup \mathcal{H}(\mathbf{y}, \mathbf{z}) - \mathcal{H}(\mathbf{z}, \mathbf{m}),$$

essentially by the definition of a median. In other words,

$$\mathcal{H}(\mathbf{x}, \mathbf{y}) = [\mathcal{H}(\mathbf{x}, \mathbf{z}) - \mathcal{H}(\mathbf{z}, \mathbf{m})] \sqcup [\mathcal{H}(\mathbf{y}, \mathbf{z}) - \mathcal{H}(\mathbf{z}, \mathbf{m})],$$

so intersecting both sides with $\mathcal{H}_{\mathbf{U}}$ and taking fio–measures gives

$$t_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = t_{\mathbf{U}}(\mathbf{x}, \mathbf{z}) + t_{\mathbf{U}}(\mathbf{y}, \mathbf{z}) - 2t_{\mathbf{U}}(\mathbf{m}, \mathbf{z}).$$

(The reader can think of the factor of 2 coming from the fact that the first two terms collectively count each halfspace separating $\mathbf{m}, \mathbf{z}$ twice.)

In particular, when $\mathbf{U}$ is $\sqsubseteq$–minimal, so $s_{\mathbf{U}} = t_{\mathbf{U}}$, we have verified that

$$s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = s_{\mathbf{U}}(\mathbf{x}, \mathbf{z}) + s_{\mathbf{U}}(\mathbf{y}, \mathbf{z}) - 2s_{\mathbf{U}}(\mathbf{m}, \mathbf{z}). \tag{†}$$

Suppose that $\mathbf{U}$ is not $\sqsubseteq$–minimal. Assume by induction on the $\sqsubseteq$–level that equation (†) holds with $\mathbf{V}$ replacing $\mathbf{U}$, for any $\mathbf{V} \sqsubsetneq \mathbf{U}$.

Then from the identities

$$t_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = t_{\mathbf{U}}(\mathbf{x}, \mathbf{z}) + t_{\mathbf{U}}(\mathbf{y}, \mathbf{z}) - 2t_{\mathbf{U}}(\mathbf{m}, \mathbf{z})$$

and

$$s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = t_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) - \sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{x}, \mathbf{y})$$

we deduce that $s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = s_{\mathbf{U}}(\mathbf{x}, \mathbf{z}) + s_{\mathbf{U}}(\mathbf{y}, \mathbf{z}) - 2s_{\mathbf{U}}(\mathbf{m}, \mathbf{z})$, i.e. equation (†) holds for arbitrary $\mathbf{U}$.

(For the reader concerned about convergence, here is the full computation. First observe that $\sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{x}, \mathbf{y})$ is absolutely convergent, where the sum is, as usual taken over the countably many nonzero terms, and the same is true with $\mathbf{x}$ replaced by $\mathbf{y}$ or $\mathbf{m}$. So from the inductive hypothesis, we have that

$$\sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{x}, \mathbf{z}) + \sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{z}, \mathbf{y}) - 2 \sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{z}, \mathbf{m}).$$

Combing this with $t_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = t_{\mathbf{U}}(\mathbf{x}, \mathbf{z}) + t_{\mathbf{U}}(\mathbf{y}, \mathbf{z}) - 2t_{\mathbf{U}}(\mathbf{m}, \mathbf{z})$ then gives

$$s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) = t_{\mathbf{U}}(\mathbf{x}, \mathbf{z}) - \sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{x}, \mathbf{z}) + t_{\mathbf{U}}(\mathbf{y}, \mathbf{z}) - \sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{z}, \mathbf{y}) - 2t_{\mathbf{U}}(\mathbf{m}, \mathbf{z}) - 2 \sum_{\mathbf{V} \sqsubsetneq \mathbf{U}} s_{\mathbf{V}}(\mathbf{z}, \mathbf{m}),$$

and this is exactly $s_{\mathbf{U}}(\mathbf{x}, \mathbf{z}) + s_{\mathbf{U}}(\mathbf{y}, \mathbf{z}) - 2s_{\mathbf{U}}(\mathbf{m}, \mathbf{z})$.)

Equation (†) immediately yields the triangle inequality for $s_{\mathbf{U}}$, since $s_{\mathbf{U}}(\mathbf{z},\mathbf{m}) \geqslant 0$, because $s_{\mathbf{U}}$ is defined as a measure of a set. So $s_{\mathbf{U}}$ is a pseudometric.

Applying the same equation to any pair of distinct elements of $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$, a computation shows that

$$s_{\mathbf{U}}(\mathbf{x},\mathbf{z}) = s_{\mathbf{U}}(\mathbf{x},\mathbf{m}) + s_{\mathbf{U}}(\mathbf{z},\mathbf{m}),$$

and similarly with either $\mathbf{x},\mathbf{z}$ replaced by $\mathbf{y}$.

Let $(\mathcal{T}^{\bullet}\mathbf{U},\mathbf{D}_{\mathbf{U}})$ be the metric quotient of $(\mathbf{X},s_{\mathbf{U}})$ (i.e. identify points at $\mathbf{s}_{\mathbf{U}}$–distance 0). Let $\pi_{\mathbf{U}}:\mathbf{X}\to\mathcal{T}^{\bullet}\mathbf{U}$ be the quotient map.

The equalities of the form

$$s_{\mathbf{U}}(\mathbf{x},\mathbf{z}) = s_{\mathbf{U}}(\mathbf{x},\mathbf{m}) + s_{\mathbf{U}}(\mathbf{z},\mathbf{m})$$

almost show that $\boldsymbol{\mu}$ is a median for the pseudometric $s_{\mathbf{U}}$, but they do not establish uniqueness of the median. In other words, we have shown that for all $\pi_{\mathbf{U}}(\mathbf{x}),\pi_{\mathbf{U}}(\mathbf{y}),\pi_{\mathbf{U}}(\mathbf{z})$, there exists

$$\bar{\mathbf{m}} = \boldsymbol{\mu}_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x}),\pi_{\mathbf{U}}(\mathbf{y}),\pi_{\mathbf{U}}(\mathbf{z})) = \pi_{\mathbf{U}}(\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z}))$$

such that

$$\mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x}),\pi_{\mathbf{U}}(\mathbf{y})) = \mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x}),\bar{\mathbf{m}}) + \mathbf{D}_{\mathbf{U}}(\bar{\mathbf{m}},\pi_{\mathbf{U}}(\mathbf{y})), \tag{††}$$

and the same holds with $\mathbf{x}$ or $\mathbf{y}$ replaced by $\mathbf{z}$. We have not yet shown that $\bar{\mathbf{m}}$ is unique with this property, which is needed to have $\mathcal{T}^{\bullet}\mathbf{U}$ be a median space. This will be verified as part of the proof that $\mathcal{T}^{\bullet}\mathbf{U}$ is an ℝ–tree, using Proposition 2.24.

Now we show that $\mathcal{T}^{\bullet}\mathbf{U}$ is a geodesic space with unique geodesics.

First, for any $\mathbf{x}\in\mathbf{X}$,

$$\mathcal{H}(\mathbf{F}_{\mathbf{U}}) \cap \mathcal{H}(\mathbf{x},\mathfrak{h}_{\mathbf{U}}(\mathbf{x})) = \varnothing,$$

so by Lemma 5.3,

$$\operatorname{fio}(\mathcal{H}(\mathbf{x},\mathfrak{h}_{\mathbf{U}}(\mathbf{x})) \cap \mathcal{H}_{\mathbf{U}}) = 0,$$

so $s_{\mathbf{U}}(\mathbf{x},\mathfrak{h}_{\mathbf{U}}(\mathbf{x})) = 0$. Thus $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathfrak{h}_{\mathbf{U}}(\mathbf{x}))$, as claimed in the statement.

By the definition of $\mathbf{s}_{\mathbf{U}}$ and the fact that $\mathfrak{h}_{\mathbf{U}}$ is 1–lipschitz, we have that $\pi_{\mathbf{U}}$ is 1–lipschitz and hence continuous. Thus, by connectedness of $\mathbf{X}$, the space $\mathcal{T}^{\bullet}\mathbf{U}$ is connected.

Next, we show that $\pi_{\mathbf{U}}$ sends geodesics to geodesics. Indeed, let $\mathbf{x},\mathbf{y}\in\mathbf{X}$ and let $\gamma:[0,L]\to\mathbf{X}$ be a geodesic (parametrised by arc length) with $\gamma(0)=\mathbf{x},\gamma(L)=\mathbf{y}$). By the definition of a median metric, for $0\leqslant r\leqslant s\leqslant t\leqslant L$, we have

$$\boldsymbol{\mu}(\gamma(r),\gamma(s),\gamma(t)) = \gamma(s).$$

Now, continuity of $\pi_{\mathbf{U}}$ implies that $\pi_{\mathbf{U}}\circ\gamma:[0,L]\to\mathcal{T}^{\bullet}\mathbf{U}$ is a continuous path joining $\pi_{\mathbf{U}}(\mathbf{x})$ to $\pi_{\mathbf{U}}(\mathbf{y})$. Moreover, we have

$$\boldsymbol{\mu}_{\mathbf{U}}(\pi_{\mathbf{U}}\circ\gamma(r),\pi_{\mathbf{U}}\circ\gamma(s),\pi_{\mathbf{U}}\circ\gamma(t)) = \pi_{\mathbf{U}}\circ\gamma(s).$$

So, by (††), we have

$$\mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}\circ\gamma(r),\pi_{\mathbf{U}}\circ\gamma(t)) = \mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}\circ\gamma(r),\pi_{\mathbf{U}}\circ\gamma(s)) + \mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}\circ\gamma(s),\pi_{\mathbf{U}}\circ\gamma(t)).$$

Applying the above to arbitrary subdivisions of $[0,L]$, if $0\leqslant a=s_0<\cdots<s_k=b\leqslant L$ is a subdivision of $[a,b]$, then for any $0\leqslant i\leqslant j\leqslant k$, we have

$$\mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}\circ\gamma(a),\pi_{\mathbf{U}}\circ\gamma(b)) = \sum_{\ell=i}^{j-1}\mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}\circ\gamma(s_\ell),\pi_{\mathbf{U}}\circ\gamma(s_{\ell+1})),$$

by repeated application of (††). This shows that the right side is independent of the subdivision of $[a,b]$, so taking suprema over subdivisions gives

$$\mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}\circ\gamma(a),\pi_{\mathbf{U}}\circ\gamma(b)) = \left|\pi_{\mathbf{U}}\circ\gamma|_{[a,b]}\right|,$$

as required. Hence $\mathcal{T}^{\bullet}\mathbf{U}$ is a geodesic space and $\pi_{\mathbf{U}}$ takes geodesics to geodesics.

We need an auxiliary claim:

Claim 5. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and let α, β be geodesics of $(\mathbf{X}, \mathsf{d}_1)$ starting at $\mathbf{x}$ and ending at $\mathbf{y}$ (and parametrised by arc length). Then $\pi_{\mathbf{U}} \circ \alpha = \pi_{\mathbf{U}} \circ \beta$.

The proof of the claim is illustrated in Figure 11.

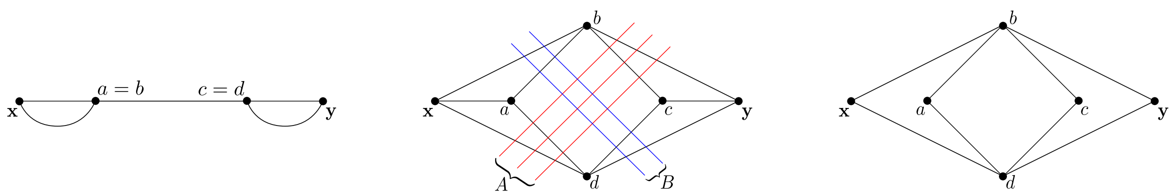

FIGURE 11. In the middle is the geodesic bigon and the sets $\mathcal{A}, \mathcal{B}$ of walls in the proof of Claim 5. If $\mathcal{A}, \mathcal{B}$ both correspond to sets of halfspaces with positive measure, then Definition 3.1 provides $\{\mathbf{U}_i\}, \{\mathbf{V}_j\}$ so that each $\mathbf{U}_i, \mathbf{V}_j$ contributes a positive-measure set of halfspaces to the rectangle $I(a, c)$, and every wall in $\mathcal{W}(a, c)$ has colour nested in some $\mathbf{U}_i, \mathbf{V}_j$, so the rectangle collapses to a point (picture on the right). But if $\mathcal{H}_{\mathcal{B}}$ has measure 0, then the rectangle is degenerate (picture on the left).

Proof of Claim 5. Since $\pi_{\mathbf{U}}$ factors through $\mathfrak{h}_{\mathbf{U}}$, we can assume for simplicity that $\mathbf{x}, \mathbf{y} \in \mathbf{F}_{\mathbf{U}}$, and hence $\alpha, \beta \subset \mathbf{F}_{\mathbf{U}}$ (by convexity of $\mathbf{F}_{\mathbf{U}}$).

If $\alpha = \beta$, we are done, so assume there exists t such that $\alpha(t) = d \in \alpha - \beta$ and $\beta(t) = b \in \beta - \alpha$. Note that b, d do not lie on a common geodesic from $\mathbf{x}$ to $\mathbf{y}$. Let $a = \boldsymbol{\mu}(\mathbf{x}, b, d)$ and let $c = \boldsymbol{\mu}(\mathbf{y}, b, d)$.

Then the sequence (a, b, c, d) is a *rectangle* in the sense of [CDH10, Definition 2.22], by [CDH10, Remark 2.23.(2)]. Hence $\mathcal{H}(a, d) = \mathcal{H}(b, c)$ and $\mathcal{H}(a, b) = \mathcal{H}(c, d)$, and every wall associated to a halfspace in the first set crosses every wall associated to a halfspace in the second set. This follows from [CDH10, Corollary 5.9]. Let $\mathcal{A}$ be the set of walls separating $\{a, b\}$ from $\{c, d\}$ and let $\mathcal{B}$ be the set of walls separating $\{a, d\}$ from $\{b, c\}$. Notice that the sets of walls $\mathcal{A}, \mathcal{B}$ are inseparable. Let $\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}$ be the sets of halfspaces associated to walls in $\mathcal{A}, \mathcal{B}$ respectively. Up to sets of fio–measure 0, we have $\mathcal{A} \subset \mathcal{W}_{\mathbf{U}}$ and $\mathcal{B} \subset \mathcal{W}_{\mathbf{U}}$.

First suppose that $\mathrm{fio}(\mathcal{H}_{\mathcal{A}}) > 0$ and $\mathrm{fio}(\mathcal{H}_{\mathcal{B}}) > 0$. Since the sets of walls $\mathcal{A}, \mathcal{B}$ are inseparable and each wall in $\mathcal{A}$ crosses each wall in $\mathcal{B}$, from Definition 3.1, there exist $\{\mathbf{U}_i\}_{i \in I}, \{\mathbf{V}_j\}_{j \in J} \sqsubseteq \mathbf{U}$ such that

- $\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i}) > 0$ for all $i \in I$, $\mathrm{fio}(\mathcal{B} \cap \mathbf{V}_j) > 0$ for all $j \in J$,
- $\mathcal{A} \subset \bigcup_{i \in I} \mathcal{W}_{\mathbf{U}_i}$, $\mathcal{B} \subset \bigcup_{j \in J} \mathcal{W}_{\mathbf{V}_i}$, up to sets of walls associated to measure-0 sets of halfspaces, and
- every wall with colour nested in $\mathbf{U}_i$ crosses every wall with colour nested in $\mathbf{V}_j$ for all $i \in I, j \in J$.

Notice that $\mathbf{V}_j \subsetneq \mathbf{U}$ for all $j \in J$. Indeed, if $\mathbf{V}_j = \mathbf{U}$ for some $j \in J$, since each wall in $\mathcal{W}_{\mathbf{U}_i}$ crosses each wall in $\mathcal{W}_{\mathbf{V}_j} = \mathcal{W}_{\mathbf{U}}$ and $\mathbf{U}_i \sqsubseteq \mathbf{U}$ we would have that each wall in $\mathcal{W}_{\mathbf{U}_i}$ crosses itself, which is a contradiction. Hence $\pi_{\mathbf{U}}(b) = \pi_{\mathbf{U}}(c)$ and $\pi_{\mathbf{U}}(a) = \pi_{\mathbf{U}}(d)$. (Indeed, the $\mathbf{U}_i$ show that, for example, $s_{\mathbf{U}}(b, c) = 0$.)

On the other hand, since $\mathbf{V}_j \subsetneq \mathbf{U}$ for all $j \in J$, we must have $\mathbf{U}_i \subsetneq \mathbf{U}$ for all $i \in I$, as argued for $\mathbf{V}_j \subsetneq \mathbf{U}$. So $\pi_{\mathbf{U}}(a) = \pi_{\mathbf{U}}(b)$ and $\pi_{\mathbf{U}}(c) = \pi_{\mathbf{U}}(d)$. In particular b, d have the same image.

Now suppose that $\mathrm{fio}(\mathcal{H}_{\mathcal{A}}) = 0$. Then $a = b$ and $c = d$. Hence b, d lie on a common geodesic from $\mathbf{x}$ to $\mathbf{y}$, a contradiction.

Hence $\pi_{\mathbf{U}} \circ \alpha(t) = \pi_{\mathbf{U}} \circ \beta(t)$ for all t, as required. □

Recall that, for any $\mathbf{x}, \mathbf{y}$, the median interval $I(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}, \mathbf{y}$ is the union of geodesics in $\mathbf{X}$ between $\mathbf{x}, \mathbf{y}$ (since $\mathbf{X}$ is a geodesic median space; see Lemma 2.7). So, by the preceding claim and the fact that geodesics map via $\pi_{\mathbf{U}}$ to geodesics, $\pi_{\mathbf{U}}(I(\mathbf{x}, \mathbf{y}))$ is (the image of) a single geodesic in $\mathcal{T}^{\bullet}\mathbf{U}$ from $\pi_{\mathbf{U}}(\mathbf{x})$ to $\pi_{\mathbf{U}}(\mathbf{y})$.

Claim 6. Let $\bar{\mathbf{x}} = \pi_{\mathbf{U}}(\mathbf{x}), \bar{\mathbf{y}} = \pi_{\mathbf{U}}(\mathbf{y})$. Suppose that $\mathbf{m} \in \mathbf{X}$ has the property that $\bar{\mathbf{m}} = \pi_{\mathbf{U}}(\mathbf{m})$ lies on a geodesic in $\mathcal{T}^{\bullet}\mathbf{U}$ from $\bar{\mathbf{x}}$ to $\bar{\mathbf{y}}$. Then there exists $\mathbf{m}' \in I(\mathbf{x}, \mathbf{y})$ such that $s_{\mathbf{U}}(\mathbf{m}, \mathbf{m}') = 0$.

Proof. Since $\mathbf{X}$ is complete and finite-rank, $I(\mathbf{x}, \mathbf{y})$ is compact (Lemma 2.8), and in particular closed in $\mathbf{X}$. As an interval, $I(\mathbf{x}, \mathbf{y})$ is convex. Hence there is a gate map $\mathfrak{g} : \mathbf{X} \to I(\mathbf{x}, \mathbf{y})$.

(More generally, intervals in a median algebra are always gated, and the gate of a point in $I(\mathbf{x}, \mathbf{y})$ is just the median of that point, $\mathbf{x}$, and $\mathbf{y}$.)

Let $\mathbf{m}' = \mathfrak{g}(\mathbf{m})$, so $\mathbf{m}' \in I(\mathbf{x}, \mathbf{y})$, and the walls separating $\mathbf{m}, \mathbf{m}'$ are exactly those that separate $\mathbf{m}$ from $I(\mathbf{x}, \mathbf{y})$. If $s_{\mathbf{U}}(\mathbf{m}, \mathbf{m}') = 0$, we are done. So, suppose that $s_{\mathbf{U}}(\mathbf{m}, \mathbf{m}') > 0$. Now, $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{m}) = \mathbf{m}'$, so by the equality established above, we have

$$s_{\mathbf{U}}(\mathbf{x}, \mathbf{m}) + s_{\mathbf{U}}(\mathbf{y}, \mathbf{m}) = s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}) + 2s_{\mathbf{U}}(\mathbf{m}, \mathbf{m}').$$

Hence

$$\mathbf{D}_{\mathbf{U}}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) < \mathbf{D}_{\mathbf{U}}(\bar{\mathbf{x}}, \bar{\mathbf{m}}) + \mathbf{D}_{\mathbf{U}}(\bar{\mathbf{y}}, \bar{\mathbf{m}}),$$

so $\bar{\mathbf{m}}$ cannot lie on a geodesic in $\mathcal{T}^{\bullet}\mathbf{U}$ from $\bar{\mathbf{x}}$ to $\bar{\mathbf{y}}$. This is a contradiction, so we are done. □

The preceding claim shows that any geodesic in $\mathcal{T}^{\bullet}\mathbf{U}$ from $\bar{\mathbf{x}}$ to $\bar{\mathbf{y}}$ lies in $\pi_{\mathbf{U}}(I(\mathbf{x}, \mathbf{y}))$, which we saw above is a single geodesic. Since $\mathbf{x}, \mathbf{y}$ were arbitrary, we conclude that any two points in $\mathcal{T}^{\bullet}\mathbf{U}$ are joined by a unique geodesic.

On the other hand, any $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}} \in \mathcal{T}^{\bullet}\mathbf{U}$ determine a geodesic triangle whose three sides all pass through $\boldsymbol{\mu}_{\mathbf{U}}(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}})$. Another application of uniqueness of geodesics now shows that $\mathcal{T}^{\bullet}\mathbf{U}$ is a 0–hyperbolic geodesic metric space, i.e. an $\mathbb{R}$–tree. In particular, $\mathcal{T}^{\bullet}\mathbf{U}$ is a median space of rank 1, and $\boldsymbol{\mu}_{\mathbf{U}}$ is the median. (See Proposition 2.24.)

In order to prove the proposition, it remains to prove the claim about $\mathbf{V} \subsetneq \mathbf{U}$ or $\mathbf{V} \pitchfork \mathbf{U}$. First suppose that $\mathbf{V} \subsetneq \mathbf{U}$, so by Lemma 5.8, we can choose $\mathbf{F}_{\mathbf{V}}$ such that $\mathbf{F}_{\mathbf{V}} \subset \mathbf{F}_{\mathbf{U}}$. If $\mathbf{x}, \mathbf{y} \in \mathbf{F}_{\mathbf{V}}$ we thus have

$$\mathrm{fio}((\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_{\mathbf{U}}) \triangle (\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_{\mathbf{V}})) = 0,$$

so $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathbf{y})$, i.e. $\pi_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}})$ is a single point. If $\mathbf{F}$ is parallel to $\mathbf{F}_{\mathbf{V}}$, then for all $\mathbf{x}, \mathbf{y} \in \mathbf{F}$, the fact that $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathfrak{h}_{\mathbf{U}}(\mathbf{x}))$ then shows that $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathbf{y})$, as required.

Now consider the case where $\mathbf{U} \pitchfork \mathbf{V}$ and consider $\pi_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}}) = \pi_{\mathbf{U}}(\mathfrak{h}_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}}))$. If the latter is a single point, we are done, so suppose that $\mathbf{x}, \mathbf{y} \in \mathbf{F}_{\mathbf{V}}$ have distinct gates on $\mathbf{F}_{\mathbf{U}}$, denoted $\bar{\mathbf{x}}, \bar{\mathbf{y}}$.

By replacing $\mathbf{x}$ with $\mathfrak{h}_{\mathbf{V}}(\bar{\mathbf{x}})$ and doing similarly for $\mathbf{y}$, we can assume that $\mathcal{H}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \mathcal{H}(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, where $\hat{\mathbf{x}} = \mathfrak{h}_{\mathbf{V}}(\bar{\mathbf{x}})$ and $\hat{\mathbf{y}} = \mathfrak{h}_{\mathbf{V}}(\bar{\mathbf{y}})$.

Now, since $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathbf{F}_{\mathbf{U}}$ we have that $\mathcal{H}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \subset \mathcal{H}(\mathbf{F}_{\mathbf{U}})$; similarly, $\mathcal{H}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \subset \mathcal{H}(\mathbf{F}_{\mathbf{V}})$. Since $\mathcal{H}(\mathbf{F}_{\mathbf{U}})$ and $\mathcal{H}(\mathbf{F}_{\mathbf{V}})$ coincide with $\mathcal{H}_{\mathbf{U}}$ and $\mathcal{H}_{\mathbf{V}}$ up to measure 0, respectively, we have that up to measure 0, $\mathcal{H}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is contained in $\mathcal{H}_{\mathbf{U}} \cap \mathcal{H}_{\mathbf{V}}$.

Let $\mathcal{A}$ be an inseparable set of walls such that $\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \triangle \mathcal{H}(\bar{\mathbf{x}}, \bar{\mathbf{y}})) = 0$ and $Col(\mathcal{A}) \subset \mathbf{U} \cap \mathbf{V}$. For example, we can take $\mathcal{A}$ to be the intersection of the inseparable set $\mathcal{H}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ with the inseparable set $\mathcal{W}_{\mathbf{U}} \cap \mathcal{W}_{\mathbf{V}}$.

Then from Definition 3.1.(III), there exists a family $\{\mathbf{W}_i\}_{i \in I} \sqsubseteq \mathbf{U}, \mathbf{V}$ such that $\mathcal{A} \subset \bigcup_{i \in I} \mathcal{W}_{\mathbf{W}_i}$ (up to a null set) and so $\mathcal{H}_{\mathcal{A}} \subset \bigcup_{i \in I} \mathcal{H}_{\mathbf{W}_i}$ (up to a null set). Moreover, each $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}_i}$ has positive measure.

Now, each $\mathbf{W}_i \sqsubseteq \mathbf{U}, \mathbf{V}$, so since $\mathbf{U}$ and $\mathbf{V}$ are not $\sqsubseteq$–related, the preceding nestings are proper. Thus the contribution of $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}_i}$ to $\mathbf{D}_{\mathbf{U}}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is 0 for each i. More precisely, we have shown that up to a set of measure 0, we have $\mathcal{A} \subset \mathcal{P}_{\mathbf{U}}$.

Hence $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathbf{y})$, i.e. $\pi_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}})$ is a single point.

Exactly as in the nested case, the fact that $\pi_{\mathbf{U}}$ factors through $\mathfrak{h}_{\mathbf{U}}$ shows that the same holds for any parallel copy of $\mathbf{F}_{\mathbf{V}}$. This completes the proof. ☐

5.1.2. *Checking the remainder of Definition 4.2.*

Proposition 5.13 (Distance formula)**.** *Let $\pi : \mathbf{X} \to \prod_{\mathbf{U} \in \mathfrak{F}^\bullet} \mathcal{T}^\bullet\mathbf{U}$ send each $\mathbf{x}$ to the tuple $(\pi_{\mathbf{U}}(\mathbf{x}))_{\mathbf{U} \in \mathfrak{F}^\bullet}$. Fixing a basepoint $\mathbf{x}_0 \in \mathbf{X}$, base each $\mathcal{T}^\bullet\mathbf{U}$ at $\mathbf{1}_{\mathbf{U}} = \pi_{\mathbf{U}}(\mathbf{x}_0)$. Then the image of π is contained in $\ell_1(\mathfrak{F}^\bullet)$, and π is an isometric embedding, and, identifying $\mathbf{X}$ with its image, each $\pi_{\mathbf{U}}$ coincides with the natural projection $\ell_1(\mathfrak{F}^\bullet) \to \mathcal{T}^\bullet\mathbf{U}$.*

Proof. The final statement, about natural projections, follows immediately from the definition of π. Let $\mathbf{x}, \mathbf{y} \in \mathcal{T}^\bullet$. We saw above that

$$\mathsf{d}_1(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{U} \in \mathfrak{F}^\bullet} s_{\mathbf{U}}(\mathbf{x}, \mathbf{y}),$$

i.e.

$$\mathsf{d}_1(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{U} \in \mathfrak{F}^\bullet} \mathbf{D}_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x}), \pi_{\mathbf{U}}(\mathbf{y})),$$

so π is an isometric embedding. In particular, since $\mathsf{d}_1(\mathbf{x}, \mathbf{x}_0) < \infty$ for all $\mathbf{x}$, the image of π is in $\ell_1(\mathfrak{F}^\bullet)$. ☐

We have now verified the first part of Definition 4.2.(5). We have also constructed the $\mathbb{R}$–trees $\mathcal{T}^\bullet\mathbf{U}$ and, for $\mathbf{V} \sqsubsetneq \mathbf{U}$ or $\mathbf{V} \pitchfork \mathbf{U}$, we have chosen a point $\rho^{\mathbf{V}}_{\mathbf{U}} \in \mathcal{T}^\bullet\mathbf{U}$ — see Proposition 5.12. As required by Definition 4.2, $\rho^{\mathbf{U}}_{\mathbf{V}}$ always lies in $\pi_{\mathbf{V}}(\mathbf{X})$, because of how it was defined.

We now check the rest of Definition 4.2.(3):

Lemma 5.14. *Let $\mathbf{U} \sqsubsetneq \mathbf{V}$ or $\mathbf{U} \perp \mathbf{V}$. Suppose $\mathbf{V} \sqsubsetneq \mathbf{W}$, or $\mathbf{V} \pitchfork \mathbf{W}$ and $\mathbf{W} \not\perp \mathbf{U}$. Then $\rho^{\mathbf{U}}_{\mathbf{W}} = \rho^{\mathbf{V}}_{\mathbf{W}}$.*

Proof. Suppose that $\mathbf{U} \sqsubsetneq \mathbf{V}$. Then we can choose $\mathbf{F}_{\mathbf{U}}$ in its parallelism class to lie in $\mathbf{F}_{\mathbf{V}}$, so by Proposition 5.12 we have $\rho^{\mathbf{U}}_{\mathbf{W}} = \rho^{\mathbf{V}}_{\mathbf{W}}$.

Next, suppose that $\mathbf{U} \perp \mathbf{V}$. Then we can choose parallel copies of $\mathbf{F}_{\mathbf{U}}, \mathbf{F}_{\mathbf{V}}$ that intersect in a point, by the definition of orthogonality. An application of Proposition 5.12 again gives $\rho^{\mathbf{U}}_{\mathbf{W}} = \rho^{\mathbf{V}}_{\mathbf{W}}$. ☐

Next, suppose that $\mathbf{U} \sqsubsetneq \mathbf{V}$. Let $\pi_{\mathbf{V}}(\mathbf{x}) \in \mathcal{T}^\bullet\mathbf{V}$. Define $\rho^{\mathbf{V}}_{\mathbf{U}}(\pi_{\mathbf{V}}(\mathbf{x}))$ to be $\pi_{\mathbf{U}}(\mathbf{x})$ if $\pi_{\mathbf{V}}(\mathbf{x}) \neq \rho^{\mathbf{U}}_{\mathbf{V}}$, and define it arbitrarily otherwise. To check that this is well-defined, we will use the following lemma, which also implies the bounded geodesic image axiom of Definition 4.2.

Lemma 5.15. *Let $\mathbf{U} \sqsubsetneq \mathbf{V}$ and let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Suppose that the geodesic in $\mathcal{T}^\bullet\mathbf{V}$ from $\pi_{\mathbf{V}}(\mathbf{x})$ to $\pi_{\mathbf{V}}(\mathbf{y})$ does not pass through $\rho^{\mathbf{U}}_{\mathbf{V}}$. Then $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathbf{y})$.*

Proof. Without loss of generality, $\mathbf{x}, \mathbf{y} \in \mathbf{F}_{\mathbf{V}}$ and $\mathbf{F}_{\mathbf{U}} \subset \mathbf{F}_{\mathbf{V}}$. So, we work entirely in $\mathbf{F}_{\mathbf{V}}$.

If $\mathfrak{h}_{\mathbf{U}}(\mathbf{x}) = \mathfrak{h}_{\mathbf{U}}(\mathbf{y})$, then $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathbf{y})$, and we are done. So, writing $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ for the gates of $\mathbf{x}, \mathbf{y}$ on $\mathbf{F}_{\mathbf{U}}$, we have

$$\text{fio}(\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}(\mathbf{F}_{\mathbf{U}})) > 0.$$

Let $a = \boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \bar{\mathbf{x}})$ and let $b = \boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \bar{\mathbf{y}})$.

So, b lies on a geodesic from $\mathbf{x}$ to $\bar{\mathbf{y}}$ (by the definition of the median and the fact that we are in a geodesic median space) and the same is true of $\bar{\mathbf{x}}$, by the definition of the gate of $\mathbf{x}$, since $\bar{\mathbf{y}} \in \mathbf{F}_{\mathbf{U}}$. In short, $b, \bar{\mathbf{x}} \in I(\mathbf{x}, \bar{\mathbf{y}})$.

Now, $\boldsymbol{\mu}(\mathbf{x}, \bar{\mathbf{x}}, b) = a$ by Definition 2.1. Also, $\boldsymbol{\mu}(b, \bar{\mathbf{x}}, \bar{\mathbf{y}}) = \bar{\mathbf{y}}$. So by [CDH10, Remark 2.23], the tuple $a, b, \bar{\mathbf{y}}, \bar{\mathbf{x}}$ (in that order) is a rectangle. So, $\mathcal{H}(a, \bar{\mathbf{x}}) = \mathcal{H}(b, \bar{\mathbf{y}})$ and $\mathcal{H}(a, b) = \mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}(\mathbf{F_U})$.

Every wall associated to a halfspace in the former set crosses every wall associated to a halfspace in the latter set. Now, if $\pi_{\mathbf{U}}(\mathbf{x}) \neq \pi_{\mathbf{U}}(\mathbf{y})$, then $\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}(\mathbf{F_U})$ has a positive-measure subset $\mathcal{A} \subset \mathcal{P}_{\mathbf{U}}$.

If $\mathcal{H}(a, \bar{\mathbf{x}})$ has measure 0, then $\pi_{\mathbf{V}}(I(\mathbf{x}, \mathbf{y}))$ intersects $\pi_{\mathbf{V}}(\mathbf{F_U}) = \rho^{\mathbf{U}}_{\mathbf{V}}$, and we are done.

Apply Definition 3.1 to find $\{\mathbf{U}_i\}_{i \in I} \sqsubseteq \mathbf{U}$ such that $\mathcal{A} \subset \bigcup_{i \in I} \mathcal{H}_{\mathbf{U}_i}$, and $\{\mathbf{V}_j\}_{j \in J} \sqsubseteq \mathbf{V}$ such that $\mathcal{H}(a, \bar{\mathbf{x}})$ is, up to a null set, contained in $\bigcup_{j \in J} \mathcal{H}_{\mathbf{V}_j}$. Moreover, every wall associated to a halfspace in $\mathcal{H}_{\mathbf{U}_i}$ crosses every wall associated to a halfspace in $\mathcal{H}_{\mathbf{V}_j}$ for all $i \in I, j \in J$, and the following hold for all i, j:

- $\operatorname{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i}) > 0$, and
- $\operatorname{fio}(\mathcal{H}(a, \bar{\mathbf{x}}) \cap \mathcal{H}_{\mathbf{V}_j}) > 0$.

Now, if $\mathbf{U}_i \subsetneq \mathbf{U}$ for some $i \in I$, then we contradict that $\mathcal{A}$ is contained in $\mathcal{P}_{\mathbf{U}}$. So, $\mathbf{U} = \mathbf{U}_i$ for all $i \in I$, whence, by Proposition 2.22 and the definition of orthogonality, we have $\mathbf{U} \perp \mathbf{V}_j$ for all $j \in J$ and a is contained in a parallel copy of $\mathbf{F_U}$. Again, since a lies on a geodesic from $\mathbf{x}$ to $\mathbf{y}$, we see that $\pi_{\mathbf{V}}(I(\mathbf{x}, \mathbf{y}))$ passes through $\pi_{\mathbf{V}}(a) = \rho^{\mathbf{U}}_{\mathbf{V}}$, as required. □

Lemma 5.16. *Let $\mathbf{U} \subsetneq \mathbf{V}$. The map $\rho^{\mathbf{V}}_{\mathbf{U}} : \mathcal{T}^\bullet\mathbf{V} \to \mathcal{T}^\bullet\mathbf{U}$ is well-defined and constant on each component of $\mathcal{T}^\bullet\mathbf{V} - \{\rho^{\mathbf{U}}_{\mathbf{V}}\}$.*

Proof. To see that $\rho^{\mathbf{V}}_{\mathbf{U}}$ is well-defined, note that Lemma 5.15 implies $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathbf{y})$ whenever $\mathbf{x}, \mathbf{y}$ have the property that $\pi_{\mathbf{V}}(\mathbf{x}) = \pi_{\mathbf{V}}(\mathbf{y}) \neq \rho^{\mathbf{U}}_{\mathbf{V}}$. This also shows that $\rho^{\mathbf{V}}_{\mathbf{U}}$ is constant on each component of $\mathcal{T}^\bullet\mathbf{V} - \{\rho^{\mathbf{U}}_{\mathbf{V}}\}$. □

It remains to verify the consistency equations from Definition 4.2.(5), and to show that $\mathbf{X}$ coincides with the set of points in $\ell_1(\mathfrak{F}^\bullet)$ satisfying the consistency equations.

Lemma 5.17. *If $\mathbf{U} \subsetneq \mathbf{V}$ and $\mathbf{x} \in \mathbf{X}$, then either $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ or $\rho^{\mathbf{V}}_{\mathbf{U}}(\pi_{\mathbf{V}}(\mathbf{x})) = \pi_{\mathbf{U}}(\mathbf{x})$.*

Proof. This is a rephrasing of the definition of $\rho^{\mathbf{V}}_{\mathbf{U}}$. □

Lemma 5.18. *Let $\mathbf{U} \pitchfork \mathbf{V}$. Let $\mathbf{x} \in \mathbf{X}$. Then either $\pi_{\mathbf{U}}(\mathbf{x}) = \rho^{\mathbf{V}}_{\mathbf{U}}$ or the same holds with the roles of $\mathbf{U}, \mathbf{V}$ reversed.*

Proof. Let $a = \mathfrak{h}_{\mathbf{U}}(\mathbf{x})$ and let $b = \mathfrak{h}_{\mathbf{V}}(\mathbf{x})$ (so, we have implicitly fixed parallel copies of $\mathbf{F_U}, \mathbf{F_V}$).

If $\hat{w}$ is a wall separating $\mathbf{x}$ from a and crossing $\mathbf{F_V}$, then $\hat{w}$ cannot separate $\mathbf{x}$ from b. If $\hat{v}$ is a wall separating $\mathbf{x}$ from b and crossing $\mathbf{F_U}$, then we therefore have that $\hat{v}, \hat{w}$ cross.

If $\pi_{\mathbf{V}}(I(\mathbf{x}, a))$ is a single point, then $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$, and we are done. So, assume not. Then

$$\operatorname{fio}(\mathcal{H}(\mathbf{x}, a) \cap \mathcal{H}_{\mathbf{V}}) > 0.$$

Similarly, if $\pi_{\mathbf{U}}(I(\mathbf{x}, b))$ is a single point, we are done, so we can assume that

$$\operatorname{fio}(\mathcal{H}(\mathbf{x}, b) \cap \mathcal{H}_{\mathbf{U}}) > 0.$$

Hence we have two positive-measure sets of halfspaces, $\mathcal{A} \subset \mathcal{H}_{\mathbf{U}}$ and $\mathcal{B} \subset \mathcal{H}_{\mathbf{V}}$, such that every wall associated to a halfspace in $\mathcal{A}$ crosses every wall in a halfspace associated to $\mathcal{B}$.

Now apply Definition 3.1 to find $\{\mathbf{U}_i\}_{i \in I} \sqsubseteq \mathbf{U}$ and $\{\mathbf{V}_j\}_{j \in J} \sqsubseteq \mathbf{V}$ such that (up to null sets of associated halfspaces) $\mathcal{A}$–halfspaces are associated to walls coloured $\mathbf{U}_i$ and $\mathcal{B}$–halfspaces are associated to walls coloured $\mathbf{V}_j$, and every wall coloured $\mathbf{U}_i$ crosses every wall coloured

$\mathbf{V}_j$. Moreover, the sets $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i}$ and $\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j}$ have positive measure. If $\mathbf{U}_i \subsetneq \mathbf{U}$ for all $i \in I$, then $\mathrm{fio}(\mathcal{A} \cap \mathcal{P}_{\mathbf{U}}) = 0$ and we get

$$\mathbf{D}_{\mathbf{U}}(\mathbf{x}, \rho^{\mathbf{V}}_{\mathbf{U}}) = 0,$$

and the same holds with $\mathbf{V}$ and $\mathbf{U}$ switching roles if $\mathbf{V}_j \subsetneq \mathbf{V}$ for all $j \in J$. We conclude that $\mathbf{U} = \mathbf{U}_{i_0}$ for some $i_0 \in I$ (and so $\{\mathbf{U}_i\} = \{\mathbf{U}_{i_0}\}$) and $\mathbf{V} = \mathbf{V}_{j_0}$ for some $j_0 \in J$ (and so $\{\mathbf{V}_j\} = \{\mathbf{V}_{j_0}\}$).

One can now use [Fio20, Corollary 3.11] and [Fio18, Proposition 2.10] to construct parallel copies $\mathbf{F}_{\mathbf{U}}, \mathbf{F}_{\mathbf{V}}$ whose convex hull in $\mathbf{X}$ is isometric to $\mathbf{F}_{\mathbf{U}} \times \mathbf{F}_{\mathbf{V}}$ (more explicitly, use Proposition 2.22), so by Definition 5.7, $\mathbf{U} \perp \mathbf{V}$, a contradiction. □

We have finished verifying the consistency conditions. To verify that the consistency conditions completely characterise the image of $\mathbf{X}$ in $\ell_1(\mathfrak{F}^\bullet)$, we prove the following.

Proposition 5.19 (Realisation). *Let $(\mathbf{x}_{\mathbf{U}})_{\mathbf{U} \in \mathfrak{F}^\bullet} \in \ell_1(\mathfrak{F}^\bullet)$ and suppose that the following hold for all $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet$:*

- *If $\mathbf{U} \pitchfork \mathbf{V}$, then*
$$\mathsf{d}_{\mathbf{U}}(\mathbf{x}_{\mathbf{U}}, \rho^{\mathbf{V}}_{\mathbf{U}}) \cdot \mathsf{d}_{\mathbf{V}}(\mathbf{x}_{\mathbf{V}}, \rho^{\mathbf{U}}_{\mathbf{V}}) = 0.$$
- *If $\mathbf{U} \subsetneq \mathbf{V}$, then*
$$\mathsf{d}_{\mathbf{V}}(\mathbf{x}_{\mathbf{V}}, \rho^{\mathbf{U}}_{\mathbf{V}}) \cdot \mathsf{d}_{\mathbf{U}}(\rho^{\mathbf{V}}_{\mathbf{U}}(\mathbf{x}_{\mathbf{V}}), \mathbf{x}_{\mathbf{U}}) = 0.$$

Then there exists $\mathbf{x} \in \mathbf{X}$ such that $\pi_{\mathbf{U}}(\mathbf{x}) = \mathbf{x}_{\mathbf{U}}$ for all $\mathbf{U} \in \mathfrak{F}^\bullet$.

Note that the point $\mathbf{x}$ provided by the proposition is necessarily unique. Before the proof, we need a general lemma about gates in $\mathbf{X}$.

Lemma 5.20. *Let $A \subset \mathbf{X}$ be a closed convex subset and let $\mathfrak{g} : \mathbf{X} \to A$ be the gate map. Then for all $\mathbf{U} \in \mathfrak{F}^\bullet$ and all $\mathbf{x} \in \mathbf{X}$, the point $\pi_{\mathbf{U}}(\mathfrak{g}(\mathbf{x})) \in \pi_{\mathbf{U}}(A)$ lies on the $\mathcal{T}^\bullet\mathbf{U}$–geodesic from $\pi_{\mathbf{U}}(\mathbf{x})$ to any point in $\pi_{\mathbf{U}}(A)$.*

Proof. Since $\pi_{\mathbf{U}}$ is continuous, $\pi_{\mathbf{U}}(A)$ is connected and hence convex in $\mathcal{T}^\bullet\mathbf{U}$. Let $p \in \pi_{\mathbf{U}}(A)$, so we can write $p = \pi_{\mathbf{U}}(\mathbf{a})$ for some $\mathbf{a} \in A$. Then $\mathfrak{g}(\mathbf{x})$ lies on a geodesic in $\mathbf{X}$ from $\mathbf{x}$ to $\mathbf{a}$, so since $\pi_{\mathbf{U}}$ sends geodesics to geodesics, $\pi_{\mathbf{U}}(\mathfrak{g}(\mathbf{x}))$ lies on the geodesic from $\pi_{\mathbf{U}}(\mathbf{x})$ to p. □

Proof of Proposition 5.19. Fix a tuple $(\mathbf{x}_{\mathbf{U}})_{\mathbf{U} \in \mathfrak{F}^\bullet} \in \ell_1(\mathfrak{F}^\bullet)$ as in the statement (i.e. a consistent tuple in $\ell_1(\mathfrak{F}^\bullet)$). Fix a basepoint $\mathbf{y} \in \mathbf{X}$.

The spaces $\mathbf{Y}_{\mathbf{U}}$: First, we study the preimage of $\mathbf{x}_{\mathbf{U}}$ under the projection $\pi_{\mathbf{U}}$.

For each $\mathbf{U} \in \mathfrak{F}^\bullet$, the set $\mathbf{Y}_{\mathbf{U}} = \pi_{\mathbf{U}}^{-1}(\mathbf{x}_{\mathbf{U}})$ is closed, since $\mathcal{T}^\bullet\mathbf{U}$ is Hausdorff and $\pi_{\mathbf{U}}$ is continuous (since it is lipschitz).

Moreover, $\mathbf{Y}_{\mathbf{U}}$ is median-convex since $\pi_{\mathbf{U}}$ is a median homomorphism. Indeed, if $\mathbf{a}, \mathbf{b} \in \mathbf{Y}_{\mathbf{U}}$ and $\mathbf{c} \in \mathbf{X}$, then $\boldsymbol{\mu}_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{a}), \pi_{\mathbf{U}}(\mathbf{b}), \pi_{\mathbf{U}}(\mathbf{c})) = \mathbf{x}_{\mathbf{U}}$, and so $\pi_{\mathbf{U}}(\boldsymbol{\mu}(\mathbf{a}, \mathbf{b}, \mathbf{c})) = \mathbf{x}_{\mathbf{U}}$, as required.

Also, $\mathbf{Y}_{\mathbf{U}} \neq \varnothing$ since $\pi_{\mathbf{U}} : \mathbf{F}_{\mathbf{U}} \to \mathcal{T}^\bullet\mathbf{U}$ is surjective (recall that $\mathcal{T}^\bullet\mathbf{U}$ is by construction a quotient of $\mathbf{F}_{\mathbf{U}}$).

Using consistency and the Helly property: We next claim that $\mathbf{Y}_{\mathbf{U}} \cap \mathbf{Y}_{\mathbf{V}} \neq \varnothing$ for all $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet$. We will use the consistency assumption.

- If $\mathbf{U} \perp \mathbf{V}$, then since $\pi_{\mathbf{U}}$ is surjective on any parallel copy of $\mathbf{F}_{\mathbf{V}}$, and the same is true reversing $\mathbf{U}$ and $\mathbf{V}$, the definition of orthogonality yields some $\mathbf{a} \in \mathbf{F}_{\mathbf{U}} \times \mathbf{F}_{\mathbf{V}}$ such that $\pi_{\mathbf{U}}(\mathbf{a}) = \mathbf{x}_{\mathbf{U}}$ and $\pi_{\mathbf{V}}(\mathbf{a}) = \mathbf{x}_{\mathbf{V}}$. So, $\mathbf{a} \in \mathbf{Y}_{\mathbf{U}} \cap \mathbf{Y}_{\mathbf{V}}$.
- If $\mathbf{V} \pitchfork \mathbf{U}$, then consistency implies, without loss of generality, that $\mathbf{x}_{\mathbf{V}} = \rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{F}_{\mathbf{U}})$. Choose $\mathbf{a} \in \mathbf{F}_{\mathbf{U}}$ such that $\pi_{\mathbf{U}}(\mathbf{a}) = \mathbf{x}_{\mathbf{U}}$. Then $\pi_{\mathbf{V}}(\mathbf{a}) = \rho^{\mathbf{U}}_{\mathbf{V}} = \mathbf{x}_{\mathbf{V}}$, so again $\mathbf{a} \in \mathbf{Y}_{\mathbf{U}} \cap \mathbf{Y}_{\mathbf{V}}$.

- If $\mathbf{U} \subsetneq \mathbf{V}$, then there are two possibilities. If $\rho^{\mathbf{U}}_{\mathbf{V}} = \mathbf{x_V}$, then we can take $\mathbf{a} \in \mathbf{F_U} \cap \mathbf{Y_U}$, and as in the transverse case we get $\mathbf{a} \in \mathbf{Y_U} \cap \mathbf{Y_V}$. Otherwise, if $\mathbf{x_V} \neq \rho^{\mathbf{U}}_{\mathbf{V}}$, then choose $\mathbf{a} \in \mathbf{F_V} \cap \mathbf{Y_V}$. Then $\pi_{\mathbf{U}}(\mathbf{a}) = \rho^{\mathbf{V}}_{\mathbf{U}}(\mathbf{x_V}) = \mathbf{x_U}$, by the consistency assumption and the definition of the map $\rho^{\mathbf{V}}_{\mathbf{U}}$.

By the Helly property for convex sets [Rol16, Theorem 2.2], for any finite set $\mathcal{S} \subset \mathfrak{F}^{\bullet}$, we therefore have

$$\mathbf{Y}_{\mathcal{S}} = \bigcap_{\mathbf{U} \in \mathcal{S}} \mathbf{Y_U} \neq \varnothing.$$

The set of relevant domains: Let $\mathcal{F}$ be the set of $\mathbf{U}$ such that $\pi_{\mathbf{U}}(\mathbf{y}) \neq \mathbf{x_U}$. Let $\mathcal{F}'_n$ be the subset of $\mathcal{F}$ consisting of those $\mathbf{U}$ for which

$$\mathbf{D_U}(\mathbf{y}, \mathbf{x_U}) > \frac{1}{n}.$$

Since $(\mathbf{x_U})_{\mathbf{U}} \in \ell_1(\mathfrak{F}^{\bullet})$ and $\mathsf{d}_1(\mathbf{y}, \mathbf{x}_0) < \infty$, we have $\sum_{\mathbf{U} \in \mathcal{F}} \mathbf{D_U}(\mathbf{y}, \mathbf{x_U}) < \infty$. So, $\mathcal{F}'_n$ is finite for each n, and $\mathcal{F}$ is countable.

If $\mathcal{F} = \varnothing$, then $\mathbf{y}$ is the desired point, and we are done. So we assume that $\mathcal{F} \neq \varnothing$.

By choosing an arbitrary enumeration of $\mathcal{F}$, we write

$$\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n,$$

where $\mathcal{F}_0 = \varnothing$, and for all $n \geqslant 1$, we have $\mathcal{F}_n = \mathcal{F}_{n-1} \cup \{\mathbf{V}_n\}$, where $\mathbf{V}_n \in \mathcal{F}$ is the n^{th} element in the enumeration.

The points $\mathbf{y}_n$: We define a sequence $(\mathbf{y}_n)_{n \geqslant 0}$ in $\mathbf{X}$ as follows. First, let $\mathbf{y}_0 = \mathbf{y}$.

For $n \geqslant 1$, using the above application of the Helly property, we have that $\mathbf{Y}_{\mathcal{F}_n} \neq \varnothing$. Moreover, $\mathbf{Y}_{\mathcal{F}_n}$ is closed and convex, as the intersection of closed convex sets. So $\mathbf{Y}_{\mathcal{F}_n}$ admits a gate map, and we declare $\mathbf{y}_n$ to be the image of $\mathbf{y}_0$ under the gate map to $\mathbf{Y}_{\mathcal{F}_n}$.

Our goal now is to show that $(\mathbf{y}_n)_n$ is a Cauchy sequence, so that by completeness of $\mathbf{X}$, this sequence converges to some point $\mathbf{x} \in \mathbf{X}$, which we will show satisfies $\pi_{\mathbf{U}}(\mathbf{x}) = \mathbf{x_U}$ for each $\mathbf{U}$.

Recall that $\mathcal{F}_n = \mathcal{F}_{n-1} \cup \{\mathbf{V}_n\}$, so that

$$\mathbf{Y}_{\mathcal{F}_n} = \mathbf{Y}_{\mathcal{F}_{n-1}} \cap \mathbf{Y}_{\mathbf{V}_n}.$$

This has two useful consequences:

(A) By the definition of the gate, $\boldsymbol{\mu}(\mathbf{y}, \mathbf{y}_n, \mathbf{y}_N) = \mathbf{y}_N$ whenever $n \geqslant N$. Hence $\mathbf{y}_N$ lies on a geodesic from $\mathbf{y}$ to $\mathbf{y}_n$ whenever $N \leqslant n$. Here, we have just used that $\mathbf{Y}_{\mathcal{F}_n} \subset \mathbf{Y}_{\mathcal{F}_N}$ when $n \geqslant N$, and both sets are closed and convex.

(B) The point $\mathbf{y}_n$ is the gate of $\mathbf{y}_{n-1}$ in $\mathbf{Y}_{\mathbf{V}_n}$. We now check this, and use the following notation: given a closed convex set A, let $\mathfrak{g}_A : \mathbf{X} \to A$ be the gate map. So, by definition,

$$\mathbf{y}_n = \mathfrak{g}_{\mathbf{Y}_{\mathcal{F}_{n-1}} \cap \mathbf{Y}_{\mathbf{V}_n}}(\mathbf{y}).$$

But $\mathbf{Y}_{\mathcal{F}n-1} \cap \mathbf{Y}_{\mathbf{V}_n} = \mathfrak{g}_{\mathbf{V}_n}(\mathbf{Y}_{\mathcal{F}n-1})$, by e.g. [Fio20, Lemma 2.2.(1)]. Hence $\mathfrak{g}_{\mathbf{Y}_{\mathcal{F}n-1} \cap \mathbf{Y}_{\mathbf{V}_n}} = \mathfrak{g}_{\mathbf{Y}_{\mathbf{V}_n}} \circ \mathfrak{g}_{\mathbf{Y}_{\mathcal{F}_{n-1}}}$, by [Fio20, Lemma 2.2.(2)]. Thus

$$\mathbf{y}_n = \mathfrak{g}_{\mathbf{Y}_{\mathbf{V}_n}}(\mathbf{y}_{n-1}).$$

Comparing projections of $\mathbf{y}, \mathbf{y}_N$: By construction, the points $\mathbf{y}, \mathbf{y}_N$ can have different projections to $\mathcal{T}^{\bullet}\mathbf{U}$ when $\mathbf{U} \in \mathcal{F}$. We now argue that elements of $\mathcal{F}$ are the only places where these projections can differ.

Claim 7. Let $N \geqslant 0$ and suppose that $\mathbf{U} \in \mathfrak{F}^{\bullet}$ satisfies $\mathbf{D_U}(\mathbf{y}, \mathbf{y}_N) > 0$. Then $\mathbf{U} \in \mathcal{F}$.

Proof. We argue by induction on N. In the base case, $N = 0$, we have $\mathbf{y} = \mathbf{y}_N$ and so the claim holds.

Hence fix $N \geqslant 1$ and suppose the claim holds for $N - 1$.

Assume that $\mathbf{U} \in \mathfrak{F}^\bullet - \mathcal{F}$. Then $\pi_{\mathbf{U}}(\mathbf{y}) = \mathbf{x}_{\mathbf{U}}$, by the definition of $\mathcal{F}$. Let $\mathbf{V} = \mathbf{V}_N$, so that $\mathcal{F}_N = \mathcal{F}_{N-1} \cup \{\mathbf{V}\}$. We saw above that $\mathbf{y}_N$ is the gate of $\mathbf{y}_{N-1}$ on $\mathbf{Y}_{\mathbf{V}}$. Also, by the induction hypothesis, $\pi_{\mathbf{U}}(\mathbf{y}_{N-1}) = \pi_{\mathbf{U}}(\mathbf{y})$, since $\mathbf{U} \notin \mathcal{F}$. Since $\mathbf{V} \in \mathcal{F}$, we have $\pi_{\mathbf{V}}(\mathbf{y}) \neq \mathbf{x}_{\mathbf{V}}$.

We now analyse four cases, according to how $\mathbf{U}$ and $\mathbf{V}$ are related:

- Suppose $\mathbf{U} \pitchfork \mathbf{V}$. Recall that $\pi_{\mathbf{U}}(\mathbf{y}) = \mathbf{x}_{\mathbf{U}}$. Suppose that $\mathbf{x}_{\mathbf{U}} \neq \rho^{\mathbf{V}}_{\mathbf{U}}$. Then by consistency, $\rho^{\mathbf{U}}_{\mathbf{V}} = \mathbf{x}_{\mathbf{V}} \neq \pi_{\mathbf{V}}(\mathbf{y})$. So the consistency of $\pi_{\mathbf{U}}(\mathbf{y})$ and $\pi_{\mathbf{V}}(\mathbf{y})$ is violated.

  Hence $\rho^{\mathbf{V}}_{\mathbf{U}} = \mathbf{x}_{\mathbf{U}}$. In other words, $\pi_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}}) = \mathbf{x}_{\mathbf{U}}$. So, we can choose $\mathbf{a} \in \mathbf{F}_{\mathbf{V}}$ such that $\mathbf{a} \in \mathbf{Y}_{\mathbf{V}}$ and $\pi_{\mathbf{U}}(\mathbf{a}) = \pi_{\mathbf{U}}(\mathbf{y}) = \pi_{\mathbf{U}}(\mathbf{y}_{N-1})$. Thus, by Lemma 5.20, the gate of $\mathbf{y}_{N-1}$ in $\mathbf{Y}_{\mathbf{V}}$ has $\mathbf{U}$–coordinate $\pi_{\mathbf{U}}(\mathbf{y})$. So $\pi_{\mathbf{U}}(\mathbf{y}) = \pi_{\mathbf{U}}(\mathbf{y}_N)$.
- Suppose $\mathbf{U} \subsetneq \mathbf{V}$. If $\pi_{\mathbf{V}}(\mathbf{y}_N)$ (which coincides with $\mathbf{x}_{\mathbf{V}}$ since $\mathbf{y}_N \in \mathbf{Y}_{\mathbf{V}}$) differs from $\rho^{\mathbf{U}}_{\mathbf{V}}$, then by consistency, we have $\pi_{\mathbf{U}}(\mathbf{y}_N) = \rho^{\mathbf{V}}_{\mathbf{U}}(\mathbf{x}_{\mathbf{V}}) = \mathbf{x}_{\mathbf{U}} = \pi_{\mathbf{U}}(\mathbf{y})$.

  Otherwise, $\pi_{\mathbf{V}}(\mathbf{y}_N) = \rho^{\mathbf{U}}_{\mathbf{V}}$. Then $\pi_{\mathbf{V}}(\mathbf{F}_{\mathbf{U}}) = \pi_{\mathbf{V}}(\mathbf{y}_N)$, so we can choose $\mathbf{a} \in \mathbf{F}_{\mathbf{U}}$ with $\pi_{\mathbf{U}}(\mathbf{a}) = \pi_{\mathbf{U}}(\mathbf{y}_{N-1})$ and $\mathbf{a} \in \mathbf{Y}_{\mathbf{V}}$. So, again by Lemma 5.20 and the induction hypothesis, $\pi_{\mathbf{U}}(\mathbf{y}_N) = \pi_{\mathbf{U}}(\mathbf{y}_{N-1}) = \pi_{\mathbf{U}}(\mathbf{y})$.
- Suppose $\mathbf{V} \subsetneq \mathbf{U}$. Recall that $\pi_{\mathbf{U}}(\mathbf{y}) = \pi_{\mathbf{U}}(\mathbf{y}_{N-1}) = \mathbf{x}_{\mathbf{U}}$. If $\mathbf{x}_{\mathbf{U}} \neq \rho^{\mathbf{V}}_{\mathbf{U}}$, then by consistency we have $\rho^{\mathbf{U}}_{\mathbf{V}}(\mathbf{x}_{\mathbf{U}}) = \mathbf{x}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{y})$, a contradiction. So $\rho^{\mathbf{V}}_{\mathbf{U}} = \mathbf{x}_{\mathbf{U}}$. Choose $\mathbf{a} \in \mathbf{F}_{\mathbf{V}}$ with $\pi_{\mathbf{V}}(\mathbf{a}) = \mathbf{x}_{\mathbf{V}}$. Then $\mathbf{a} \in \mathbf{Y}_{\mathbf{V}}$ and $\pi_{\mathbf{U}}(\mathbf{a}) = \pi_{\mathbf{U}}(\mathbf{y}) = \pi_{\mathbf{U}}(\mathbf{y}_{N-1})$, so by Lemma 5.20, $\pi_{\mathbf{U}}(\mathbf{y}_N) = \pi_{\mathbf{U}}(\mathbf{y})$.
- Suppose $\mathbf{U} \perp \mathbf{V}$. Then there exists $\mathbf{a} \in \mathbf{F}_{\mathbf{U}} \times \mathbf{F}_{\mathbf{V}}$ such that $\pi_{\mathbf{V}}(\mathbf{a}) = \mathbf{x}_{\mathbf{V}}$ and $\pi_{\mathbf{U}}(\mathbf{a}) = \mathbf{x}_{\mathbf{U}} = \pi_{\mathbf{U}}(\mathbf{y}_{N-1})$. So by Lemma 5.20, $\pi_{\mathbf{U}}(\mathbf{y}_N) = \pi_{\mathbf{U}}(\mathbf{y})$.

This completes the proof of the claim. □

Conclusion: Recall that $\mathbf{y}_N$ lies on the geodesic from $\mathbf{y}$ to $\mathbf{y}_n$ when $n \geqslant N$.

For any $\mathbf{U} \in \mathfrak{F}^\bullet$, one of the following holds. If $\mathbf{U} \in \mathcal{F}$, then for all sufficiently large n, we have $\mathbf{y}_n \in \mathbf{Y}_{\mathbf{U}}$, so $\pi_{\mathbf{U}}(\mathbf{y}_n) = \mathbf{x}_{\mathbf{U}}$. If $\mathbf{U} \notin \mathcal{F}$, then by the Claim, we have that $\pi_{\mathbf{U}}(\mathbf{y}_n) = \pi_{\mathbf{U}}(\mathbf{y}) = \mathbf{x}_{\mathbf{U}}$ for all n.

Since $\pi_{\mathbf{U}}$ takes geodesics to geodesics, $\mathbf{D}_{\mathbf{U}}(\mathbf{y}, \mathbf{y}_N) \leqslant \mathbf{D}_{\mathbf{U}}(\mathbf{y}, \mathbf{y}_n)$ for $n \geqslant N$. For sufficiently large n, we have $\pi_{\mathbf{U}}(\mathbf{y}_n) = \mathbf{x}_{\mathbf{U}}$. So $\mathbf{D}_{\mathbf{U}}(\mathbf{y}, \mathbf{y}_N) \leqslant \mathbf{D}_{\mathbf{U}}(\mathbf{y}, \mathbf{x}_{\mathbf{U}})$.

From the Claim, we thus get:

$$\mathsf{d}_1(\mathbf{y}, \mathbf{y}_N) \leqslant \sum_{\mathbf{U} \in \mathcal{F}} \mathbf{D}_{\mathbf{U}}(\mathbf{y}, \mathbf{x}_{\mathbf{U}}) = \mathsf{d}_1(\mathbf{y}, (\mathbf{x}_{\mathbf{U}})_{\mathbf{U}}) < \infty,$$

and $\mathsf{d}_1(\mathbf{y}, \mathbf{y}_N)$ is nondecreasing in N. So the sequence $\mathsf{d}_1(\mathbf{y}, \mathbf{y}_N)$ is convergent, and in particular $\mathsf{d}_1(\mathbf{y}_m, \mathbf{y}_n) \to 0$ as $m, n \to 0$.

Thus $(\mathbf{y}_n)_n$ is a Cauchy sequence, which must converge to some $\mathbf{x} \in \mathbf{X}$ by completeness of $\mathbf{X}$. Finally, we saw that the $\mathbf{y}$ and $\mathbf{y}_n$ projections to $\mathcal{T}^\bullet\mathbf{U}$ can only differ if $\mathbf{U} \in \mathcal{F}$, and for such $\mathbf{U}$, we have $\pi_{\mathbf{U}}(\mathbf{y}_n) = \mathbf{x}_{\mathbf{U}}$ for all sufficiently large n. Hence, by continuity of $\pi_{\mathbf{U}}$, we have $\pi_{\mathbf{U}}(\mathbf{x}) = \mathbf{x}_{\mathbf{U}}$ for each $\mathbf{U}$, as required. □

Lemma 5.21. *The real cubing* $(\mathbf{X}, \mathfrak{F}^\bullet)$ *has nonempty products.*

Proof. Let $\mathbf{U} \in \mathfrak{F}^\bullet$. Consider the parallelism class of subspaces $\mathbf{F}_{\mathbf{U}}$. By construction, $\mathbf{F}_{\mathbf{U}} \neq \varnothing$. Let $\mathbf{F}_{\mathbf{U}}$ be a fixed representative. Choose $\mathbf{x} \in \mathbf{F}_{\mathbf{U}}$. Let $\mathbf{V} \in \mathfrak{F}^\bullet$ satisfy $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$. Then by definition, $\pi_{\mathbf{V}}(\mathbf{F}_{\mathbf{U}}) = \pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$, which verifies Definition 4.9, i.e. the nonempty products property. □

This finishes the proof of Theorem 5.1. In summary, $(\mathbf{X}, \mathfrak{F}^\bullet)$ is an $\mathbb{R}$–cubing, where

- $\mathfrak{F}^\bullet \subset \mathfrak{F}^\bullet_1$ is the index set;

- $\mathcal{T}^\bullet\mathbf{U}$ is the metric quotient of the pseudometric space $(\mathbf{X}, s_{\mathbf{U}})$, the projection map $\pi_{\mathbf{U}}$ is the quotient map, and it is a median homomorphism;
- the nesting relation on $\mathfrak{F}^\bullet$ is the original one coming from the poset-colouring;
- $\mathbf{U}\perp\mathbf{V}$ if and only if, up to parallelism, $\mathbf{F_U}, \mathbf{F_V}$ span a convex product region (and dealing with orthogonality is why we restricted to $\mathcal{T}^\bullet$);
- if $\mathbf{U} \sqsubsetneq \mathbf{V}$ or $\mathbf{U}\pitchfork\mathbf{V}$, then $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{U}}(\mathbf{F_U})$;
- if $\mathbf{U} \sqsubsetneq \mathbf{V}$, then for all $\mathbf{x} \in \mathbf{X}$, either $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ or $\rho^{\mathbf{V}}_{\mathbf{U}}(\pi_{\mathbf{V}}(\mathbf{x})) = \pi_{\mathbf{U}}(\mathbf{x})$.

In particular, the subspaces $\mathbf{F_U}$ defined using the tangible filter $\sigma_{\mathbf{U}}$ coincides with the subspace $F_{\mathbf{U}}$ defined in terms of the real cubing structure in Section 4.10. Indeed, let $\mathbf{F_U}$ be the subspace defined in terms of $\sigma_{\mathbf{U}}$. As noted above, if $\mathbf{U} \sqsubsetneq \mathbf{V}$ or $\mathbf{U}\pitchfork\mathbf{V}$, then $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{F_U})$, so $\mathbf{F_U} \subset \mathbf{P_U}$. Moreover, if $\mathbf{V}\perp\mathbf{U}$, then since $\pi_{\mathbf{V}}$ factors through the gate map to $\mathbf{F_V}$, the definition of orthogonality implies that $\pi_{\mathbf{V}}$ is constant on $\mathbf{F_U}$. So, letting $\mathbf{P_U} = F_{\mathbf{U}} \times E_{\mathbf{U}}$ as in Proposition 4.37, we have a parallel copy $F_{\mathbf{U}}$ with $\mathbf{F_U} \subset F_{\mathbf{U}}$. On the other hand, if $p \in F_{\mathbf{U}}$, then the gate $\bar{p}$ on $\mathbf{F_U}$ has the same $\mathbf{V}$–coordinate as p whenever $\mathbf{V} \not\sqsubseteq \mathbf{U}$. On the other hand, if $\mathbf{V} \sqsubseteq \mathbf{U}$, then $\mathbf{F_V}$ is parallel to a subset of $\mathbf{F_U}$, so no wall separating $p, \bar{p}$ has colour $\mathbf{V}$. Hence $p = \bar{p}$ and $F_{\mathbf{U}} = \mathbf{F_U}$. So from now on we use the notation $\mathbf{F_U}$ for $F_{\mathbf{U}}$ in a real cubing since, if that real cubing happened to come from a poset-colouring (as we are about to show is always the case), there would be no ambiguity.

Remark 5.22. A given real cubing can admit many real cubing structures, and a given median space can admit many finite-depth tangible poset-colourings. This is clear even when the space is $\mathbb{R}$ — it has a trivial real cubing structure as a real tree, corresponding to a poset-colouring in which all walls have the same colour. On the other hand, by subdividing it into 1–cubes, we get a different real cubing structure from Example 4.25, corresponding to a poset-colouring with countably many colours. This is in contrast to Proposition 3.27, which says that among *orthogonal* poset-colourings, there is essentially a unique one. □

5.2. ☺ **Real cubings have finite depth tangible poset-colourings.** Conversely, we now provide a poset-colouring for the walls in a real cubing:

Proposition 5.23. *Any real cubing is a complete, connected median metric space of finite rank which admits a finite-depth poset-colouring that satisfies the tangible filter condition.*

Proof. Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be a real cubing. We invoke Lemma 4.5 and assume without loss of generality, that $\mathfrak{F}^\bullet$ has a unique $\sqsubseteq$–maximal element.

Moreover, for convenience, we apply Proposition 4.10 and henceforth assume that $(\mathbf{X}, \mathfrak{F}^\bullet)$ has nonempty products.

From Lemma 4.7 and Lemma 4.13, the real cubing $(\mathbf{X}, \mathfrak{F}^\bullet)$ is a connected median metric space of finite rank and by definition, it is complete.

Let $\mathcal{W}$ be the set of walls. For all $\mathbf{U} \in \mathfrak{F}^\bullet$, let $\mathbf{F_U}$ be the subspace defined (up to parallelism) in Proposition 4.37. Define $\mathfrak{F}^\bullet_p \subset \mathfrak{F}^\bullet$ to be the subset $\{\mathbf{U} \in \mathfrak{F}^\bullet \mid \mathsf{diam}(\mathbf{F_U}) > 0\}$.

Observe that if $\mathbf{U} \notin \mathfrak{F}^\bullet_p$, that is $\mathbf{F_U}$ is a point, and $\mathbf{V} \sqsubseteq \mathbf{U}$, then $\mathbf{V} \notin \mathfrak{F}^\bullet_p$ as $\mathbf{F_V}$ is (parallel to) a subspace of $\mathbf{F_U}$ by Proposition 4.37. Furthermore, if $\mathbf{U} \notin \mathfrak{F}^\bullet_p$, then there is no wall that crosses $\mathbf{F_U}$ as $w \cap \mathbf{F_U}$ and $W^* \cap \mathbf{F_U}$ can not be both non-empty.

Now, $\mathbf{X}$ is a point if and only if both $\mathfrak{F}^\bullet_p$ and $\mathcal{W}$ are empty. In this case, the poset-colouring map $Col : \mathcal{W} \to \{\mathbf{S}\}$ trivially satisfies the definition.

We now assume that $\mathbf{X}$ is not a point and so the $\sqsubseteq$-maximal $\mathbf{S} \in \mathfrak{F}^\bullet_p$ and $\mathcal{W} \neq \varnothing$.

We define the map:
$$Col : \mathcal{W} \to \mathfrak{F}^\bullet_p$$
where, for a wall $\hat{w} \in \mathcal{W}$, $Col(\hat{w})$ is the $\sqsubseteq$–minimal element $\mathbf{U}$ of $\mathfrak{F}^\bullet_p$ such that $\hat{w}$ crosses $\mathbf{F_U}$, that is, both halfpsaces w, w^* intersect $\mathbf{F_U}$.

We now show that the map is well-defined. First, every wall crosses some $\mathbf{F_U}$, for $\mathbf{U} \in \mathfrak{F}_p^\bullet$. Indeed, suppose that the wall $\hat{w}$ separates $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Since $\mathbf{x}, \mathbf{y} \in \mathbf{F_S}$, where $\mathbf{S}$ is the unique $\sqsubseteq$–maximal element, $\hat{w}$ crosses $\mathbf{F_S} = \mathbf{X}$.

We next show that the minimal element $\mathbf{U}$ is well-defined. For that, it suffices to show that if $\hat{w}$ crosses $\mathbf{F_U}$ and $\mathbf{F_V}$, then there exists $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$ such that $\hat{w}$ crosses $\mathbf{F_W}$. Then, from the fact that $\sqsubseteq$-chains are finite, the existence of a $\sqsubseteq$-minimal such $\mathbf{U}$ follows. Notice that without loss of generality, we can assume that $\mathbf{U} \pitchfork \mathbf{V}$. Indeed, since there is a wall which crosses both $\mathbf{F_U}$ and $\mathbf{F_V}$, $\mathbf{U}$ and $\mathbf{V}$ cannot be orthogonal, because a halfspace must intersect $\mathbf{F_U} \times \mathbf{F_V}$ in a convex set. If one of them is nested in the other one, say $\mathbf{U} \sqsubseteq \mathbf{V}$, then it suffices to take $\mathbf{W}$ to be $\mathbf{U}$. So assume $\mathbf{U} \pitchfork \mathbf{V}$.

If $\hat{w}$ crosses $\mathbf{F_U}$ and $\mathbf{F_V}$, then there exist $\mathbf{x}, \mathbf{y} \in \mathfrak{g}_{\mathbf{F_V}}(\mathbf{F_U})$ such that $\hat{w}$ separates them. Since $\rho^{\mathbf{V}}_{\mathbf{U}}$ is a point and $\mathbf{x}, \mathbf{y} \in \mathbf{F_V}$, we have that $\pi_{\mathbf{U}}(\mathbf{x}) = \pi_{\mathbf{U}}(\mathbf{y}) = \rho^{\mathbf{V}}_{\mathbf{U}}$; symmetrically, we have that $\pi_{\mathbf{V}}(\mathbf{x}) = \pi_{\mathbf{V}}(\mathbf{y}) = \rho^{\mathbf{U}}_{\mathbf{V}}$. Since $\mathbf{x} \neq \mathbf{y}$, by Lemma 4.6 there exists $\mathbf{W} \in \mathfrak{F}^\bullet$ such that $\pi_{\mathbf{W}}(\mathbf{x}) \neq \pi_{\mathbf{W}}(\mathbf{y})$. In particular $\mathbf{F_W}$ is not a point and so $\mathbf{W} \in \mathfrak{F}_p^\bullet$. Furthermore from consistency, we have that $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$ as required. Therefore, the map Col is well-defined. In both the existence and uniqueness proofs above, we have used that $\sqsubseteq$–chains in $\mathfrak{F}^\bullet$ have bounded length, by Definition 4.2.(4).

We next show that Col is a poset-colouring as in Definition 3.1.

Since $\mathfrak{F}^\bullet$ is an index set for a real cubing structure, $(\mathfrak{F}_p^\bullet, \sqsubseteq)$ is a partially ordered set and by assumption, it has a $\sqsubseteq$–maximal element.

Verifying Definition 3.1.(I): Assume that $\hat{u}$ separates $\hat{h}, \hat{v}$ and $Col(\hat{h}), Col(\hat{v}) \sqsubseteq \mathbf{U} \in \mathfrak{F}_p^\bullet$. From the definition of the map Col, we have that $\hat{h}$ and $\hat{v}$ cross $\mathbf{F_U}$. Since $\hat{u}$ separates $\hat{h}$ and $\hat{v}$, it follows that $\hat{u}$ crosses $\mathbf{F_U}$, in view of convexity of the latter. From the minimality of Col, we have that $Col(\hat{u}) \sqsubseteq \mathbf{U}$, by the same argument as was used to show that Col is well-defined.

Verifying Definition 3.1.(II): Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}_p^\bullet$ and suppose that each wall in $\mathcal{W}_{\mathbf{U}}$ crosses each wall in $\mathcal{W}_{\mathbf{V}}$, up to measure 0 sets of halfspaces. Then, from the definition of the poset-colouring, we have that each wall that crosses $\mathbf{F_U}$ crosses every wall that crosses $\mathbf{F_V}$ (a priori, up to measure 0 halfspaces). Since real cubings are connected median metric spaces of finite rank and $\mathbf{F_U}$ and $\mathbf{F_V}$ are closed, median–convex subspaces, it follows from Proposition 2.22, that there is an isometric embedding of $\mathbf{F_U} \times \mathbf{F_V}$ into $\mathbf{X}$ with median–convex image. Since by assumption $\mathbf{F_U}$ and $\mathbf{F_V}$ are not points, it follows Proposition 4.37 that $\mathbf{U}$ and $\mathbf{V}$ are not $\sqsubseteq$–comparable.

Verifying Definition 3.1.(III): Suppose we have an inseparable set $\mathcal{A}$ of walls such that $Col(\mathcal{A}) \sqsubseteq \mathbf{U}, \mathbf{V}$ and the set $\mathcal{H}_{\mathcal{A}}$ of halfspaces associated to $\mathcal{A}$ has positive fio–measure.

Let $D = \mathfrak{h}_{\mathbf{F_U}}(\mathbf{F_V})$. Then every wall in $\mathcal{A}$ crosses D, since by assumption any $\hat{a}$ in $\mathcal{A}$ has $Col(\hat{a})$ nested in $\mathbf{U}$ and $\mathbf{V}$ and hence crossing $\mathbf{F_U}$ and $\mathbf{F_V}$ since $\mathbf{F}_{Col(\hat{a})}$ is (up to parallelism) contained in $\mathbf{F_U}$ and $\mathbf{F_V}$.

Let $\mathbf{x}, \mathbf{y} \in D$ be such that $\mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_A$ has positive fio-measure (such $\mathbf{x}, \mathbf{y}$ exist because of how fio is defined in [Fio20, Section 3]).

For each such $\mathbf{x}, \mathbf{y}$, all elements of $Rel(\mathbf{x}, \mathbf{y}) = \{\mathbf{W} \in \mathfrak{F}^\bullet \mid \pi_{\mathbf{W}}(\mathbf{x}) \neq \pi_{\mathbf{W}}(\mathbf{y})\} \subset \mathfrak{F}_p^\bullet$ are nested in $\mathbf{U}$ and $\mathbf{V}$.

Consider the subset $MaxRel(\mathbf{x}, \mathbf{y})$ of $\sqsubseteq$–maximal elements of $Rel(\mathbf{x}, \mathbf{y})$. Notice that there are countably many elements in $Rel(\mathbf{x}, \mathbf{y})$, since the sum of the distances between the projections of $\mathbf{x}, \mathbf{y}$ into the real trees associated to elements in $Rel(\mathbf{x}, \mathbf{y})$ converges to the distance between $\mathbf{x}$ and $\mathbf{y}$. Hence, there are also countably many elements in $MaxRel(\mathbf{x}, \mathbf{y})$.

Moreover, the median interval $I(\mathbf{x}, \mathbf{y})$ is the union of closed convex subsets, each of which is a nontrivial median subinterval, and for each of which there is some $\mathbf{W} \in MaxRel(\mathbf{x}, \mathbf{y})$ so that the gate map to $\mathbf{F_W}$ is an isometric embedding on the given subinterval. (These subintervals may overlap.) Nontriviality means each such $\mathbf{W}$ has the property that $\mathcal{H}_{\mathbf{W}} \cap \mathcal{H}(\mathbf{x},\mathbf{y})$ has positive measure. Moreover, up to a null set, $\mathcal{H}(\mathbf{x},\mathbf{y})$ is contained in the union of the sets $\mathcal{H}_{\mathbf{W}} \cap \mathcal{H}(\mathbf{x},\mathbf{y})$. (Indeed, for each such $\mathbf{W}$ we have

$$\text{fio}(\mathcal{H}_{\mathbf{W}} \cap \mathcal{H}(\mathbf{x},\mathbf{y})) = \sum_{\mathbf{V} \sqsubseteq \mathbf{W}} \mathsf{d}_{\mathbf{V}}(\mathbf{x},\mathbf{y}),$$

so since $\mathsf{d}_{\mathbf{V}}(\mathbf{x},\mathbf{y}) = 0$ unless $\mathbf{V} \sqsubseteq \mathbf{W}$ for some such $\mathbf{W}$, we get

$$\text{fio}(\bigcup_{\mathbf{W} \in MaxRel(\mathbf{x},\mathbf{y})} \mathcal{H}_{\mathbf{W}} \cap \mathcal{H}(\mathbf{x},\mathbf{y})) = \sum_{\mathbf{V} \in Rel(\mathbf{x},\mathbf{y})} \mathsf{d}_{\mathbf{V}}(\mathbf{x},\mathbf{y}) = \mathsf{d}_{\mathbf{X}}(\mathbf{x},\mathbf{y}) = \text{fio}(\mathcal{H}(\mathbf{x},\mathbf{y}),$$

as required. The first equality follows from the inclusion-exclusion principle, countability of $Rel(\mathbf{x},\mathbf{y})$, and the fact that each $\mathbf{V}$ is nested in finitely many (and at least one) of the $\mathbf{W}$ by consistency.)

We define the family $\{\mathbf{W}_i\}_{i \in I}$ from Definition 3.1.(III) to be the set of elements $\mathbf{W}_i \in MaxRel(\mathbf{x},\mathbf{y})$ such that both

$$\text{fio}(\mathcal{H}(\mathbf{x},\mathbf{y}) \cap \mathcal{H}_{\mathbf{W}_i}) > 0$$

and

$$\text{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}_i}) > 0.$$

For the given $\mathbf{x},\mathbf{y}$, let $\mathcal{A}_{bad}(\mathbf{x},\mathbf{y})$ be the set of walls that separate $\mathbf{x},\mathbf{y}$, belong to $\mathcal{A}$, and do not cross one of the $\mathbf{F}_{\mathbf{W}_i}$ for $\mathbf{W}_i$ in our family. Since $\mathcal{A}_{bad}(\mathbf{x},\mathbf{y}) \cap \mathcal{W}_{\mathbf{T}}$ is associated to a set of halfspaces of measure 0 for each $\mathbf{T} \in MaxRel(\mathbf{x},\mathbf{y})$ and $MaxRel(\mathbf{x},\mathbf{y})$ is countable, we have that $\mathcal{H}_{\mathcal{A}_{bad}(\mathbf{x},\mathbf{y})}$ is a countable union of measure 0 sets, so it has measure 0.

Let $\mathcal{A}_{bad}$ be the set of walls in $\mathcal{A}$ that do not have colour nested in one of the $\mathbf{W}_i$ in our family. Then for any $\mathbf{x},\mathbf{y} \in D$, we have $\mathcal{A}_{bad} \cap \mathcal{W}(\mathbf{x},\mathbf{y}) = \mathcal{A}_{bad}(\mathbf{x},\mathbf{y})$. Now, from the definition of the measure fio (see [Fio20, p. 19]), this implies that $\mathcal{H}_{\mathcal{A}_{bad}}$ has measure 0. Indeed, $\mathcal{H}_{\mathcal{A}_{bad}}$ has positive measure only if it has positive-measure intersection with some set of halfspaces of the form $\mathcal{H}(\mathbf{x},\mathbf{y})$, and since all walls in $\mathcal{A}$ cross D, we can restrict to points $\mathbf{x},\mathbf{y}$ in D. This shows that Condition (III) is satisfied.

Verifying Definition 3.1.(IV): Suppose that there exist inseparable sets $\mathcal{A},\mathcal{B}$ of walls such that $Col(\mathcal{A}) \sqsubseteq \mathbf{U}, Col(\mathcal{B}) \sqsubseteq \mathbf{V}$ and every wall in $\mathcal{A}$ crosses every wall in $\mathcal{B}$. Suppose that the sets $\mathcal{H}_{\mathcal{A}},\mathcal{H}_{\mathcal{B}}$ of halfspaces associated to $\mathcal{A}$ and $\mathcal{B}$ have positive fio–measure.

Consider families $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_j\}$ satisfying Definition 3.1.(III) for $\mathcal{A},\mathbf{U}$ and $\mathcal{B},\mathbf{V}$ respectively.

Specifically, each $\mathbf{U}_i \sqsubseteq \mathbf{U}$, each $\mathbf{U}_i$ satisfies $\text{fio}(\mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}_{\mathcal{A}}) > 0$, and all but a measure 0 subset of $\mathcal{H}_{\mathcal{A}}$ is contained in $\cup_i \mathcal{H}_{\mathbf{U}_i}$. Analogous properties hold for $\mathcal{B},\mathbf{V},\{\mathbf{V}_i\}$.

Among sets $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_j\}$ with the given properties, choose these sets so that the levels of the $\mathbf{U}_i,\mathbf{V}_j$ are as small as possible; more precisely, suppose that $\{\mathbf{U}_i\}$ cannot be replaced by a set with the preceding properties by replacing some $\mathbf{U}_j$ by a set $\{\mathbf{U}'_j\}$ with each $\mathbf{U}'_j \sqsubsetneq \mathbf{U}_j$; such a choice is possible by finite complexity.

Note that the first, second, fourth, and fifth bullet points from Definition 3.1.(IV) are satisfied by these sets.

We will show that $\mathbf{U}_i \perp \mathbf{V}_j$ for all i,j. This then provides a product region $\mathbf{F}_{\mathbf{U}_i} \times \mathbf{F}_{\mathbf{U}_j}$, in view of Proposition 4.37, which in turn implies that all walls crossing the first factor — i.e. all walls in $\mathcal{W}_{\mathbf{U}_i}$ — cross all walls crossing the second factor — i.e. all walls in $\mathcal{W}_{\mathbf{V}_j}$. Hence, to verify the third bullet point in Definition 3.1.(IV), it indeed suffices to those that $\mathbf{U}_i,\mathbf{V}_j$ are orthogonal.

Fix $i \in I, j \in J$. Since $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i}$ has positive fio–measure, there exist $\mathbf{x},\mathbf{y} \in \mathbf{X}$ such that

$$\text{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{x},\mathbf{y})) > 0.$$

We call $\mathbf{x}, \mathbf{y}$ a pair of *test points* for $\mathbf{U}_i$ and $\mathcal{A}$. As explained in [Fio20, Section 3], a set of halfspaces has positive measure only if it has positive-measure intersection with $\mathcal{H}(\mathbf{x}, \mathbf{y})$ for some $\mathbf{x}, \mathbf{y}$. So test points exist. Likewise, we can choose $\mathbf{w}, \mathbf{z} \in \mathbf{X}$ such that

$$\mathrm{fio}(\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j} \cap \mathcal{H}(\mathbf{w}, \mathbf{z})) > 0.$$

Moreover, by minimality of the levels of $\mathbf{U}_i, \mathbf{V}_j$, we can assume that $\mathbf{U}_i \in Rel(\mathbf{x}, \mathbf{y})$ and $\mathbf{V}_j \in Rel(\mathbf{w}, \mathbf{z})$. Indeed, if not, then for all pairs $\mathbf{x}, \mathbf{y}$ of test points for $\mathbf{U}_i$, we have $\pi_{\mathbf{U}_i}(\mathbf{x}) = \pi_{\mathbf{U}_i}(\mathbf{y})$, and by taking gates on $\mathbf{F}_{\mathbf{U}_i}$, we can assume that every element of $Rel(\mathbf{x}, \mathbf{y})$ is properly nested in $\mathbf{U}_i$.

It follows that $\mathbf{F}_{\mathbf{U}_i} \cap I(\mathbf{x}, \mathbf{y})$ is contained in the union of subsets obtained by projecting $\mathbf{F}_{\mathbf{W}}$ to $I(\mathbf{x}, \mathbf{y})$, where $\mathbf{W} \subsetneq \mathbf{U}_i$ is one of the countably many elements of $Rel(\mathbf{x}, \mathbf{y})$. Now, if $\hat{w} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$ is a wall crossing $\mathbf{F}_{\mathbf{U}_i}$, then (after excluding a set of walls associated to a measure–0 set of halfspaces) $\hat{w}$ must cross some such $\mathbf{F}_{\mathbf{W}}$. So there is a set $T(\mathbf{x}, \mathbf{y})$ of $\mathbf{W} \subsetneq \mathbf{U}_i$ such that $\mathrm{fio}(\mathcal{H}_{\mathbf{W}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_{\mathcal{A}}) > 0$ for $\mathbf{W} \in T(\mathbf{x}, \mathbf{y})$ and $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ is contained in the union of the $\mathcal{H}_{\mathbf{W}}$.

Now replace $\mathbf{U}_i$ by the union of the $T(\mathbf{x}, \mathbf{y})$ as $\mathbf{x}, \mathbf{y}$ vary over the test points for $\mathbf{U}_i, \mathcal{A}$. This yields a new family $\{\mathbf{U}_i'\}$ with the same properties mentioned above, but where $\mathbf{U}_i$ has been replaced by lower-level elements, contradicting our minimal level choice.

Hence we can assume that, for our given $i \in I, j \in J$ and test pairs $\mathbf{x}, \mathbf{y}$ and $\mathbf{w}, \mathbf{z}$ for $\mathbf{U}_i, \mathcal{A}$ and $\mathbf{V}_j, \mathcal{B}$ respectively, we have $\mathbf{U}_i \in Rel(\mathbf{x}, \mathbf{y})$ and $\mathbf{V}_j \in Rel(\mathbf{w}, \mathbf{z})$.

Since all walls in $\mathcal{A}$ cross all walls in $\mathcal{B}$, we thus have $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbf{X}$ such that

- $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, in that order, form a median rectangle in $\mathbf{X}$, and
- $\mathcal{H}(\{\mathbf{a}, \mathbf{d}\}, \{\mathbf{b}, \mathbf{c}\})$ contains $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ and $\mathbf{U}_i \in Rel(\mathbf{a}, \mathbf{b})$, and
- $\mathcal{H}(\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{c}, \mathbf{d}\})$ contains $\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j} \cap \mathcal{H}(\mathbf{w}, \mathbf{z})$ and $\mathbf{U}_i \in Rel(\mathbf{a}, \mathbf{d})$.

Indeed, we can take $\mathbf{x} = \mathbf{a}, \mathbf{y} = \mathbf{b}$, and then obtain $\mathbf{d}, \mathbf{c}$ by moving $\mathbf{a}, \mathbf{b}$, respectively, across the walls in $\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j} \cap \mathcal{H}(\mathbf{w}, \mathbf{z})$. If necessary, we use [CDH10, Remark 2.23.(2)] to tighten to a median rectangle. See Figure 12.

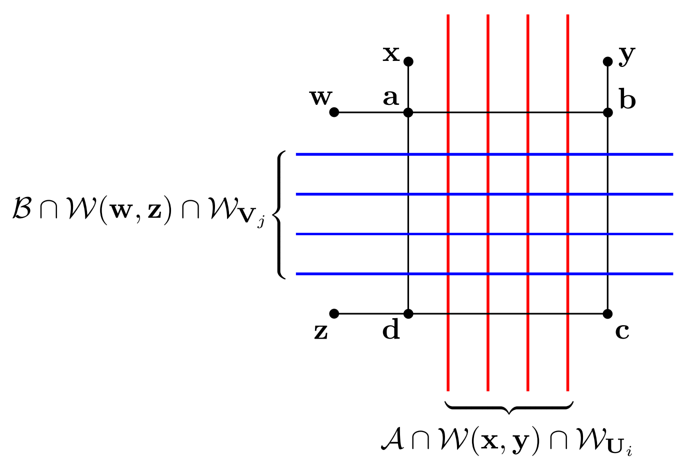

FIGURE 12. Verifying Definition 3.1.(IV) in a real cubing.

Now, by applying consistency to the median rectangle, exactly as in the proof Lemma 4.13, we get that $\mathbf{U}_i \perp \mathbf{V}_j$.

This completes the verification of Definition 3.1.(IV).

Finite depth and tangibility: We have shown that Col is a poset-colouring. Since the index set has finite depth so does the poset-colouring. We are left to check that it satisfies the tangible condition.

Observe that if $\hat{w}$ is a wall crossing $\mathbf{F_U}$, then by definition $Col(\hat{w}) \sqsubseteq \mathbf{U}$ so $\hat{w} \in \mathcal{W}_\mathbf{U}$. Conversely, if $Col(\hat{w}) \sqsubseteq \mathbf{U}$, then $\mathbf{F}_{Col(\hat{w})}$ is, up to parallelism, contained in $\mathbf{F_U}$, so $\hat{w}$ crosses $\mathbf{F_U}$. Thus $\mathcal{W}_\mathbf{U}$ is exactly the set of walls crossing $\mathbf{F_U}$.

By construction, $\sigma_\mathbf{U}$ is therefore the set of halfspaces containing $\mathbf{F_U}$, which is tangible since, for any choice of basepoint $\mathbf{x}_0 \in \mathbf{X}$), we have

$$\mathrm{fio}(\sigma_\mathbf{U} \triangle \sigma_{\mathbf{x}_0}) = \mathbf{d}_1(\mathbf{x}_0, \mathbf{F_U}) < \infty,$$

where $\mathbf{d}_1$ is the distance in $\mathbf{X}$, as needed. □

Combining Proposition 5.23 with Theorem 5.1 gives:

Corollary 5.24 (Characterisation of real cubing among median metric spaces)**.** *A complete, connected median metric space $(\mathbf{X}, d_1, \boldsymbol{\mu})$ of finite rank admits a finite depth poset-colouring which satisfies the tangible filter condition if and only if there is a real cubing $(\mathbf{Y}, \mathfrak{F}^\bullet)$ and a median-preserving isometry $\mathbf{X} \to \mathbf{Y}$, where $\mathbf{Y}$ is equipped with the metric from Definition 4.2 and the median from Lemma 4.7.*

Example 5.25. In Example 4.27 we discussed that the trees of flats $\widetilde{X}$ and $\widetilde{X'}$ are complete, connected median spaces of finite rank and the orthogonal poset-colourings satisfy the tangible condition. What are the real trees associated to the real cubing structures described in Theorem 5.1, applied to the orthogonal poset-colourings? □

6. ☺ Characterisation of real cubings with clean containers and wedges among median spaces

We have characterised real cubings as complete, connected, finite-rank median spaces whose walls admit a finite-depth poset-colouring satisfying the tangible filter property.

On the other hand, in Section 3.3, we have proven that the orthogonal poset-colouring always exists for median spaces. In this section, we prove that the finite depth of the canonical orthogonal poset-colouring characterises complete, connected median space of finite rank that admit a real cubing structure with clean containers and wedges.

6.1. ☺ An index set where $^\perp$ is an involution.

In this section, we show that if a real cubing has an index set with clean containers and wedges, then the real cubing admits a different real cubing structure where the index is an orthogonal set in the sense of Definition 3.7. Furthermore, the poset-colouring defined by the orthogonal index set is a finite depth tangible orthogonal poset-colouring.

First, there is a relationship between clean containers and the *orthogonality determined nesting* condition in Definition 3.7 that can be used to check whether the latter condition holds.

In this section, given a real cubing $(\mathbf{X}, \mathfrak{F}^\bullet)$ with clean containers, we denote by $\mathbf{U}^\perp$ the orthogonal complement of $\mathbf{U}$, which is, by definition, defined whenever there is some $\mathbf{V}$ with $\mathbf{V} \perp \mathbf{U}$.

Proposition 6.1 (Double orthogonal)**.** *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be a real cubing such that $\mathfrak{F}^\bullet$ has clean containers. Assume that $\mathfrak{F}^\bullet$ has a unique $\sqsubseteq$–maximal element $\mathbf{S}$.*

Then the following are equivalent:

- ***Complementation is an involution**. For all $\mathbf{U} \in \mathfrak{F}^\bullet - \{\mathbf{S}\}$, we have that $\mathbf{U}^\perp$ is defined and $\mathbf{U}^{\perp\perp} = \mathbf{U}$.*
- ***Orthogonality determines nesting**. For all $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet - \{\mathbf{S}\}$, we have $\mathbf{U} \subsetneq \mathbf{V}$ if and only if $\mathbf{V}^\perp \subsetneq \mathbf{U}^\perp$.*

Proof. Suppose that $\mathbf{U}^{\perp\perp} = \mathbf{U}$ for all $\mathbf{U} \sqsubsetneq \mathbf{S}$. Suppose that $\mathbf{U} \sqsubsetneq \mathbf{V} \sqsubsetneq \mathbf{S}$. Then $\mathbf{V}^{\perp} \sqsubsetneq \mathbf{U}^{\perp}$. The properness of the nesting follows since, if $\mathbf{V}^{\perp} = \mathbf{U}^{\perp}$, then, taking orthogonal complements and applying the assumption, we would have $\mathbf{U} = \mathbf{V}$. Conversely, suppose that $\mathbf{V}^{\perp} \sqsubsetneq \mathbf{U}^{\perp}$. Take orthogonal complements to get $\mathbf{U} \sqsubsetneq \mathbf{V}$.

Now suppose that orthogonality determines nesting. Just from clean containers, we already have for all $\mathbf{U}$ that $\mathbf{U} \sqsubseteq \mathbf{U}^{\perp\perp}$. But since $\mathbf{U}^{\perp} = \mathbf{U}^{\perp\perp\perp}$ (again, just from clean containers), if the nesting was proper, we would have from our assumption that $\mathbf{U}^{\perp} \sqsubsetneq \mathbf{U}^{\perp}$, which is impossible. Hence $\mathbf{U} = \mathbf{U}^{\perp\perp}$. (See also Lemma 3.8.) □

Remark 6.2 (Double orthogonal in hierarchically hyperbolic spaces)**.** In this remark (which the reader not interested in later applications to hierarchically hyperbolic spaces should skip), we look ahead to the definition of a hierarchically hyperbolic space, Definition 10.1, where one can define the clean containers property for the HHS nesting and orthogonality relations exactly as for real cubings (see Definition 35.3), and likewise define the "orthogonality determines nesting" property in exactly the same way as for real cubings. If these properties hold, then Proposition 6.1 holds also — the proof used only the relations on the index set, so works as well in the hierarchically hyperbolic setting as it does in the real cubing setting. In some cases (see e.g. [HS20, Corollary 3.4]), naturally occurring hierarchically hyperbolic structures have the property that the orthogonal complementation operator is an involution.

However, there are naturally occurring hierarchically hyperbolic structures *with clean containers* where orthogonal complementation is not an involution and so orthogonality does not determine nesting. This is the case for the standard HHS structure on the mapping class group of a surface S; see Section 36.1 for a description. Figure 13 shows a surface S and two subsurfaces U, V such that $U \sqsubsetneq (U^{\perp})^{\perp}$, because we disregard pants. Later, when we

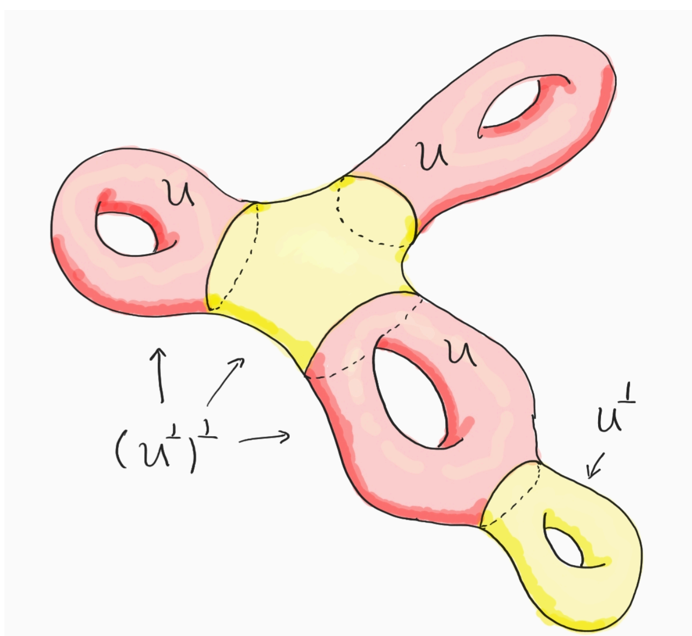

FIGURE 13. The subsurface U consists of two (red) open one-holed tori and a (red) open two-holed torus, and their boundary annuli. Its topological complement is the yellow (open) subsurface, but since we disregard pants, $U^{\perp}$ is just the yellow once-punctured torus. So $U \sqsubsetneq (U^{\perp})^{\perp}$.

work with HHSes, we often assume clean containers but not that orthogonality determines nesting. □

We also need an auxiliary lemma to allow us to assume our real cubings have nonempty products.

Lemma 6.3. *Let* $(\mathbf{X}, \mathfrak{F}^\bullet)$ *be a real cubing and suppose that* $\mathfrak{F}^\bullet$ *has clean containers. Let* $\mathfrak{F}_1^\bullet \subset \mathfrak{F}^\bullet$ *be the set of* $\mathbf{U}$ *such that* $\mathbf{P_U} \neq \varnothing$. *Then* $(\mathbf{X}, \mathfrak{F}_1^\bullet)$ *is a real cubing with clean containers.*

Proof. By Proposition 4.10, $(\mathbf{X}, \mathfrak{F}_1^\bullet)$ is a real cubing. So to conclude, it suffices to show that $\mathbf{P_U} \neq \varnothing$ implies $\mathbf{P}_{\mathbf{U}^\perp} \neq \varnothing$ when $\mathbf{U}^\perp$ is defined, i.e. when $\mathbf{U}$ is orthogonal to at least one element. But this follows since the latter must contain the former. Indeed, if $\mathbf{x} \in P_\mathbf{U}$, then $\pi_\mathbf{V}(\mathbf{x}) = \rho_\mathbf{V}^\mathbf{U}$ whenever $\mathbf{U} \subsetneq \mathbf{V}$ or $\mathbf{U} \pitchfork \mathbf{V}$. Suppose that $\mathbf{V} \pitchfork \mathbf{U}^\perp$ or $\mathbf{U}^\perp \subsetneq \mathbf{V}$. Then we cannot have $\mathbf{V} \perp \mathbf{U}$ (which would force $\mathbf{V} \sqsubseteq \mathbf{U}^\perp$) and we cannot have $\mathbf{V} \sqsubseteq \mathbf{U}$ (which would force $\mathbf{V} \perp \mathbf{U}^\perp$). So $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$. So $\pi_\mathbf{V}(\mathbf{x}) = \rho_\mathbf{V}^\mathbf{U} = \rho_\mathbf{V}^{\mathbf{U}^\perp}$, with the latter equality coming from Definition 4.2.(3). Hence $\mathbf{x} \in \mathbf{P}_{\mathbf{U}^\perp}$, as required. □

Definition 6.4 (Isoorthogonal)**.** Suppose that $\mathbf{U}, \mathbf{V}$ have the property that, for all $\mathbf{W} \in \mathfrak{F}^\bullet$, we have $\mathbf{U} \perp \mathbf{W}$ if and only if $\mathbf{V} \perp \mathbf{U}$. Then we say that $\mathbf{U}, \mathbf{V}$ are *isoorthogonal.* □

Remark 6.5. Notice that if the index set $\mathfrak{F}^\bullet$ has clean containers, then two elements $\mathbf{U}$ and $\mathbf{V}$ are isoorthogonal if and only if either $\mathbf{U}^\perp = \mathbf{V}^\perp$ or each of $\mathbf{U}$ and $\mathbf{V}$ is not orthogonal to any element of the index set.

Indeed, suppose that $\mathbf{U}$ and $\mathbf{V}$ are isoorthogonal and they are orthogonal to an element $\mathbf{W}$. Then $\mathbf{U}^\perp$ is orthogonal to $\mathbf{V}$, so $\mathbf{U}^\perp$ is nested in $\mathbf{V}^\perp$, and symmetrically $\mathbf{V}^\perp$ is nested in $\mathbf{U}^\perp$, so they are equal. Conversely, if $\mathbf{U}^\perp = \mathbf{V}^\perp$, then $\mathbf{W} \perp \mathbf{U}$ implies $\mathbf{W} \sqsubseteq \mathbf{U}^\perp = \mathbf{V}^\perp$ which implies $\mathbf{W} \perp \mathbf{V}$, and conversely. □

The main proposition is:

Proposition 6.6. *Let* $(\mathbf{X}, \mathfrak{F}^\bullet)$ *be a real cubing with clean containers. Then* $\mathbf{X}$ *admits a real cubing structure* $(\mathbf{X}, \mathfrak{F}_r^\bullet)$ *such that*

- $(\mathbf{X}, \mathfrak{F}_r^\bullet)$ *has nonempty products;*
- *there is a unique* $\sqsubseteq$*–maximal element* $\mathbf{S}$*;*
- *no two distinct elements of* $\mathfrak{F}_r^\bullet$ *are isoorthogonal;*
- $\mathbf{U} = \mathbf{U}^{\perp\perp}$, *and* $\mathbf{U}^\perp$ *is defined, for all* $\mathbf{U} \neq \mathbf{S}$*;*
- *for all* $\mathbf{U}, \mathbf{V} \subsetneq \mathbf{S}$, *we have* $\mathbf{U} \subsetneq \mathbf{V}$ *if and only if* $\mathbf{V}^\perp \subsetneq \mathbf{U}^\perp$.

In particular, every $\mathbf{U} \in \mathfrak{F}_r^\bullet$ *is orthogonal to some element* $\mathbf{U}^\perp$, *unless* $\mathbf{U} = \mathbf{S}$.

Finally, if $\mathfrak{F}^\bullet$ *has wedges, then so does* $\mathfrak{F}_r^\bullet$.

Proof. By Lemma 4.5, we assume without loss of generality that the index set $\mathfrak{F}^\bullet$ has a unique maximal element, say $\mathbf{S}$.

We also invoke Lemma 6.3 and assume that $(\mathbf{X}, \mathfrak{F}^\bullet)$ has nonempty products; in particular, Lemma 4.12 (ρ–consistency) applies.

We define $\mathfrak{F}_r^\bullet \subset \mathfrak{F}^\bullet$ to be the subset of elements of the form $\mathbf{U}^{\perp\perp}$ together with the maximal element $\mathbf{S}$. The relations in $\mathfrak{F}_r^\bullet$ are induced from the relations in $\mathfrak{F}^\bullet$.

Notice that $\mathbf{U}$ and $\mathbf{U}^{\perp\perp}$ are isoorthogonal. Furthermore, $\mathbf{U}^{\perp\perp}$ is the unique $\sqsubseteq$-maximal element in the isoorthogonality class of $\mathbf{U}$, and all the elements of the class are nested in $\mathbf{U}^{\perp\perp}$. Indeed, if $\mathbf{V}$ and $\mathbf{U}$ are isoorthogonal in $\mathfrak{F}^\bullet$, then, from Remark 6.5 we have that $\mathbf{U}^\perp = \mathbf{V}^\perp$ and so $\mathbf{U}^{\perp\perp} = \mathbf{V}^{\perp\perp}$. Furthermore, since $\mathbf{V} \sqsubseteq \mathbf{V}^{\perp\perp}$, we have that $\mathbf{V} \sqsubseteq \mathbf{U}^{\perp\perp}$.

Our goal is to prove that $\mathbf{X}$ admits a real cubing structure with index set $\mathfrak{F}_r^\bullet$, with the given relations.

From Proposition 5.23, we have that $\mathbf{X}$ admits a poset-colouring

$$Col : \mathcal{W} \to \mathfrak{F}_p^\bullet,$$

where $\mathfrak{F}_p^\bullet$ is the set of $\mathbf{U}$ for which $\mathbf{F_U}$ is not a single point.

We define the map:

$$Col_r : \mathcal{W} \to \mathfrak{F}_r^\bullet \cap \mathfrak{F}_p^\bullet$$

as $Col_r(\hat{w}) = Col(\hat{w})^{\perp\perp}$ if $Col(\hat{w})^\perp$ is defined, and $Col(\hat{w}) = \mathbf{S}$ if $Col(\hat{w})$ is isoorthogonal to $\mathbf{S}$. To see that this is well-defined, note that if $\mathbf{U} \in \mathfrak{F}_p^\bullet$, then $\mathbf{F}_{\mathbf{U}^{\perp\perp}}$, which contains a parallel copy of $\mathbf{F_U}$, cannot be a point, and so $\mathbf{U}^{\perp\perp} \in \mathfrak{F}_p^\bullet$.

We need the following lemma:

Lemma 6.7. *In the notation above, Col_r is a poset-colouring that satisfies the tangible filter condition and it has finite depth.*

Proof. Let us begin with an observation. Suppose $\mathbf{U}^\perp$ is defined, i.e. $\mathbf{U}$ is not isoorthogonal to $\mathbf{S}$.

Let $\hat{w} \in \mathcal{W}_{\mathbf{U}^{\perp\perp}}$. Then by definition, $Col(\hat{w}) \sqsubseteq \mathbf{U}^{\perp\perp}$. By definition of Col_r we have that $Col_r(\hat{w}) = Col(\hat{w})^{\perp\perp}$. As a consequence of the fact that $\mathbf{U}^{\perp\perp\perp} = \mathbf{U}^\perp$, we deduce that $Col(\hat{w})^{\perp\perp} \sqsubseteq \mathbf{U}^{\perp\perp}$. Indeed, since $Col(\hat{w}) \sqsubseteq \mathbf{U}^{\perp\perp}$, it follows that $\mathbf{U}^{\perp\perp\perp} = \mathbf{U}^\perp \sqsubseteq Col(\hat{w})^\perp$ and so $Col(\hat{w})^{\perp\perp} \sqsubseteq \mathbf{U}^{\perp\perp}$. It follows that the sets $\mathcal{W}_{\mathbf{U}^{\perp\perp}}$ coincide for the two maps Col and Col_r.

Our goal is to show that Col_r is a poset-colouring, see Definition 3.1. In order to do so, we use the fact that Col is a poset-colouring and the observation that the sets $\mathcal{W}_{\mathbf{U}^{\perp\perp}}$ coincide for the two maps.

Item (I) holds since the set $\mathcal{W}_{\mathbf{U}^{\perp\perp}}$ for Col_r is the same as for Col, and since Col is a poset-colouring, then that set is inseparable.

Item (II) is an immediate consequence from the fact that $\mathfrak{F}_r^\bullet$ is a subset of $\mathfrak{F}^\bullet$ and the relations are induced. Indeed if $\mathbf{U}^{\perp\perp}, \mathbf{V}^{\perp\perp} \in \mathfrak{F}_r^\bullet$ satisfy the conditions of Item II, they do so as elements of $\mathfrak{F}^\bullet$ and since Col is a poset-colouring, it follows that $\mathbf{U}$ and $\mathbf{V}$ are $\sqsubseteq$-incomparable in $\mathfrak{F}^\bullet$ and so in $\mathfrak{F}_r^\bullet$.

Let us now prove Item (III). Let $\mathbf{U}^{\perp\perp}, \mathbf{V}^{\perp\perp} \in \mathfrak{F}_r^\bullet \subset \mathfrak{F}^\bullet$ satisfy the hypothesis with respect to some $\mathcal{A}$ as in Item (III).

Let $\{\mathbf{W}_i\}$ be the family of elements of $\mathfrak{F}^\bullet$ provided by Item (III) for the poset-colouring Col. We claim that the family $\{\mathbf{W}_i^{\perp\perp}\}$ of elements of $\mathfrak{F}_r^\bullet$ satisfies the requirements. The elements $\mathbf{W}_i^{\perp\perp}$ exist since $\mathbf{W}_i \sqsubseteq U^{\perp\perp}$, so $\mathbf{W}_i \perp \mathbf{U}^\perp$, so $\mathbf{W}_i^\perp$ and hence $\mathbf{W}_i^{\perp\perp}$ is defined.

Indeed, since $\mathbf{W}_i \sqsubseteq \mathbf{W}_i^{\perp\perp}$, we have that $\mathcal{H}_{\mathbf{W}_i} \subset \mathcal{H}_{\mathbf{W}_i^{\perp\perp}}$. Then since $\mathrm{fio}(\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{H}_\mathcal{A}) > 0$ by properties of the family $\{\mathbf{W}_i\}$, we have that $\mathrm{fio}(\mathcal{H}_{\mathbf{W}_i^{\perp\perp}} \cap \mathcal{H}_\mathcal{A}) > 0$. Similarly, since $\mathrm{fio}\,(\mathcal{H}_\mathcal{A} - \bigcup_i (\mathcal{H}_{\mathbf{W}_i} \cap \mathcal{H}_\mathcal{A})) = 0$, we have that $\mathrm{fio}\left(\mathcal{H}_\mathcal{A} - \bigcup_i (\mathcal{H}_{\mathbf{W}_i^{\perp\perp}} \cap \mathcal{H}_\mathcal{A})\right) = 0$.

Since $\mathbf{W}_i \sqsubseteq \mathbf{U}^{\perp\perp}, \mathbf{V}^{\perp\perp}$, as shown above, we have that $\mathbf{W}_i^{\perp\perp} \sqsubseteq \mathbf{U}^{\perp\perp}, \mathbf{V}^{\perp\perp}$.

This proves that Item (III) holds.

Item (IV) is proven analogously - one considers the families $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_i\}$ provided by the poset-colouring Col and checks that the families $\{\mathbf{U}_i^{\perp\perp}\}$ and $\{\mathbf{V}_i^{\perp\perp}\}$ satisfy the required conditions.

This shows that Col_r is indeed a poset-colouring, and since $\mathfrak{F}^\bullet$ (and so $\mathfrak{F}_r^\bullet$) has finite depth, so does Col_r.

We are left to check that it satisfies the tangible condition. Observe that if $\hat{w}$ is a wall crossing $\mathbf{F_U}$, then by definition $Col(\hat{w}) \sqsubseteq \mathbf{U}$ so $\hat{w} \in \mathcal{W}_\mathbf{U}$. Conversely, if $Col(\hat{w}) \sqsubseteq \mathbf{U}$, then $\mathbf{F}_{Col(\hat{w})}$ is, up to parallelism, contained in $\mathbf{F_U}$, so $\hat{w}$ crosses $\mathbf{F_U}$. Thus $\mathcal{W}_\mathbf{U}$ is exactly the set of walls crossing $\mathbf{F_U}$. From the first observation, $\mathcal{W}_{\mathbf{U}^{\perp\perp}}$ coincide for Col and Col_r and so $\mathcal{W}_{\mathbf{U}^{\perp\perp}}$ is exactly the set of walls crossing $\mathbf{F}_{\mathbf{U}^{\perp\perp}}$.

By construction, $\sigma_{\mathbf{U}^{\perp\perp}}$ is therefore the set of halfspaces containing $\mathbf{F}_{\mathbf{U}^{\perp\perp}}$, which is tangible since, for any choice of basepoint $\mathbf{x}_0 \in \mathbf{X}$, we have

$$\mathrm{fio}(\sigma_{\mathbf{U}^{\perp\perp}} \triangle \sigma_{\mathbf{x}_0}) = \mathbf{d}_1(\mathbf{x}_0, \mathbf{F}_{\mathbf{U}^{\perp\perp}}) < \infty,$$

where $\mathbf{d}_1$ is the distance in $\mathbf{X}$, as needed.

This finishes the proof that Col_r is a poset-colouring of finite depth satisfying the tangible filter condition. □

We now proceed with the proof of Proposition 6.6.

We apply Theorem 5.1 with the new poset-colouring Col_r and give $\mathbf{X}$ a real cubing structure with index set $\mathfrak{F}^\bullet_r$. Notice that $\mathfrak{F}^\bullet_r$ has clean containers because $(\mathbf{U}^{\perp\perp})^\perp = (\mathbf{U}^\perp)^{\perp\perp} \in \mathfrak{F}^\bullet_r$.

By construction, there is a unique $\sqsubseteq$–maximal $\mathbf{S}$, every other element $\mathbf{U}$ has a well-defined orthogonal complement and satisfies $\mathbf{U}^{\perp\perp} = \mathbf{U}$, and so by Proposition 6.1, orthogonality determines nesting.

Nonempty products holds because it held for $\mathfrak{F}^\bullet$. More precisely, the product region $\mathbf{P}_{\mathbf{U}^{\perp\perp}}$ for the new real cubing structure coincides with the product region for the old real cubing structure, which was nonempty by assumption.

Finally, the wedge property persists since, in $\mathfrak{F}^\bullet$, we have

$$\mathbf{U}^{\perp\perp} \wedge \mathbf{V}^{\perp\perp} = (\mathbf{U}^\perp \vee \mathbf{V}^\perp)^\perp = (\mathbf{U} \wedge \mathbf{V})^{\perp\perp}.$$

This concludes the proof. □

Remark 6.8 (Real trees got blown up)**.** The proof of Theorem 5.1 shows how the real trees change when passing from $(\mathbf{X}, \mathfrak{F}^\bullet)$ to $(\mathbf{X}, \mathfrak{F}^\bullet_r)$ in the preceding proof. For each $\mathbf{U}^{\perp\perp}$, the new real tree is obtained from the original real tree by "blowing up" the point $\rho^{\mathbf{V}}_{\mathbf{U}^{\perp\perp}}$ to a subtree (isometric to $\mathcal{T}^\bullet\mathbf{V}$) for each $\mathbf{V}$ isoorthogonal to $\mathbf{U}^{\perp\perp}$. The new real tree for $\mathbf{S}$ is obtained from $\mathcal{T}^\bullet\mathbf{S}$ by blowing up $\rho^{\mathbf{V}}_{\mathbf{S}}$ to a copy of $\mathcal{T}^\bullet\mathbf{S}$ whenever $\mathbf{V}$ is isoorthogonal to $\mathbf{S}$ (i.e. nothing is orthogonal to $\mathbf{V}$). This is reminiscent of a very similar modification to HHS structures introduced by Abbott-Behrstock-Durham in [ABD21]. The blow-ups aren't done explicitly, rather they are an alternate way of looking at the real trees produced by the construction used to prove Theorem 5.1.

Another way to look at it: for each isoorthogonality class, we discarded all but the unique $\sqsubseteq$–maximal element $\mathbf{V}$ in the class. Then we considered the image of $\mathbf{X}$ in the product $\prod_{\mathbf{U}^\perp = \mathbf{V}^\perp} \mathcal{T}^\bullet\mathbf{U}$, and noted that this image is a median subalgebra whose rank must be 1 since no two of the factors of the product correspond to orthogonal elements. Hence this image is a real tree, and we use it as our new $\mathcal{T}^\bullet\mathbf{U}$.

The advantage of the latter construction, and the version above using poset-colourings, is that it's easier to construct ℝ–trees as quotients of $\mathbf{X}$ (or some $\mathbf{F}_{\mathbf{U}}$) than it is to literally blow up points in a real tree to obtain a large real tree. This is the same sort of insight used in [ABD21] in the HHS case. □

6.2. ☺ **Characterisation of real cubings with clean containers and wedges.** Now we come to the main goal of this section:

Corollary 6.9. *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete, connected median space. Then the following are equivalent:*

(1) $(\mathbf{X}, \mathfrak{F}^\bullet)$ is a real cubing where $\mathfrak{F}^\bullet$ is an index set with clean containers and wedges;
(2) $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ has finite rank and admits an orthogonal poset-colouring of finite depth;
(3) $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ has finite rank and the canonical orthogonal poset-colouring has finite depth.

Proof. Clearly (3) implies (2). On the other hand, from Proposition 3.27 it follows that if *an* orthogonal poset-colouring has finite depth, so does *the* canonical orthogonal poset-colouring and since the canonical poset-colouring exists, by Proposition 3.25, it follows that (2) implies (3).

Let us show that (3) implies (1). Assume that $\mathbf{X}$ is a complete, connected median space of finite rank and that the canonical orthogonal poset-colouring has finite depth. From Theorem 3.13, we deduce that the orthogonal poset-colouring satisfies the tangible filter condition. Recall that the orthogonal poset-colouring has clean containers and wedges.

In order to give a real cubing structure to the median space with clean containers and wedges, one would like to invoke Theorem 5.1 applied to the orthogonal poset-colouring and argue that the index set inherits the relations from the orthogonal poset-colouring which by definition has clean containers and wedges. However, the real cubing structure from Theorem 5.1 is obtained from a poset-colouring, not necessarily an orthogonal one, and in particular, in Definition 5.7, we defined the orthogonality between elements of the index set precisely when there is a median-preserving isometric embedding $\mathbf{F_U} \times \mathbf{F_V} \to \mathbf{X}$. Therefore, to endow the index set with clean containers and wedges, one would like to relate the orthogonality in the orthogonal poset-colouring with the orthogonality defined in the index set.

The key point in relating the two notions of orthogonality is via the relation between the set of walls $\mathcal{W}_\mathbf{U}$ nested in a colour and the set of walls that cross the corresponding subspace $\mathbf{F_U}$, which as a consequence of Theorem 2.18, differ in general by a set of measure 0.

In order to establish the correspondence between the two different notions of orthogonality, we will modify the domain of the orthogonal poset-colouring by removing what we call *redundant* elements, see definition below, and show that with this new poset-colouring, the two notions of orthogonality coincide.

Let $(\mathbf{X}, (\mathfrak{F}^\bullet, \sqsubseteq, \perp_2))$ be the real cubing obtained from Theorem 5.1 applied to the orthogonal poset-colouring $Col_1 : \mathcal{W} \to (\mathfrak{F}^\bullet, \sqsubseteq, \perp_1)$. From the definition, the nesting relations coincide in the index set and the orthogonal poset-colouring but the orthogonality relations $\perp_1$ and $\perp_2$ may differ.

From Proposition 5.23, the real cubing $(\mathbf{X}, (\mathfrak{F}^\bullet, \sqsubseteq, \perp_2))$ (which has wedges and clean containers) defines a poset-colouring $Col_2 : \mathcal{W} \to (\mathfrak{F}^\bullet, \sqsubseteq, \perp_2)$, where for all $\hat{h} \in \mathcal{W}$, $Col_2(\hat{h})$ is defined to be the $\sqsubseteq$-minimal element $\mathbf{U}$ in $\mathfrak{F}^\bullet$ such that $\hat{h}$ crosses $\mathbf{F_U}$.

Definition 6.10 (Redundant)**.** We say that $\mathbf{U} \in \mathfrak{F}^\bullet$ is *redundant* if there exists $\mathbf{V} \sqsubsetneq \mathbf{U}$ such that $\mathbf{F_V} = \mathbf{F_U}$.

We let $\mathfrak{F}^\bullet_+$ denote the set of $\mathbf{U} \in \mathfrak{F}^\bullet$ such that

- $\mathbf{U}$ is not redundant, and
- $\mathbf{F_U}$ is not reduced to a single point.

□

Lemma 6.11. $Col_2(\mathcal{W}) \subset \mathfrak{F}^\bullet_+$.

Proof. Let $\hat{w}$ be a wall and let $\mathbf{U} = Col_2(\hat{w})$. Since by definition $\hat{w}$ crosses $\mathbf{F_U}$, the latter space cannot be a single point. Moreover, if $\mathbf{V} \sqsubsetneq \mathbf{U}$ and $\mathbf{F_V} = \mathbf{F_U}$, then $\hat{w}$ crosses $\mathbf{F_V}$, contradicting that $Col_2(\hat{w}) = \mathbf{U}$. So $\mathbf{U}$ must be non-redundant, whence $\mathbf{U} \in \mathfrak{F}^\bullet_+$. □

Lemma 6.12. *Let $\mathbf{U} \in (\mathfrak{F}^\bullet, \sqsubseteq, \perp_2)$ and suppose that $\mathbf{F_U}$ is not a single point. Then there exists $\mathbf{U}_1 \sqsubseteq \mathbf{U}$ such that*

- $\mathbf{U}_1 \in \mathfrak{F}^\bullet_+$, *and*
- $\mathbf{F}_{\mathbf{U}_1} = \mathbf{F_U}$, *and*
- $\mathbf{U}_1$ *is the unique $\sqsubseteq$–minimal element of $\mathfrak{F}^\bullet$ such that $Col(\hat{w}) \sqsubseteq \mathbf{U}_1$ whenever $\hat{w}$ crosses $\mathbf{F_U}$.*

Proof. Let $\{\mathbf{V}_i\}_{i \in I}$ be the set of elements of the form $Col_2(\hat{w})$ where $\hat{w}$ is a wall crossing $\mathbf{F_U}$. Let $\mathbf{U}_1$ be $\sqsubseteq$–minimal among the set of all $\mathbf{V} \in \mathfrak{F}^\bullet$ such that $\mathbf{V}_i \sqsubseteq \mathbf{V}$ for all i.

To see that $\mathbf{U}_1$ exists, it suffices to observe that $\mathbf{V}_i \sqsubseteq \mathbf{U}$ for all i, since each wall $\hat{w}$ crossing $\mathbf{F}_{\mathbf{U}}$ has colour nested in $\mathbf{U}$.

Recall that by definition the orthogonal poset-colouring has wedges and since the nesting relation in the index set, see Definition 5.6, coincides with nesting relation in the orthogonal poset-colouring, the index set $(\mathfrak{F}^\bullet, \sqsubseteq, \perp_2)$ inherits the wedge property. Then, uniqueness of $\mathbf{U}_1$ follows from the wedge property in the index set. It also follows from the wedge property that $\mathbf{U}_1 \sqsubseteq \mathbf{U}$. Indeed, for any $\hat{w}$ with $Col_2(\hat{w}) \sqsubseteq \mathbf{U}$, we have $Col_2(\hat{w}) \sqsubseteq \mathbf{U}_1 \wedge \mathbf{U}$, so by minimality, $\mathbf{U}_1 = \mathbf{U}_1 \wedge \mathbf{U} \sqsubseteq \mathbf{U}$.

Hence $\mathbf{F}_{\mathbf{U}_1} \subset \mathbf{F}_{\mathbf{U}}$ (for appropriately chosen parallel copies). Now, suppose that this containment is proper. Then there exists a wall $\hat{w}$ such that $\hat{w}$ crosses $\mathbf{F}_{\mathbf{U}}$ but not $\mathbf{F}_{\mathbf{U}_1}$, so $Col_2(\hat{w}) = \mathbf{U}$, a contradiction. Thus $\mathbf{F}_{\mathbf{U}} = \mathbf{F}_{\mathbf{U}_1}$.

Finally, if $\mathbf{F}_{\mathbf{V}} = \mathbf{F}_{\mathbf{U}_1}$, then each $\hat{w}$ crossing $\mathbf{F}_{\mathbf{V}}$ has colour nested in $\mathbf{V}$, so $\mathbf{V} \not\sqsubseteq \mathbf{U}_1$, whence $\mathbf{U}_1$ is non-redundant and hence in $\mathfrak{F}^\bullet_+$. □

Lemma 6.13. *Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet_+ \subset \mathfrak{F}^\bullet$. Then $\mathbf{U} \perp_1 \mathbf{V}$ in the orthogonal poset-colouring if and only if $\mathbf{U} \perp_2 \mathbf{V}$ in the index set $(\mathfrak{F}^\bullet, \sqsubseteq, \perp_2)$.*

Proof. Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet_+$ be such that $\mathbf{U} \perp_1 \mathbf{V}$ and let $\mathbf{F}_{\mathbf{U}}$ and $\mathbf{F}_{\mathbf{V}}$ the the associated closed, median-convex subspaces. From the definition of the orthogonal poset-colouring, we have that each wall in $\mathcal{W}^1_{\mathbf{U}}$ crosses each wall in $\mathcal{W}^1_{\mathbf{V}}$.

From Theorem 2.18, we have that for all $\mathbf{U} \in \mathfrak{F}^\bullet$, the set of halfspaces $\mathcal{H}(\mathbf{U})$ associated to walls with colour nested in $\mathbf{U}$ and the set $\mathcal{H}(\mathbf{F}_{\mathbf{U}})$ of halfspaces that cross $\mathbf{F}_{\mathbf{U}}$ differ by a set of measure 0. Therefore, the sets $\mathcal{H}(\mathbf{U}) \cap \mathcal{H}(\mathbf{F}_{\mathbf{U}})$ and $\mathcal{H}(\mathbf{V}) \cap \mathcal{H}(\mathbf{F}_{\mathbf{V}})$ differ from $\mathcal{H}(\mathbf{F}_{\mathbf{U}})$ and $\mathcal{H}(\mathbf{F}_{\mathbf{V}})$ correspondingly by a set of measure 0 and each wall in $\mathcal{W}^1_{\mathbf{U}} \cap \mathcal{W}_{\mathbf{F}_{\mathbf{U}}}$ crosses each wall in $\mathcal{W}^1_{\mathbf{V}} \cap \mathcal{W}_{\mathbf{F}_{\mathbf{V}}}$. Then, from Proposition 2.22, it follows that there exists a median-preserving isometric embedding from $\mathbf{F}_{\mathbf{U}} \times \mathbf{F}_{\mathbf{V}}$ to $\mathbf{X}$ and from the definition of the orthogonality in the index set, we have that $\mathbf{U} \perp_2 \mathbf{V}$.

Suppose now that $\mathbf{U} \perp_2 \mathbf{W}$, i.e. there is a median-preserving isometric embedding from $\mathbf{F}_{\mathbf{U}} \times \mathbf{F}_{\mathbf{W}}$ to $\mathbf{X}$ and so the set of walls $\mathcal{W}_{\mathbf{F}_{\mathbf{U}}}$ crosses the set of walls $\mathcal{W}_{\mathbf{F}_{\mathbf{W}}}$.

From the definition of the orthogonal poset-colouring, the set of walls that cross $\mathcal{W}_{\mathbf{F}_{\mathbf{U}}}$ coincides with $\mathcal{W}^1_{\mathbf{V}}$ for some $\mathbf{V} \in (\mathfrak{F}^\bullet, \sqsubseteq, \perp_1)$. Since the orthogonal poset-colouring has clean containers, we have that $\mathcal{W}_{\mathbf{F}_{\mathbf{U}}} \subset \mathcal{W}^1_{\mathbf{V}^\perp}$.

Since $\mathcal{W}^1_{\mathbf{U}} \cap \mathcal{W}_{\mathbf{F}_{\mathbf{U}}} \subset \mathcal{W}^1_{\mathbf{U}} \cap \mathcal{W}^1_{\mathbf{V}^\perp} = \mathcal{W}^1_{\mathbf{U} \wedge \mathbf{V}^\perp} \subset \mathcal{W}^1_{\mathbf{U}}$ and $\mathcal{W}^1_{\mathbf{U}} \wedge \mathcal{W}_{\mathbf{F}_{\mathbf{U}}}$ differs from $\mathcal{W}^1_{\mathbf{U}}$ by a set of measure 0, so do $\mathcal{W}^1_{\mathbf{U} \wedge \mathbf{V}^\perp}$ and $\mathcal{W}^1_{\mathbf{U}}$. It follows that $\mathbf{F}_{\mathbf{U} \wedge \mathbf{V}^\perp} = \mathbf{F}_{\mathbf{U}}$. Since by assumption $\mathbf{U} \in \mathfrak{F}^\bullet_+$, $\mathbf{U}$ is non-redundant and so $\mathbf{U} \wedge \mathbf{V}^\perp = \mathbf{U}$.

Since each wall in $\mathcal{W}_{\mathbf{F}_{\mathbf{W}}}$ crosses each wall in $\mathcal{W}_{\mathbf{F}_{\mathbf{U}}}$, from the definition of $\mathbf{V}$ we have that $\mathcal{W}_{\mathbf{F}_{\mathbf{W}}} \subset \mathcal{W}_{\mathbf{V}}$. A similar argument as the one above, shows that the sets of halfspaces associated to the sets of walls $\mathcal{W}^1_{\mathbf{W} \wedge \mathbf{V}}$ and $\mathcal{W}_{\mathbf{F}_{\mathbf{W}}}$ differ by a set of measure 0 and so $\mathbf{F}_{\mathbf{W} \wedge \mathbf{V}} = \mathbf{F}_{\mathbf{W}}$ and from the non-redundancy of $\mathbf{W}$ we deduce that $\mathbf{W} = \mathbf{W} \wedge \mathbf{V}$. Now, since $\mathbf{W} \wedge \mathbf{V} \sqsubseteq \mathbf{V}$, $\mathbf{U} \wedge \mathbf{V}^\perp \sqsubseteq \mathbf{V}^\perp$ and $\mathbf{V} \perp_1 \mathbf{V}^\perp$, we deduce that $\mathbf{W} = (\mathbf{W} \wedge \mathbf{V}) \perp_1 (\mathbf{U} \wedge \mathbf{V}^\perp) = \mathbf{U}$, that is $\mathbf{W} \perp_1 \mathbf{U}$. □

Lemma 6.14. *Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be the real cubing as above. Then $Col_2 : \mathcal{W} \to \mathfrak{F}^\bullet_+$ is a poset-colouring with wedges and clean containers. Furthermore, $(\mathbf{X}, \mathfrak{F}^\bullet_+)$ is a real cubing with clean containers and wedges.*

Proof. By Lemma 6.11, $Col_2(\mathfrak{F}^\bullet) \subset \mathfrak{F}^\bullet_+$ and by Lemma 6.13, the index set $(\mathfrak{F}^\bullet_+, \sqsubseteq, \perp_2)$ satisfies that the nesting and orthogonality coincide with the restriction of $\mathfrak{F}^\bullet_+ \subset (\mathfrak{F}^\bullet, \sqsubseteq, \perp_1)$.

Let us show that $\mathfrak{F}^\bullet_+$ has wedges. Let $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V} \in \mathfrak{F}^\bullet_+$. Since $\mathfrak{F}^\bullet_+ \subset \mathfrak{F}^\bullet$ and $\mathfrak{F}^\bullet$ has wedges, there exists $\mathbf{U} \wedge \mathbf{V} \in \mathfrak{F}^\bullet$. Let $(\mathbf{U} \wedge \mathbf{V})_1 \sqsubseteq (\mathbf{U} \wedge \mathbf{V})$ be as in Lemma 6.12. We show that $(\mathbf{U} \wedge \mathbf{V})_1$ is the wedge of $\mathbf{U}, \mathbf{V}$ in $\mathfrak{F}^\bullet_+$. Let $\mathbf{W} \in \mathfrak{F}^\bullet_+$ be such that $\mathbf{W} \sqsubseteq \mathbf{U} \wedge \mathbf{V}$. Since

$\mathbf{W} \sqsubseteq \mathbf{U} \wedge \mathbf{V}$, from Lemma 5.8 we have that $\mathbf{F_W} \subset \mathbf{F_{U\wedge V}} = \mathbf{F}_{(\mathbf{U}\wedge\mathbf{V})_1}$ and so $\mathcal{H}(\mathbf{F_W}) \subset \mathcal{H}(\mathbf{F}_{(\mathbf{U}\wedge\mathbf{V})_1})$.

Since from Theorem 2.18, we have that $\mathcal{H}(\mathbf{F_W})$ and $\mathcal{H}(\mathbf{W})$ differ by a set of measure 0, it follows that $\mathcal{H}(\mathbf{W} \wedge (\mathbf{U} \wedge \mathbf{V})_1)$ and $\mathcal{H}(\mathbf{F_U}) \cap \mathcal{H}(\mathbf{F}_{(\mathbf{U}\wedge\mathbf{V})_1}) = \mathcal{H}(\mathbf{F_W})$ differ by a set of measure 0 and since $\mathcal{H}(\mathbf{F_W})$ and $\mathcal{H}(\mathbf{W})$ also differ by a set of measure 0, we conclude that $\mathcal{H}(\mathbf{W} \wedge (\mathbf{U} \wedge \mathbf{V})_1)$ and $\mathcal{H}(\mathbf{W})$ differ by a set of measure 0. It follows that $\mathbf{F}_{\mathbf{W}\wedge(\mathbf{U}\wedge\mathbf{V})_1} = \mathbf{F_W}$ and since $\mathbf{W}$ is non-redundant, we have that $\mathbf{W} \wedge (\mathbf{U} \wedge \mathbf{V})_1 = \mathbf{W}$ and so $\mathbf{W} \sqsubseteq (\mathbf{U} \wedge \mathbf{V})_1$. Therefore, $(\mathbf{U} \wedge \mathbf{V})_1$ is the wedge in $\mathfrak{F}^{\bullet}_{+}$.

Let us now show that it has clean containers. Let $\mathbf{U} \in \mathfrak{F}^{\bullet}_{+}$. Let $\mathbf{U}^{\perp} \in \mathfrak{F}^{\bullet}$ and let $(\mathbf{U}^{\perp})_1 \in \mathfrak{F}^{\bullet}_{+}$ be as in Lemma 6.12. Then $(\mathbf{U}^{\perp})_1$ is the clean container in $\mathfrak{F}^{\bullet}_{+}$. Indeed, since $(\mathbf{U}^{\perp})_1 \sqsubseteq \mathbf{U}^{\perp}$ and $\mathbf{U} \perp \mathbf{U}^{\perp}$ we have that $\mathbf{U} \perp (\mathbf{U}^{\perp})_1$. Suppose that $\mathbf{W} \in \mathfrak{F}^{\bullet}_{+}$ satisfies that $\mathbf{W} \perp \mathbf{U}$, then $\mathbf{W} \sqsubseteq \mathbf{U}^{\perp}$. An argument similar to the one proven for the wedge shows that $\mathbf{F_W} = \mathbf{F}_{\mathbf{W}\wedge(\mathbf{U}^{\perp})_1}$ and from non-redundancy, we deduce that $\mathbf{W} = \mathbf{W} \wedge (\mathbf{U}^{\perp})_1 \sqsubseteq (\mathbf{U}^{\perp})_1$.

Finally, given the poset-colouring $Col_2 : \mathcal{W} \to \mathfrak{F}^{\bullet}_{+}$ and applying Theorem 5.1, we obtain that $(\mathbf{X}, \mathfrak{F}^{\bullet}_{+})$ is a real cubing and by construction, the orthogonality in the index set and in the poset-colouring Col_2 is given by a median-preserving isometric embedding of $\mathbf{F_U} \times \mathbf{F_V}$ in to $\mathbf{X}$ and so the two notions of orthogonality coincide. Since we have seen that $\mathfrak{F}^{\bullet}_{+}$ has wedges and clean containers, so does the real cubing. This concludes the proof. □

This concludes the proof that (3) implies (1).

To finish proving Corollary 6.9, we are left to show that (1) implies (2), that is, if $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is a real cubing with clean containers and wedges, then it has an orthogonal poset-colouring of finite depth, as follows.

Proposition 6.6 allows us to assume that $(\mathbf{X}, \mathfrak{F}^{\bullet})$ has wedges and clean containers, $\mathfrak{F}^{\bullet}$ has a unique $\sqsubseteq$–maximal element, and orthogonality determines nesting, i.e. $(\mathfrak{F}^{\bullet}, \sqsubseteq, \perp)$ is an orthogonal set in the sense of Definition 3.7. The finite complexity assumption implies that the depth is finite.

Now, since $(\mathbf{X}, \mathfrak{F}^{\bullet}_{\perp})$ is a real cubing, from Proposition 5.23, it admits a poset-colouring map $Col : \mathcal{W} \to \mathfrak{F}^{\bullet}_{\perp}$ that sends $\hat{u} \in \mathcal{W}$ to the $\sqsubseteq$-minimal $\mathbf{U} \in \mathfrak{F}^{\bullet}_{\perp}$ such that $\hat{u}$ crosses $F_{\mathbf{U}}$. In fact, this poset-colouring was used in the proof of Proposition 6.6.

We are left to show that this colouring is an orthogonal poset-colouring. This amounts to proving the following claim.

Claim 8. If $\hat{h}, \hat{v}$ are walls, then they cross if and only if $Col(\hat{h}) \perp Col(\hat{v})$.

Proof of Claim 8. Suppose $\hat{h}, \hat{v}$ cross.

Choose $a \in h \cap v, b \in h \cap v^*, c \in h^* \cap v^*, d \in h^* \cap v$, which is possible since $\hat{h}$ and $\hat{v}$ cross. Applying [CDH10, Lemma 2.26], we can assume that a, b, c, d is a median rectangle.

Let $\mathrm{Rel}(a, b)$ be the set of $\mathbf{V} \in \mathfrak{F}^{\bullet}_{\perp}$ such that $\pi_{\mathbf{V}}(a) \neq \pi_{\mathbf{V}}(b)$ and define $\mathrm{Rel}(a, d)$ analogously. Now, $\{a, b, c, d\}$ is the image of an embedding of $\{0, 1\}^2$ in $\mathbf{X}$ as a median subalgebra (this is exactly what it means to be a median rectangle), so by the proof of Lemma 4.13 in the case $n = 2$, we have that $\mathbf{A} \perp \mathbf{B}$ whenever $\mathbf{A} \in \mathrm{Rel}(a, b)$ and $\mathbf{B} \in \mathrm{Rel}(a, d)$.

Let $\mathbf{U} = \bigvee_{\mathbf{A}\in\mathrm{Rel}(a,b)} \mathbf{A}$ and let $\mathbf{V} = \bigvee_{\mathbf{B}\in\mathrm{Rel}(a,d)} \mathbf{B}$.

Observe that $\hat{h}$ crosses $F_{\mathbf{U}}$ and so $Col(\hat{h}) \sqsubseteq \mathbf{U}$. Similarly, $Col(\hat{v}) \sqsubseteq \mathbf{V}$. So, to conclude that $Col(\hat{h}) \perp Col(\hat{v})$, it suffices to show that $\mathbf{U} \perp \mathbf{V}$.

We have that $\mathbf{B} \sqsubseteq \bigwedge_{\mathbf{A}\in\mathrm{Rel}(a,b)} \mathbf{A}^{\perp}$ whenever $\mathbf{B} \in \mathrm{Rel}(a, d)$. From the definitions of wedges, joins, and orthogonal complements, it follows that $\mathbf{B} \perp \mathbf{U}$. But then $\mathbf{U} \sqsubseteq \bigwedge_{\mathbf{B}\in\mathrm{Rel}(a,d)} \mathbf{B}^{\perp}$, so $\mathbf{U} \perp \mathbf{V}$.

Conversely, suppose that $Col(\hat{h}) \perp Col(\hat{v})$. Then $\hat{h}, \hat{v}$ respectively cross $\mathbf{F}_{Col(\hat{h})}, \mathbf{F}_{Col(\hat{v})}$ and Proposition 4.37 provides a convex subset $\mathbf{F}_{Col(\hat{h})} \times \mathbf{F}_{Col(\hat{v})}$, so $\hat{h}$ and $\hat{v}$ cross. This proves the claim. □

We have proven the equivalence and so we conclude the proof of Corollary 6.9. □

Example 6.15. Notice that the staircase, see Example 3.11, is a median space of finite rank. As discussed in the Example, the staircase can be given a real cubing structure (and so a finite depth tangible poset-colouring). However, since the orthogonal poset-colouring of the staircase has infinite depth, it cannot admit an index set with clean containers and wedges (although the orthogonal poset-colouring is tangible).

Similarly, the 3-dimensional staircase, see Example 3.6, is a CAT(0) cube complex and so it can be given a real cubing structure, see Example 4.25. In this case, the orthogonal poset-colouring has infinite depth and it is not tangible. In particular, the 3-dimensional staircase cannot be given a real cubing structure with clean containers and wedges.

Also recall that there are median spaces of finite rank and infinite depth orthogonal poset-colouring that cannot be given a real cubing structure; see Proposition 4.28. □

6.3. **Automorphisms of $(\mathbf{X}, \mathfrak{F}^\bullet)$ from colour-preserving isometries.** In this subsection, we consider a complete, connected, finite-rank median space $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ and a finite-depth poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ of the walls of $\mathbf{X}$, satisfying the tangible filter condition. Theorem 5.1 provides a real cubing structure $(\mathbf{X}, \mathfrak{F}^\bullet)$.

Let Γ be a group and let $\Gamma \to \mathrm{Isom}(\mathbf{X})$ be an action by (necessarily median-preserving) isometries. This gives a natural Γ–action on $\mathcal{W}$. Suppose also that Γ acts on $\mathfrak{F}^\bullet$ in such a way that Col is Γ–equivariant and the action of Γ preserves the partial ordering $\sqsubseteq$.

For each $g \in \Gamma$ and each $\mathbf{U} \in \mathfrak{F}^\bullet$, observe that $g\mathbf{F}_\mathbf{U} = \mathbf{F}_{g\mathbf{U}}$.

It then follows that Γ preserves the relation $\perp$, and hence the relation $\pitchfork$.

Given $g \in \Gamma$ and $\mathbf{U} \in \mathfrak{F}^\bullet$, we define an isometry $g : \mathcal{T}^\bullet\mathbf{U} \to \mathcal{T}^\bullet g\mathbf{U}$ as follows. For each $p \in \mathcal{T}^\bullet\mathbf{U}$, choose $\mathbf{x} \in \mathbf{F}_\mathbf{U}$ projecting to p, and let $g(p) = \pi_{g\mathbf{U}}(g\mathbf{x})$. Since g preserves colours of walls and measures of sets of halfspaces, this is independent of the choice of $\mathbf{x}$, and defines an isometry. By construction, $g\pi_\mathbf{U}(\mathbf{x}) = \pi_{g\mathbf{U}}(g\mathbf{x})$ for all $g, \mathbf{U}, \mathbf{x}$. From this, and the definition of the point $\rho^\mathbf{U}_\mathbf{V}$, we have $g\rho^\mathbf{U}_\mathbf{V} = \rho^{g\mathbf{U}}_{g\mathbf{V}}$ whenever $\mathbf{U} \sqsubsetneq \mathbf{V}$ or $\mathbf{U} \perp \mathbf{V}$. This verifies the properties demanded by Definition 4.30, so:

Proposition 6.16 ($\mathbb{R}$–cubing automorphisms from colour preserving isometries)**.** *Let $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ be a complete connected finite-rank median space and let $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ be a finite-depth poset-colouring of the walls satisfying the tangible filter condition. Let $\Gamma \to \mathrm{Isom}(\mathbf{X})$ be a group action such that, giving $\mathcal{W}$ the natural Γ action and equipping $\mathfrak{F}^\bullet$ with an order-preserving Γ action, the poset-colouring Col is Γ–equivariant. Then the action of Γ on $\mathbf{X}$ is an action by $\mathbb{R}$–cubing automorphisms on the $\mathbb{R}$–cubing structure from Theorem 5.1.*

We will use Theorem 5.1 and Proposition 6.16 in the study of asymptotic cones of hierarchically hyperbolic spaces.

6.3.1. ☺ *Automorphisms and orthogonal sets.* In the next theorem, we show that if the real cubing $\mathbf{X}$ has an index set $\mathfrak{F}^\bullet$ such that $(\mathfrak{F}^\bullet, \sqsubseteq, \perp)$ is an orthogonal set, then any isometry $f : \mathbf{X} \to \mathbf{X}$ induces an automorphism of the real cubing $(\mathbf{X}, \mathfrak{F}^\bullet)$. Notice that by Corollary 6.9, the existence of an index set with clean containers and wedges assures the existence of an orthogonal index set of finite depth. In particular, if a real cubing has an index set which is the canonical orthogonal set, then this structure is minimal and preserved by the group of isometries of $\mathbf{X}$.

Theorem 6.17. *Let* $(\mathbf{X}, \mathfrak{F}^\bullet)$ *be a real cubing and assume that* $(\mathfrak{F}^\bullet, \sqsubseteq, \perp)$ *is the canonical orthogonal set, see Section 3.3.*

If $f : \mathbf{X} \to \mathbf{X}$ *is an isometry, then* f *induces an automorphism of the real cubing* $(\mathbf{X}, \mathfrak{F}^\bullet)$*, see Definition 4.30.*

Proof. Since $(\mathfrak{F}^\bullet, \sqsubseteq, \perp)$ is the orthogonal set, in particular it has clean containers and wedges and so by Corollary 6.9 it has an orthogonal poset-colouring of finite depth, $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ that sends a wall $\hat{u} \in \mathcal{W}$ to the $\sqsubseteq$-minimal $\mathbf{U} \in \mathfrak{F}^\bullet$ such that $\hat{u}$ crosses $F_\mathbf{U}$. Recall that for $\mathbf{U} \in \mathfrak{F}^\bullet$ we denote by $\mathcal{W}_\mathbf{U}$ the subset of walls $\hat{u}$ with colour $Col(\hat{u})$ nested in $\mathbf{U}$. Notice that by construction $\mathcal{W}_\mathbf{U}$ is non-empty, since in Definition 3.15 we only consider non-empty sets of walls and the index set is defined as a quotient of a subset of that set.

Observe that, since $\mathbf{X}$ is a median space, the isometry $f : \mathbf{X} \to \mathbf{X}$ preserves the median structure and so it has to preserve the set of walls. In particular, the isometry f induces a bijection on the set of walls, which abusing the notation we denote again by $f : \mathcal{W} \to \mathcal{W}$.

We next show that the isometry f induces a bijection on the index set $\mathfrak{F}^\bullet$ which makes the map Col equivariant. More precisely, we claim that for all $\mathbf{U} \in \mathfrak{F}^\bullet$, there exists $\mathbf{V} \in \mathfrak{F}^\bullet$ such that $f(\mathcal{W}_\mathbf{U})$ is $\mathcal{W}_\mathbf{V}$ (up to a subset of walls of measure 0).

Suppose towards contradiction that this is not the case, then up to taking f^{-1} if necessary, we can assume that $f(\mathcal{W}_\mathbf{U})$ is not contained (up to measure 0) in any $\mathcal{W}_\mathbf{V}$. Then the set $\{\mathbf{V}_i\} \subset \mathfrak{F}^\bullet$ such that $f(\mathcal{W}_\mathbf{U}) \cap \mathcal{W}_{\mathbf{V}_i}$ has positive measure has at least 2 elements. Let $\mathbf{U}$ be $\sqsubseteq$-minimal with this property.

Each wall in $f(\mathcal{W}_\mathbf{U})$ crosses each wall in $f(\mathcal{W}_{\mathbf{U}^\perp})$. If $f(\mathcal{W}_{\mathbf{U}^\perp}) \not\subset \mathcal{W}_{\mathbf{V}_i^\perp}$, then Item III in Definition 3.1 and Remark 3.2, there exists $\mathbf{V}_i' \sqsubseteq \mathbf{V}_i$ such that $f(\mathcal{W}_\mathbf{U}) \cap \mathcal{W}_{\mathbf{V}_i} \subset \mathcal{W}_{\mathbf{V}_i'}$ and $f(\mathcal{W}_{\mathbf{U}^\perp}) \subset \mathcal{W}_{\mathbf{V}_i'^\perp}$. Hence without loss of generality, it suffices to consider $\mathbf{V}_i$ in the set such that $f(\mathcal{W}_{\mathbf{U}^\perp}) \subset \mathcal{W}_{\mathbf{V}_i^\perp}$.

Since $\mathfrak{F}^\bullet$ is an orthogonal set, in particular elements have different clean containers and so $\mathbf{V}_i^\perp \neq \mathbf{V}_j^\perp$ for $i \neq j$. Furthermore, since orthogonality determines nesting and $f(\mathcal{W}_{\mathbf{U}^\perp}) \subset \mathcal{W}_{\mathbf{V}_i^\perp}$, we have that $f(\mathcal{W}_{\mathbf{U}^\perp}) \subsetneq \mathcal{W}_{\mathbf{V}_i^\perp}$.

It follows that the each wall in $f(\mathcal{W}_\mathbf{U}) \cap \mathcal{W}_{\mathbf{V}_i}$ crosses each wall in $\mathcal{W}_{\mathbf{V}_i^\perp}$ and $f(\mathcal{W}_{\mathbf{U}^\perp}) \subsetneq \mathcal{W}_{\mathbf{V}_i^\perp}$. Again from Item III in the definition of poset-colouring, there exists $\mathbf{U}_i \sqsubseteq \mathbf{U}$ such that $f(\mathcal{W}_\mathbf{U}) \cap \mathcal{W}_{\mathbf{V}_i} \subset f(\mathcal{W}_{\mathbf{U}_i})$. Since $f(\mathcal{W}_\mathbf{U})$ is contained in the union of $\mathcal{W}_{\mathbf{V}_i}$, it follows that $\mathbf{U} = \bigvee_{i \in I} \mathbf{U}_i$. Since $\mathbf{U}$ is chosen to be the nest-minimal with the property that $f(\mathcal{W}_\mathbf{U})$ does not coincide up to measure 0 with $\mathcal{W}_\mathbf{V}$ and $\mathbf{U}_i \sqsubseteq \mathbf{U}$, we have that $f(\mathcal{W}_{\mathbf{U}_i})$ is $\mathcal{W}_{\mathbf{W}_i}$ for some $\mathbf{W}_i$. From the above it follows that $f(\mathcal{W}_\mathbf{U}) = f(\mathcal{W}_{\bigvee_{i \in I} \mathbf{U}_i})$ coincides up to measure 0 with $\mathcal{W}_{\bigvee_{i \in I} \mathbf{W}_i}$ - a contradiction.

Therefore, the isometry f induces a bijection on the index set that respects nesting and orthogonality and it makes the map Col equivariant. Therefore, it induces an automorphism of the real cubing $(\mathbf{X}, \mathfrak{F}^\bullet)$. □

7. ☺ Characterisation of real cubings with systems of equations and inequalities

In this subsection, we give yet another characterisation of real cubings. We will work with Definition 4.2 later in the paper, since that definition is more visibly related to asymptotic cones of hierarchically hyperbolic spaces (a relationship that becomes more apparent if one compares Definition 4.2 to Definition 10.1). However, we feel it clarifies matters to have the following definition in terms of *semialgebraic sets*.

The are two (independent) ideas that are at the core of this characterisation. Firstly, in a real cubing, we associate real trees to elements of the index set. However, as we will see below, real trees can in turn be given a real cubing structure where all the trees associated to the elements of the index set are connected subspaces of $\mathbb{R}$. This allows one to think of real cubings as subspaces of $\ell_1(\mathbb{R})$. Secondly, real cubings are precisely the set of consistent points, see Definition 4.2. Consistency is a property that involves only pairs of elements of the index set and can hence be checked by examining the projection of the candidate real cubing into planes coming from pairs of coordinates. In Definition 7.1 below, we give an algebraic description in terms of semialgebraic sets of how this projections into pairs should look like, see also Figure 14. Roughly speaking, we show that having semialgebraic projections into pairs characterises a real cubing.

Given a set $\mathfrak{F}^\bullet$, let $\ell_1(\mathfrak{F}^\bullet)$ be the set of functions $f : \mathfrak{F}^\bullet \to \mathbb{R}$ such that

$$\|f\|_1 = \sum_{\mathbf{U}\in\mathfrak{F}^\bullet} |f(\mathbf{U})| < \infty.$$

We emphasise that $\mathfrak{F}^\bullet$ has no restriction on cardinality, but that any $f \in \ell_1(\mathfrak{F}^\bullet)$ has countable support, as before. This is consistent with Notation 4.1; we are considering the special case where all the $\mathbb{R}$–trees are copies of $\mathbb{R}$ based at 0.

We usually denote a function $f : \mathfrak{F}^\bullet \to \mathbb{R}$ by a tuple $(x_{\mathbf{U}})_{\mathbf{U}\in\mathfrak{F}^\bullet}$, where $x_{\mathbf{U}} = f(\mathbf{U})$.

Definition 7.1 (Cubical system, $\mathbf{U},\mathbf{V}$–semialgebraic set)**.** Given distinct $\mathbf{U},\mathbf{V} \in \mathfrak{F}^\bullet$, and two connected subspaces $I(\mathbf{U})$ and $I(\mathbf{V})$ of $\mathbb{R}$, a $(\mathbf{U},\mathbf{V})$–*cubical system* (relative to $I(\mathbf{U}), I(\mathbf{V})$) is a system of equations and inequalities in the variables $x_{\mathbf{U}}, x_{\mathbf{V}}$ of one of the following forms:

- the set of at most four inequalities stating that $x_{\mathbf{V}} \in I(\mathbf{V})$ and $x_{\mathbf{U}} \in I(\mathbf{U})$, in which case we say $\mathbf{U},\mathbf{V}$ are *independent*;
- the following additional condition: for some $a \in I(\mathbf{U})$, $b \in I(\mathbf{V})$
  - $x_{\mathbf{U}} - a = 0$ and $x_{\mathbf{V}} \in I(\mathbf{V})$ or
  - $x_{\mathbf{V}} - b = 0$ and $x_{\mathbf{U}} \in I(\mathbf{U})$.

  In this case we say that $\mathbf{U},\mathbf{V}$ are *quadratically related*;
- or the following alternate additional condition: for some $a \in I(\mathbf{V})$ and some $b, c \in I(\mathbf{U})$, where $b \neq c$,
  - $x_{\mathbf{V}} - a = 0$ and $x_{\mathbf{U}} \in I(\mathbf{U})$ or
  - $x_{\mathbf{U}} - b = 0$ and $x_{\mathbf{V}} \in I(\mathbf{V})$ and $x_{\mathbf{V}} \leqslant a$ or
  - $x_{\mathbf{U}} - c = 0$ and $x_{\mathbf{V}} \in I(\mathbf{V})$ and $x_{\mathbf{V}} \geqslant a$.

  In this case, we say that $\mathbf{V}$ *dominates* $\mathbf{U}$.

Given a $(\mathbf{U},\mathbf{V})$–cubical system relative to $I(\mathbf{U}), I(\mathbf{V})$, let $\mathbf{X}(\mathbf{U},\mathbf{V})$ be its solution set in $\ell_1(\{\mathbf{U},\mathbf{V}\}) \cong \mathbb{R}^2$. This is a *semialgebraic set* in $\mathbb{R}^2$, in the sense of real algebraic geometry, see [BCR98]. See Figure 14. □

In other words, $\mathbf{X}(\mathbf{U},\mathbf{V})$ is one of three types of median subalgebra of $I(\mathbf{U}) \times I(\mathbf{V})$.

Remark 7.2. In Definition 7.1 above, when we define domination, the requirement that b and c be distinct can be avoided. Indeed, in the case that $b = c$ the definition of "$\mathbf{V}$ dominates $\mathbf{U}$" degenerates into "$\mathbf{V}$ and $\mathbf{U}$ are quadratically related". However, abusing the definition, in the case when $b = c$, sometimes it will be convenient for us to declare that $\mathbf{V}$ dominates $\mathbf{U}$. This does not cause problems provided that we take care that only one of the two alternatives takes place: either $\mathbf{V}$ dominates $\mathbf{U}$ or $\mathbf{U}$ dominates $\mathbf{V}$. □

Definition 7.3 (Cubical semialgebraic set)**.** For each $\mathbf{U} \in \mathfrak{F}^\bullet$, let $I(\mathbf{U})$ be a fixed connected subspace of $\mathbb{R}$.

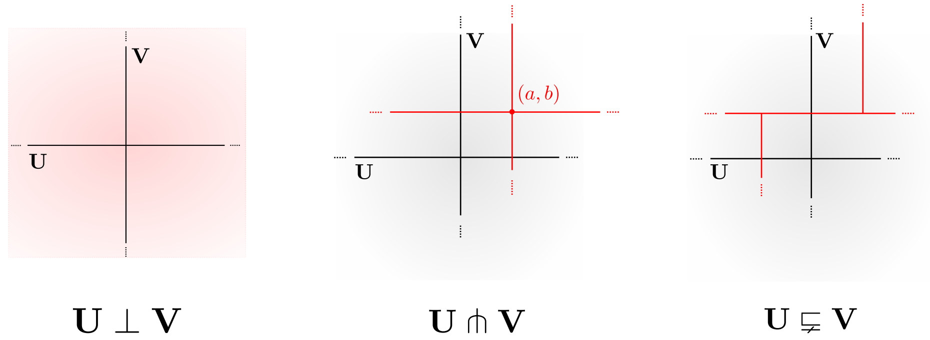

FIGURE 14. The semialgebraic set $\mathbf{X}(\mathbf{U},\mathbf{V})$ in the three cases. Left to right: independence, quadratic relation, domination.

Suppose that for each distinct $\mathbf{U},\mathbf{V} \in \mathfrak{F}^\bullet$, we have chosen *exactly one* cubical system relative to $I(\mathbf{U}), I(\mathbf{V})$, whose solution set is the semialgebraic set $\mathbf{X}(\mathbf{U},\mathbf{V})$. Let $P_{\mathbf{U},\mathbf{V}} : \ell_1(\mathfrak{F}^\bullet) \to \ell_1(\{\mathbf{U},\mathbf{V}\})$ be the natural projection.

Then

$$\mathbf{X} = \bigcap_{\mathbf{U}\neq\mathbf{V}} P_{\mathbf{U},\mathbf{V}}^{-1}(\mathbf{X}(\mathbf{U},\mathbf{V}))$$

is a *cubical semialgebraic set* in $\ell_1(\mathfrak{F}^\bullet)$ provided that $P_{\mathbf{U},\mathbf{V}}(\mathbf{X}) = \mathbf{X}(\mathbf{U},\mathbf{V})$ for each distinct pair $\mathbf{U},\mathbf{V} \in \mathfrak{F}^\bullet$. □

Definition 7.4 (Finite depth, finite dimension)**.** The cubical semialgebraic set $\mathbf{X} \subset \ell_1(\mathfrak{F}^\bullet)$ has *finite dimension* if there is a bound on the cardinality of pairwise independent sets.

A cubical semialgebraic set $\mathbf{X} \subset \ell_1(\mathfrak{F}^\bullet)$ has *finite depth* if there exists N such that any $N+1$ distinct elements of $\mathfrak{F}^\bullet$ contain either a quadratically related or independent pair.

As we shall see in Lemma 7.10 below, domination is a strict partial order on $\mathfrak{F}^\bullet$, hence in fact we require that there be a bound on the length of chains in the domination order.

□

Definition 7.5 (ℝ–cubing)**.** A cubical semialgebraic set $\mathbf{X} \subset \ell_1(\mathfrak{F}^\bullet)$ satisfying the finite depth and finite dimension conditions is a *real cubing* (or ℝ–cubing). □

Remark 7.6. Notice that by construction, the real cubing $\mathbf{X}$ (in the sense of Definition 7.5) is obtained from $\ell_1(\mathfrak{F}^\bullet)$ by deleting a collection of quarterspaces and halfspaces. Recall that a $\mathbf{U}$–*halfspace* in $\mathbf{X}$ is a subset of the form $P_{\mathbf{U}}^{-1}(R)$, where $\mathbf{U} \in \mathfrak{F}^\bullet$, and $P_{\mathbf{U}} : \ell_1(\mathfrak{F}^\bullet) \to \ell_1(\{\mathbf{U}\}) \cong \mathbb{R}$ is the natural projection (which is surjective), and R is a ray in $\ell_1(\{\mathbf{U}\})$.

In our situation, the various $I(\mathbf{U})$ will all be closed in $\mathbb{R}$, so we can take each R to be an open ray.

A $(\mathbf{U},\mathbf{V})$–*quarterspace* is the intersection of a $\mathbf{U}$–halfspace and a $\mathbf{V}$–halfspace for $\mathbf{U} \neq \mathbf{V}$. By continuity of natural projections (they are lipschitz), halfspaces and hence quarterspaces are open in $\ell_1(\mathfrak{F}^\bullet)$. The viewpoint of real cubings as complements in a product of collections of quarterspaces is partly inspired by the notion of the *Guirardel core* of a product of trees [Gui05], and by the fact that any median subalgebra of the 0–skeleton of a CAT(0) cube complex is obtained by deleting halfspaces and quarterspaces. □

We next aim to prove that this definition is equivalent to Definition 4.2. Along the way, we also compare the class of cubical semialgebraic sets to the class of median metric spaces.

Lemma 7.7 (Cubical semialgebraic implies complete)**.** *Let $\mathbf{X} \subset \ell_1(\mathfrak{F}^\bullet)$ be a cubical semialgebraic set and suppose that $I(\mathbf{U})$ is closed for all $\mathbf{U} \in \mathfrak{F}^\bullet$. Then $\mathbf{X}$, equipped with the subspace metric d_1, is complete.*

Proof. We first show that $\ell_1(\mathfrak{F}^\bullet)$ is complete.

For each $n \in \mathbb{N}$, let $f^n : \mathfrak{F}^\bullet \to \mathbb{R}$ be a function such that $\sum_{\mathbf{U}} |f^n(\mathbf{U})| < \infty$, and such that $(f^n)_n$ is a Cauchy sequence in the metric d_1.

For each $n, m \in \mathbb{N}$, let $\mathfrak{F}^\bullet_{n,m}$ be the set of $\mathbf{U} \in \mathfrak{F}^\bullet$ for which $f^n(\mathbf{U}) \neq f^m(\mathbf{U})$. Then since $\mathsf{d}_1(f^m, f^n) < \infty$, the set $\mathfrak{F}^\bullet_{n,m}$ is countable. So $\mathfrak{F}^\bullet_* = \bigcup_{m,n} \mathfrak{F}^\bullet_{m,n}$ is countable. Thus $(f^n)_n$ is a Cauchy sequence in the isometrically embedded copy $\ell_1(\mathfrak{F}^\bullet_*)$ of $\ell_1(\mathbb{N})$ in $\ell_1(\mathfrak{F}^\bullet)$. Since $\ell_1(\mathbb{N})$ is complete, it follows that (f^n) has a limit in $\ell_1(\mathfrak{F}^\bullet)$, as required.

To deduce completeness of $\mathbf{X}$, we use the structure of $\mathbf{X}$ described in Remark 7.6. Since halfspaces and hence quarterspaces are open in $\ell_1(\mathfrak{F}^\bullet)$ and by construction, $\mathbf{X}$ is obtained from $\ell_1(\mathfrak{F}^\bullet)$ by deleting a collection of quarterspaces and halfspaces, it follows that $\mathbf{X}$ is closed in $\ell_1(\mathfrak{F}^\bullet)$. Since the latter is complete, so is $\mathbf{X}$. □

We observe that $\ell_1(\mathfrak{F}^\bullet)$ is a median metric space. Indeed, given summable functions $f, g, h : \mathfrak{F}^\bullet \to \mathbb{R}$, we can define $m : \mathfrak{F}^\bullet \to \mathbb{R}$ by letting $m(\mathbf{U})$ be the median of $f(\mathbf{U}), g(\mathbf{U}), h(\mathbf{U}) \in \ell_1(\{\mathbf{U}\}) \cong \mathbb{R}$. It is routine to check that this *product median*, which we denote $\boldsymbol{\mu}$, makes $\ell_1(\mathfrak{F}^\bullet)$ a median metric space; see Section 2.5.

Lemma 7.8 (Cubical semialgebraic implies median)**.** *Let $\mathbf{X} \subset \ell_1(\mathfrak{F}^\bullet)$ be a cubical semialgebraic set. Then $\mathbf{X}$ with the metric inherited from the ℓ_1 metric on $\ell_1(\mathfrak{F}^\bullet)$ is a median subalgebra of $\ell_1(\mathfrak{F}^\bullet)$. Hence $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ is a median metric space.*

Proof. It is immediate to check that for all $\mathbf{U}, \mathbf{V}$ the semialgebraic set $\mathbf{X}(\mathbf{U}, \mathbf{V})$ is a median subalgebra and so is $P^{-1}_{\mathbf{U},\mathbf{V}}(\mathbf{X}(\mathbf{U}, \mathbf{V}))$. Indeed, this amounts to saying that the complement in $\mathbb{R}^2$ (with the product median and ℓ_1 metric) of a collection of four quarterspaces and at most four halfspaces is a median subalgebra.

By definition the cubical semialgebraic set $\mathbf{X}$ is therefore the intersection of median subalgebras and hence is a median subalgebra. Since d_1 and $\boldsymbol{\mu}$ make $\ell_1(\mathfrak{F}^\bullet)$ a median metric space, the same holds for any median subalgebra (with the subspace metric), as required. □

Proposition 7.9. *Let $\mathbf{X} \subset \ell_1(\mathfrak{F}^\bullet)$ be a cubical semialgebraic set with all $I(\mathbf{U})$ closed. Then $\mathbf{X}$ is connected. Hence $(\mathbf{X}, \mathsf{d}_1, \boldsymbol{\mu})$ is a complete geodesic median metric space.*

Proof. Lemma 7.8 showed that $\mathbf{X}$ is a median metric space, and Lemma 7.7 showed that it is complete. Once we prove connectedness, Lemma 4.6 of [Bow16b] will imply that the metric on $\mathbf{X}$ is geodesic.

Claim 9 (Finite cube case)**.** If $\mathfrak{F}^\bullet$ is finite, then $\mathbf{X}$ is connected.

Proof of Claim 9. We use induction on the number k of sets in the intersection $\mathbf{X} = \bigcap_{\mathbf{U}\neq\mathbf{V}} P^{-1}_{\mathbf{U},\mathbf{V}}(\mathbf{X}(\mathbf{U}, \mathbf{V}))$ to show that $\mathbf{X}$ is a connected median subalgebra. Note that k is always finite since so is $\mathfrak{F}^\bullet$.

Since $P^{-1}_{\mathbf{U},\mathbf{V}}(\mathbf{X}(\mathbf{U}, \mathbf{V}))$ is connected, we have the base of induction. By induction suppose that if the number of sets in the intersection is less than k, then $\mathbf{X}$ is a connected median subalgebra of $\ell_1(\mathfrak{F}^\bullet)$.

To simplify the notation, write $\mathbf{X} = \bigcap_{i=1,\ldots,k} \mathbf{X}_i$. Suppose that $\mathbf{X}$ is not connected and let $\mathbf{x}, \mathbf{y} \in \bigcap_{i=1,\ldots,k} \mathbf{X}_i$ be so that $P_{\mathbf{U},\mathbf{V}}(\mathbf{x}) = P_{\mathbf{U},\mathbf{V}}(\mathbf{y})$, where $\mathbf{X}(\mathbf{U}, \mathbf{V}) = \mathbf{X}_k$.

Since $\mathbf{x}, \mathbf{y} \in \bigcap_{i=1,\ldots,k-1} \mathbf{X}_i$ and $\bigcap_{i=1,\ldots,k-1} \mathbf{X}_i$ is a connected median subalgebra, it follows that the interval $I_{k-1}(\mathbf{x}, \mathbf{y})$ is a connected subspace of $\bigcap_{i=1,\ldots,k-1} \mathbf{X}_i$. The projection $P_{\mathbf{U},\mathbf{V}}$ is median preserving and $P_{\mathbf{U},\mathbf{V}}(\mathbf{x}) = P_{\mathbf{U},\mathbf{V}}(\mathbf{y})$, hence for any $\mathbf{z}$ in $I_{k-1}(\mathbf{x}, \mathbf{y})$, we have $P_{\mathbf{U},\mathbf{V}}(\mathbf{z}) = P_{\mathbf{U},\mathbf{V}}(\mathbf{x}) = P_{\mathbf{U},\mathbf{V}}(\mathbf{y})$. It follows that the interval $I_{k-1}(\mathbf{x}, \mathbf{y}) \subseteq P^{-1}_{\mathbf{U},\mathbf{V}}(P_{\mathbf{U},\mathbf{V}}(\mathbf{x}))$ and the fibres of points under $P_{\mathbf{U},\mathbf{V}}$ are connected. Hence, if $\mathbf{X}$ is disconnected, so is $P_{\mathbf{U},\mathbf{V}}(\mathbf{X})$, which is $\mathbf{X}(\mathbf{U}, \mathbf{V})$ by definition - a contradiction. □

Claim 10 (Reduction to finite cube case)**.** $\mathbf{X}$ is connected.

Proof of Claim 10. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Let $I(\mathbf{x}, \mathbf{y})$ be the median interval in $\ell_1(\mathfrak{F}^\bullet)$ between $\mathbf{x}, \mathbf{y}$.

Observe that

$$I(\mathbf{x}, \mathbf{y}) = \prod_{\mathbf{U} \in \mathcal{U}} [\mathbf{x}_{\mathbf{U}}, \mathbf{y}_{\mathbf{U}}],$$

where $\mathcal{U}$ is the (necessarily countable) set of $\mathbf{U}$ for which $\mathbf{x}_{\mathbf{U}} \neq \mathbf{y}_{\mathbf{U}}$. The above product, equipped with the ℓ_1 metric inherited from $\ell_1(\mathfrak{F}^\bullet)$, is compact — since

$$\sum_{\mathbf{U} \in \mathcal{U}} |\mathbf{x}_{\mathbf{U}} - \mathbf{y}_{\mathbf{U}}| < \infty,$$

the ℓ_1–metric topology on $I(\mathbf{x}, \mathbf{y})$ coincides with the Tychonov topology.

For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the set of $\mathbf{U} \in \mathcal{U}$ with $|\mathbf{x}_{\mathbf{U}} - \mathbf{y}_{\mathbf{U}}| > \frac{1}{n}$, which is finite.

Let

$$I_n = \prod_{\mathbf{U} \in \mathcal{U}_n} [\mathbf{x}_{\mathbf{U}}, \mathbf{y}_{\mathbf{U}}],$$

and let $p_n : I(\mathbf{x}, \mathbf{y}) \to I_n$ be the natural projection.

Let $\mathbf{x}_n = p_n(\mathbf{x}), \mathbf{y}_n = p_n(\mathbf{y})$. Let $\mathbf{X}_n$ be the cubical semialgebraic set in $I_n = \ell_1(\mathcal{U}_n)$ obtained by imposing the defining conditions from Definition 7.3 used to define $\mathbf{X}$ on the pairs $\mathbf{U}, \mathbf{V} \in \mathcal{U}_n$.

Note that $p_n^{-1}(\mathbf{X}_n) \subset p_m^{-1}(\mathbf{X}_m)$ when $n \geqslant m$, and

$$\mathbf{X} \cap I(\mathbf{x}, \mathbf{y}) = \bigcap_{n \geqslant 1} p_n^{-1}(\mathbf{X}_n),$$

and that

$$p_n^{-1}(\mathbf{X}_n) = \mathbf{X}_n \times \prod_{\mathbf{U} \notin \mathcal{U}_n} [\mathbf{x}_{\mathbf{U}}, \mathbf{y}_{\mathbf{U}}].$$

Now, by Claim 9, there exists a geodesic $\gamma_n : [0, L_n] \to \mathbf{X}_n$ joining $\mathbf{x}_n$ to $\mathbf{y}_n$. The product structure of $p_n^{-1}(\mathbf{X}_n)$ mentioned above allows us to lift γ_n to a geodesic $\alpha_n : [0, M_n] \to I(\mathbf{x}, \mathbf{y})$ such that

- $\alpha_n(0) = \mathbf{x}, \alpha_n(M_n) = \mathbf{y}$;
- the image of α_n lies in $p_n^{-1}(\mathbf{X}_n)$.

Since $I(\mathbf{x}, \mathbf{y})$ is compact, after passing to a subsequence, the functions α_n converge uniformly to a function $\alpha : [0, M] \to I(\mathbf{x}, \mathbf{y})$ such that $\alpha(0) = \mathbf{x}, \alpha(M) = \mathbf{y}$. Now, for any n and any $m \geqslant n$, we have that $\operatorname{im}(\alpha_m) \subset p_n^{-1}(\mathbf{X}_n)$, so $\operatorname{im}(\alpha) \subset \mathbf{X} \cap I(\mathbf{x}, \mathbf{y})$, i.e. $\mathbf{x}, \mathbf{y}$ are joined by a path in $\mathbf{X} \cap I(\mathbf{x}, \mathbf{y})$. □

The preceding claim completes the proof. □

Lemma 7.10. *Let $\mathbf{X} \subset \ell_1(\mathfrak{F}^\bullet)$ be a cubical semialgebraic set. Then*

- *domination is a strict partial order on $\mathfrak{F}^\bullet$;*
- *if $\mathbf{V}$ dominates $\mathbf{U}$ and $\mathbf{W}$ and $\mathbf{V}$ are independent, then $\mathbf{U}$ and $\mathbf{W}$ are also independent.*

Proof. By definition, $\mathbf{U}$ does not dominate $\mathbf{U}$ and if $\mathbf{V}$ dominates $\mathbf{U}$, then $\mathbf{U}$ does not dominate $\mathbf{V}$. Here we are using that each pair $(\mathbf{U}, \mathbf{V})$ is assigned exactly one of the sets of equations/inequalities, and that in the event of domination, $b \neq c$.

We next address transitivity: if $\mathbf{V}$ dominates $\mathbf{U}$ and $\mathbf{W}$ dominates $\mathbf{V}$, then $\mathbf{W}$ dominates $\mathbf{U}$.

First we define some notation, $\mathbf{Y}(\mathbf{U}, \mathbf{V})$ and $\mathbf{Z}(\mathbf{U}, \mathbf{V})$, according to how $\mathbf{U}, \mathbf{V}$ are related.

Suppose that $\mathbf{V}$ dominates $\mathbf{U}$. By definition, then we have the following system $a, b, c \in \mathbb{R}$, either $x_{\mathbf{V}} - a = 0$, or $x_{\mathbf{V}} - a \geqslant 0$ and $x_{\mathbf{U}} - b = 0$, or $x_{\mathbf{V}} - a \leqslant 0$ and $x_{\mathbf{U}} - c = 0$, see Figure

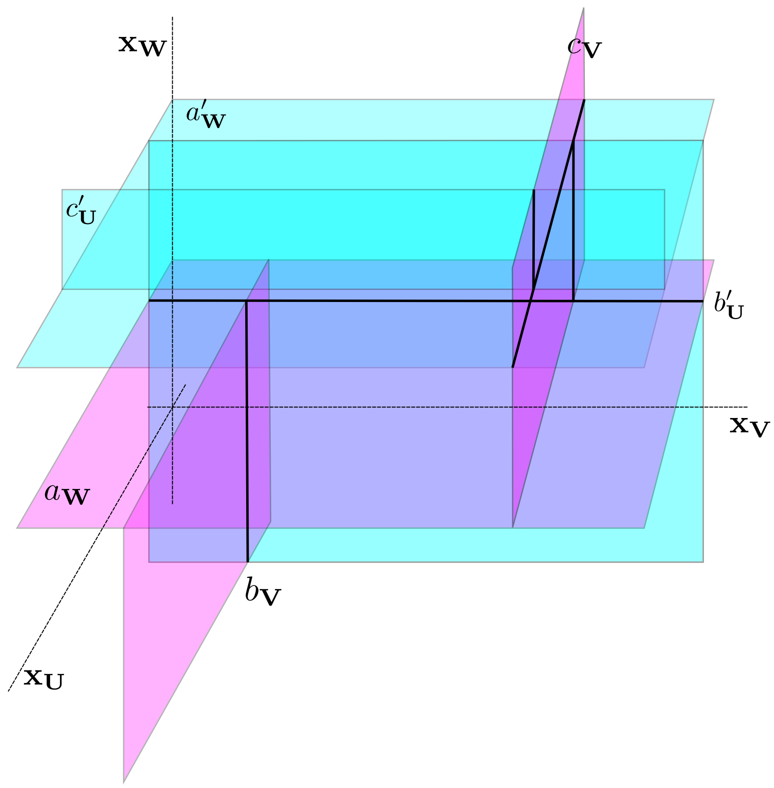

FIGURE 15. Transitivity of domination.

15. Recall that we denote its solution set by $\mathbf{X}(\mathbf{U},\mathbf{V})$. In this case, we decompose $\mathbf{X}(\mathbf{U},\mathbf{V})$ as a union of sets $\mathbf{Y}(\mathbf{U},\mathbf{V})$ and $\mathbf{Z}(\mathbf{U},\mathbf{V})$. Define

$$\begin{aligned}\mathbf{Y}(\mathbf{U},\mathbf{V}) &= \{(x_{\mathbf{U}},x_{\mathbf{V}}) \mid x_{\mathbf{V}} > a, x_{\mathbf{U}} = b\} \cup \{(x_{\mathbf{U}},x_{\mathbf{V}}) \mid x_{\mathbf{V}} < a, x_{\mathbf{U}} = c\};\\ \mathbf{Z}(\mathbf{U},\mathbf{V}) &= \{(x_{\mathbf{U}},x_{\mathbf{V}}) \mid x_{\mathbf{V}} = a\}.\end{aligned}$$

Clearly, $\mathbf{X}(\mathbf{U},\mathbf{V})$ is the union of $\mathbf{Y}(\mathbf{U},\mathbf{V})$ and $\mathbf{Z}(\mathbf{U},\mathbf{V})$.

Suppose that $\mathbf{U}$ and $\mathbf{V}$ are quadratically related. As above, we decompose $\mathbf{X}(\mathbf{U},\mathbf{V})$ as a union of $\mathbf{Y}(\mathbf{U},\mathbf{V}) = \mathbf{X_U} = \{(x_{\mathbf{U}},x_{\mathbf{V}}) \mid x_{\mathbf{U}} = a\}$ and $\mathbf{Z}(\mathbf{U},\mathbf{V}) = \mathbf{X_V} = \{(x_{\mathbf{U}},x_{\mathbf{V}}) \mid x_{\mathbf{V}} = b\}$.

Finally, suppose that $\mathbf{V}$ and $\mathbf{U}$ are independent. For convenience, we introduce the following notation set $\mathbf{Y}(\mathbf{U},\mathbf{V}) = \mathbf{Z}(\mathbf{U},\mathbf{V}) = \mathbf{X}(\mathbf{U},\mathbf{V})$.

We are now ready to check transitivity. Suppose that $\mathbf{W}$ dominates $\mathbf{V}$, which in turn dominates $\mathbf{U}$. We show that in this case $\mathbf{W}$ dominates $\mathbf{U}$.

By definition, we have that for some $a,b,c \in \mathbb{R}$, either $x_{\mathbf{V}} - a = 0$, or $x_{\mathbf{V}} - a \geqslant 0$ and $x_{\mathbf{U}} - b = 0$, or $x_{\mathbf{V}} - a \leqslant 0$ and $x_{\mathbf{U}} - c = 0$; and for some $a',b',c' \in \mathbb{R}$, either $x_{\mathbf{W}} - a' = 0$, or $x_{\mathbf{W}} - a' \geqslant 0$ and $x_{\mathbf{V}} - b' = 0$, or $x_{\mathbf{W}} - a' \leqslant 0$ and $x_{\mathbf{V}} - c' = 0$. It is immediate to check that $P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{U},\mathbf{V}}(\mathbf{X}(\mathbf{U},\mathbf{V})) \cap P^{-1}_{\mathbf{V},\mathbf{W}}(\mathbf{X}(\mathbf{V},\mathbf{W})))$ is the semialgebraic set $\mathbf{X}'(\mathbf{U},\mathbf{W})$ defined by the collection $x_{\mathbf{W}} - a' = 0$, or $x_{\mathbf{W}} - a' \geqslant 0$ and $x_{\mathbf{U}} - b = 0$, or $x_{\mathbf{W}} - a' \leqslant 0$ and $x_{\mathbf{U}} - c = 0$. Define $\mathbf{Y}'(\mathbf{U},\mathbf{W})$ and $\mathbf{Z}'(\mathbf{U},\mathbf{W})$ analogously to $\mathbf{Y}(\mathbf{U},\mathbf{W})$ and $\mathbf{Z}(\mathbf{U},\mathbf{W})$, correspondingly. By definition $\mathbf{X}'(\mathbf{U},\mathbf{W}) = \mathbf{Y}'(\mathbf{U},\mathbf{W}) \cup \mathbf{Z}'(\mathbf{U},\mathbf{W})$.

We show that $\mathbf{X}(\mathbf{U},\mathbf{W}) = \mathbf{X}'(\mathbf{U},\mathbf{W})$. We first establish that

$$\mathbf{X}(\mathbf{U},\mathbf{W}) = \bigcap_{\mathbf{R},\mathbf{S}\in\mathfrak{F}^{\bullet}} P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{R},\mathbf{S}}(\mathbf{X}(\mathbf{R},\mathbf{S}))).$$

Indeed, by definitions of $\mathbf{X}$ and $\mathbf{X}(\mathbf{U},\mathbf{W})$ we have the following equalities, while the last inclusion holds in general

$$\mathbf{X}(\mathbf{U},\mathbf{W}) = P_{\mathbf{U},\mathbf{W}}(\mathbf{X}) = P_{\mathbf{U},\mathbf{W}}(\bigcap_{\mathbf{R},\mathbf{S}\in\mathfrak{F}^\bullet} P^{-1}_{\mathbf{R},\mathbf{S}}(\mathbf{X}(\mathbf{R},\mathbf{S}))) \subseteq \bigcap_{\mathbf{R},\mathbf{S}\in\mathfrak{F}^\bullet} P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{R},\mathbf{S}}(\mathbf{X}(\mathbf{R},\mathbf{S}))).$$

Since the intersection is taken over all $\mathbf{R},\mathbf{S}\in\mathfrak{F}^\bullet$, setting $(\mathbf{R},\mathbf{S}) = (\mathbf{U},\mathbf{W})$ we have that the last intersection contains the term $P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{U},\mathbf{W}}(\mathbf{X}(\mathbf{U},\mathbf{W}))) = \mathbf{X}(\mathbf{U},\mathbf{W})$, hence

$$\bigcap_{\mathbf{R},\mathbf{S}\in\mathfrak{F}^\bullet} P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{R},\mathbf{S}}(\mathbf{X}(\mathbf{R},\mathbf{S}))) \subseteq \mathbf{X}(\mathbf{U},\mathbf{W})$$

which establishes the desired equality.

Since $\mathbf{X}'(\mathbf{U},\mathbf{W}) = P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{U},\mathbf{V}}(\mathbf{X}(\mathbf{U},\mathbf{V}))\cap P^{-1}_{\mathbf{V},\mathbf{W}}(\mathbf{X}(\mathbf{V},\mathbf{W})))$, it follows that $\mathbf{X}(\mathbf{U},\mathbf{W})\subseteq \mathbf{X}'(\mathbf{U},\mathbf{W})$. We prove the converse, write

$$\bigcap_{\mathbf{R},\mathbf{S}\in\mathfrak{F}^\bullet} P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{R},\mathbf{S}}(\mathbf{X}(\mathbf{R},\mathbf{S}))) = \bigcap_{\mathbf{R},\mathbf{S}\in\mathfrak{F}^\bullet} P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{R},\mathbf{S}}(\mathbf{Y}(\mathbf{R},\mathbf{S}))\cup P_{\mathbf{U},\mathbf{W}}(P^{-1}_{\mathbf{R},\mathbf{S}}(\mathbf{Z}(\mathbf{R},\mathbf{S}))).$$

Rewrite the (infinite) intersection of (single) unions as (infinite) union of (infinite) intersections. In this union, consider the term corresponding to

- $\mathbf{Y}(\mathbf{U},\mathbf{V})$, where $\{\mathbf{R},\mathbf{S}\} = \{\mathbf{U},\mathbf{V}\}$;
- $\mathbf{Y}(\mathbf{V},\mathbf{W})$, where $\{\mathbf{R},\mathbf{S}\} = \{\mathbf{V},\mathbf{W}\}$;
- $\mathbf{X}_{\mathbf{R}}$, whenever $\mathbf{U} = \mathbf{S}$ and they are quadratically related;
- $\mathbf{Z}(\mathbf{R},\mathbf{S})$, if $\mathbf{R}$ dominates $\mathbf{S} = \mathbf{U}$;
- $\mathbf{Y}(\mathbf{R},\mathbf{S})$, where $\mathbf{R} = \mathbf{U}$ dominates $\mathbf{S}$;
- either $\mathbf{Y}(\mathbf{R},\mathbf{S})$ or $\mathbf{Z}(\mathbf{R},\mathbf{S})$ in all other cases (i.e. whenever $\mathbf{R},\mathbf{S}\notin\{\mathbf{U},\mathbf{V}\}$ or whenever $\mathbf{R}$ and $\mathbf{S}$ are independent);

It is routine to check that this term is exactly $\mathbf{Y}'(\mathbf{U},\mathbf{W})$.

The construction for the term containing $\mathbf{Z}'(\mathbf{U},\mathbf{W})$ is analogous. Therefore, $\mathbf{X}(\mathbf{U},\mathbf{W}) = \mathbf{X}'(\mathbf{U},\mathbf{W})$.

Finally, we show that if $\mathbf{V}$ dominates $\mathbf{U}$, and $\mathbf{W}$ and $\mathbf{V}$ are independent, then so are $\mathbf{U}$ and $\mathbf{W}$. As above, we have that for some $a,b,c\in\mathbb{R}$, either $x_{\mathbf{V}} - a = 0$, or $x_{\mathbf{V}} - a \geqslant 0$ and $x_{\mathbf{U}} - b = 0$, or $x_{\mathbf{V}} - a \leqslant 0$ and $x_{\mathbf{U}} - c = 0$. Since $\mathbf{V}$ and $\mathbf{W}$ are independent, the projection of $\mathbf{X}$ on to $\ell_1(\{\mathbf{U},\mathbf{V},\mathbf{W}\})$ contains points of the form (b,a,p) as well as (c,a,p), where p is arbitrary.

By Lemma 7.8, $\mathbf{X}$ is a connected median subalgebra of $\ell_1(\mathfrak{F}^\bullet)$ and since the points (b,a,p) and (c,a,p) belong to the projection of $\mathbf{X}$ to $\ell_1(\{\mathbf{U},\mathbf{V},\mathbf{W}\})$, it follows that so does the geodesic connecting the two points. Since p is arbitrary, we conclude that any point of the form $x_{\mathbf{W}} = p$, $x_{\mathbf{V}} = a$ and $x_{\mathbf{U}} = q$ also belongs to the projection of $\mathbf{X}$ to $\ell_1(\{\mathbf{U},\mathbf{V},\mathbf{W}\})$, here q is arbitrary and hence $\mathbf{U}$ and $\mathbf{W}$ are independent; see Figure 16. This concludes the proof. □

Proposition 7.11. *Let $\mathbf{X}\subset\ell_1(\mathfrak{F}^\bullet)$ be a cubical semialgebraic set satisfying the finite depth and finite dimension conditions, with each $I(\mathbf{U})$ closed in $\mathbb{R}$. Then $\mathbf{X}$, with the inherited ℓ_1 metric, is an $\mathbb{R}$–cubing in the sense of Definition 4.2. In particular, $\mathbf{X}$ is a complete geodesic space.*

Proof. The underlying index set is $\mathfrak{F}^\bullet$, and the $\mathbb{R}$–tree $\mathcal{T}^\bullet$ associated to $\mathbf{U}$ is an isometric copy of $\mathbb{R}$. So, $\ell_1(\mathfrak{F}^\bullet)$ is naturally isometric to the ℓ_1 space from Notation 4.1. We define relations $\sqsubseteq$ and $\perp$ on $\mathfrak{F}^\bullet$ by letting $\sqsubseteq$ denote domination, and $\perp$ denote independence. For $\sqsubseteq$–incomparable, $\perp$–incomparable $\mathbf{U},\mathbf{V}$, we write $\mathbf{U}\pitchfork\mathbf{V}$ (this corresponds to a quadratically related pair).

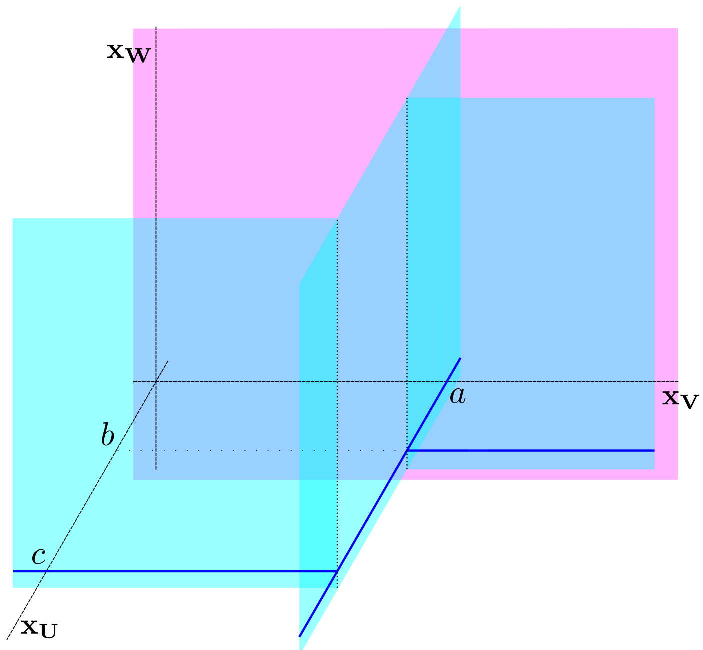

FIGURE 16. If $\mathbf{V}$ dominates $\mathbf{U}$, and $\mathbf{W}$ and $\mathbf{V}$ are independent, then so are $\mathbf{U}$ and $\mathbf{W}$.

By Lemma 7.10, $\sqsubseteq$ is a partial order, and by hypothesis, there is a uniform bound on the length of chains. Again, by hypothesis, there is a uniform bound on the size of pairwise-independent sets.

Thus far, we have produced the complete geodesic space $(\mathbf{X}, \mathsf{d}_1)$, the index set $\mathfrak{F}^\bullet$, the associated ℝ–trees, and the relations from Definition 4.2. We have verified finite complexity (Definition 4.2.(4)). We have also verified that the metric on $\mathbf{X}$ is just the subspace metric inherited from the ℓ_1 metric on $\ell_1(\mathfrak{F}^\bullet)$, as demanded by Definition 4.2, item (5).

To verify Definition 4.2, item (2), observe that the relation $\perp$ is symmetric and anti-reflexive by definition. Moreover, by Lemma 7.10, if $\mathbf{U}, \mathbf{V}, \mathbf{W} \in \mathfrak{F}^\bullet$ satisfy $\mathbf{U} \sqsubseteq \mathbf{V}$ and $\mathbf{V} \perp \mathbf{W}$, then $\mathbf{U} \perp \mathbf{W}$.

If $\mathbf{U} \pitchfork \mathbf{V}$, then $\mathbf{X}(\mathbf{U}, \mathbf{V})$ is, by definition, the solution set of $(\mathbf{x_U} - a_\mathbf{U})(\mathbf{x_V} - a_\mathbf{V})$. Let $\rho^\mathbf{U}_\mathbf{V} = a_\mathbf{V}$ and $\rho^\mathbf{V}_\mathbf{U} = a_\mathbf{U}$.

Similarly, if $\mathbf{U} \sqsubseteq \mathbf{V}$, then we can recover the value $\rho^\mathbf{U}_\mathbf{V}$, and the function $\rho^\mathbf{V}_\mathbf{U}$ (constant on each component of $\mathbb{R} - \rho^\mathbf{U}_\mathbf{V}$) from the $(\mathbf{U}, \mathbf{V})$–cubical system (see Definition 7.1). So, we have verified Definition 4.2.(1).

Claim 11. Suppose that $\mathbf{U} \sqsubsetneq \mathbf{V}$ or $\mathbf{U} \perp \mathbf{V}$, and $\rho^\mathbf{U}_\mathbf{W}$ and $\rho^\mathbf{V}_\mathbf{W}$ are defined and single points, then $\rho^\mathbf{U}_\mathbf{W} = \rho^\mathbf{V}_\mathbf{W}$.

Proof of Claim 11. We consider the case $\mathbf{U} \sqsubsetneq \mathbf{V}$, the argument in all the other cases is the same.

Since $\rho^\mathbf{U}_\mathbf{W}$ and $\rho^\mathbf{V}_\mathbf{W}$ are defined and single points, then either $\mathbf{U} \sqsubseteq \mathbf{V} \sqsubseteq \mathbf{W}$ or $\mathbf{V} \pitchfork \mathbf{W}$ and either $\mathbf{U} \pitchfork \mathbf{W}$ or $\mathbf{U} \sqsubsetneq \mathbf{W}$.

We first consider the case $\mathbf{U} \sqsubseteq \mathbf{V} \sqsubseteq \mathbf{W}$. We have the following system:

$$(1) \quad \begin{cases} \mathbf{U} \sqsubseteq \mathbf{V}: & \mathbf{x_V} = a_\mathbf{V} \vee (\mathbf{x_V} < a_\mathbf{V} \wedge \mathbf{x_U} = b_\mathbf{U}) \vee (\mathbf{x_V} > a_\mathbf{V} \wedge \mathbf{x_U} = c_\mathbf{U}); \\ \mathbf{V} \sqsubseteq \mathbf{W}: & \mathbf{x_W} = a_\mathbf{W} \vee (\mathbf{x_W} < a_\mathbf{W} \wedge \mathbf{x_V} = b_\mathbf{V}) \vee (\mathbf{x_W} > a_\mathbf{W} \wedge \mathbf{x_V} = c_\mathbf{V}); \\ \mathbf{U} \sqsubseteq \mathbf{W}: & \mathbf{x_W} = a'_\mathbf{W} \vee (\mathbf{x_W} < a'_\mathbf{W} \wedge \mathbf{x_V} = b'_\mathbf{U}) \vee (\mathbf{x_W} > a'_\mathbf{W} \wedge \mathbf{x_V} = c'_\mathbf{U}). \end{cases}$$

We have $\rho^{\mathbf{U}}_{\mathbf{V}} = (0, a_{\mathbf{V}}, 0)$, $\rho^{\mathbf{V}}_{\mathbf{W}} = (0, 0, a_{\mathbf{W}})$, $\rho^{\mathbf{U}}_{\mathbf{W}} = (0, 0, a'_{\mathbf{W}})$ in $(\mathbf{x_U}, \mathbf{x_V}, \mathbf{x_W})$-coordinates. Suppose towards contradiction that $a_{\mathbf{W}} \neq a'_{\mathbf{W}}$.

In this case, the solution set of the system formed by last two collections of system (1) is formed by 2 rays and 2 lines and has 2 connected components, namely, lines (in $\ell_1(\{\mathbf{U}, \mathbf{V}, \mathbf{W}\})$) defined by $\mathbf{x_W} = a'_{\mathbf{W}}, \mathbf{x_V} = c_{\mathbf{V}}$ and $\mathbf{x_W} = a_{\mathbf{W}}, \mathbf{x_U} = b'_{\mathbf{U}}$ and rays defined by $\mathbf{x_W} > a'_{\mathbf{W}}, \mathbf{x_V} = c_{\mathbf{V}}, \mathbf{x_U} = c'_{\mathbf{U}}$ and $\mathbf{x_W} < a_{\mathbf{W}}, \mathbf{x_V} = b_{\mathbf{V}}, \mathbf{x_U} = b'_{\mathbf{U}}$.

Since $a_{\mathbf{W}} \neq a'_{\mathbf{W}}$, this set has 2 connected components. It is now easy to check that the solution set of the entire system (1) has 2 connected components.

Hence, the image of $\mathbf{X}$ under the projection of $\ell_1(\mathfrak{F}^{\bullet})$ onto $\ell_1(\{\mathbf{U}, \mathbf{V}, \mathbf{W}\})$ is disconnected, which is impossible since $\mathbf{X}$ is connected and the projection is a continuous map, thus $a_{\mathbf{W}} = a'_{\mathbf{W}}$ and $\rho^{\mathbf{U}}_{\mathbf{W}} = \rho^{\mathbf{V}}_{\mathbf{W}}$. □

This completes the verification of Definition 4.2.(3).

The remaining part of Definition 4.2.(5) ("Consistency and realisation") follows from how we chose the various points/maps $\rho^{\mathbf{U}}_{\mathbf{V}}$ together with Definition 7.3.

It remains to verify the "Bounded geodesic image" property, Definition 4.2.(6). Let $\mathbf{V} \subsetneq \mathbf{W}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. If $\mathbf{x_V} \neq \mathbf{y_V}$, then by Definition 7.3, we have that $\mathbf{x_W}$ and $\mathbf{y_W}$ are separated by $a_{\mathbf{W}} = \rho^{\mathbf{V}}_{\mathbf{W}}$, as required. (Figure 14 illustrates this.)

Therefore, the cubical semialgebraic set is a real cubing in the sense of Definition 4.2. □

Having shown that cubical semialgebraic sets with all $I(\mathbf{U})$ closed are real cubings, we proceed to the converse. For this we need the following lemma about embedding real trees in products of copies of $\mathbb{R}$ so that the image is a real cubing.[6]

Lemma 7.12. *Let T be any real tree, then T can be given a structure of a real cubing $(\mathbf{X}, \mathfrak{F}^{\bullet})$ whose index set $\mathfrak{F}^{\bullet}$ has the following properties:*

- *for all $\mathbf{W} \in \mathfrak{F}^{\bullet}$, the real tree $\mathcal{T}\mathbf{W}$ is a closed segment, line, or ray, and $\mathcal{T}\mathbf{W}$ is a single point if and only if T is;*
- *for all $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$ we have $\mathbf{U} \pitchfork \mathbf{V}$.*

Moreover, either all of the segments involved are nontrivial, or T is a point, and $\mathfrak{F}^{\bullet}$ consists of a single element.

Proof. Let T be a real tree. If T is a single point, just take $\mathfrak{F}^{\bullet}$ to be a singleton whose associate real tree is a point, and we are done. So assume that T is nontrivial.

We define inductively a family of subtrees T_i of T such that $\bigcup_{i \in I} T_i = T$ and index sets $\mathfrak{F}^{\bullet}_i$ that satisfy the properties of the statement and such that $(T_i, \mathfrak{F}^{\bullet}_i)$ is a real cubing. We define $\mathfrak{F}^{\bullet} = \bigcup_{i \in I} \mathfrak{F}^{\bullet}_i$ and by transfinite induction we show that $(T, \mathfrak{F}^{\bullet})$ is a real cubing with the desired properties.

For the base of induction, let $\mathbf{b} \in T$ be any point which we fix and refer to as the basepoint. We choose any point $\mathbf{y} \in T$ distinct from the basepoint and consider the subtree $T_1 = [\mathbf{b}, \mathbf{y}]$ corresponding to the convex hull of $\mathbf{b}$ and $\mathbf{y}$ in T. We define the index set $\mathfrak{F}^{\bullet}_1$ to have a unique element $\mathbf{U}_1$ with associated tree the segment $[\mathbf{b}, \mathbf{y}]$. Clearly, $(T_1, \mathfrak{F}^{\bullet}_1)$ is a real cubing with the required properties.

Induction hypothesis. Let α be any ordinal and suppose that T_α is defined as the closed convex hull of α points; in particular, T_α is closed. Suppose that the index set $\mathfrak{F}^{\bullet}_\alpha$ is defined and satisfies the properties of the statement and that $(T_\alpha, \mathfrak{F}^{\bullet}_\alpha)$ is a real cubing.

[6]The embedding part seems to be well-known, although we are not clear on the correct reference. Yves de Cornulier provides a proof here: `https://mathoverflow.net/questions/226049/embedding-of-real-trees-into-ell-1-gamma`. However, we need a bit more than an isometric embedding, namely a characterisation of points in the image as consistent tuples.

Induction step. We choose any point $\mathbf{z} \in T \smallsetminus T_\alpha$ and define $T_{\alpha+1}$ to be the union of T_α and the segment $[\mathbf{z}, \mathbf{p}]$, where $\mathbf{p}$ is the (unique) closest point projection of $\mathbf{z}$ onto T_α. Notice that $\mathbf{p}$ is well-defined since by induction hypothesis T_α is closed. The corresponding index set $\mathfrak{F}^\bullet_{\alpha+1}$ is the union of $\mathfrak{F}^\bullet_\alpha$ and the element $\mathbf{U}_{\alpha+1}$ which is transverse to all the other elements of the index set and the associated real tree is the segment $[\mathbf{z}, \mathbf{p}]$. The ρ maps are defined naturally as $\rho^{\mathbf{U}_i}_{\mathbf{U}_{\alpha+1}} = \mathbf{p}$ and $\rho^{\mathbf{U}_\alpha}_{\mathbf{U}_i} = \pi_{\mathbf{U}_i}(\mathbf{p})$. It is easy to check that the tree $(T_{\alpha+1}, \mathfrak{F}^\bullet_{\alpha+1})$ is a real cubing and satisfies all the conditions required.

Suppose now that α is a limit ordinal and that T_β and $\mathfrak{F}^\bullet_\beta$ are defined for all $\beta < \alpha$. Let $T'_\alpha = \bigcup_{\beta<\alpha} T_\beta$ and define T_α to be the closure of T'_α in T. We define $\mathfrak{F}^\bullet_\alpha$ to be the union $\bigcup\limits_{\beta<\alpha} \mathfrak{F}^\bullet_\beta$. We now show that $(T_\alpha, \mathfrak{F}^\bullet_\alpha)$ is a real cubing. From the definition and the induction hypothesis, it suffices to prove consistency for points $\mathbf{x}$ in the closure of T'_α and realization. Let $\mathbf{x} \in T_\alpha \backslash T'_\alpha$ and let $(\mathbf{x}_n) \in T'_\alpha$ be a sequence of points that converges to $\mathbf{x}$ in T (note that in particular $\{\mathbf{x}_n\}$ is Cauchy).

For all $\mathbf{U}_i \in \mathfrak{F}^\bullet_\alpha$, we define the projection $\pi_{\mathbf{U}_i}(\mathbf{x})$ to be the limit of $\pi_{\mathbf{U}_i}(\mathbf{x}_n)$. Notice that since the sequence $\{\mathbf{x}_n\}$ is Cauchy, so is the sequence $\{\pi_{\mathbf{U}_i}(\mathbf{x}_n)\}$, and since the tree associated to $\mathbf{U}_i$ is a closed segment, its limit belongs to $\mathbf{U}_i$. From the definition of $\pi_{\mathbf{U}_i}$ and the consistency conditions for all points in T_β (for all $\beta < \alpha$), we deduce the consistency condition at $\mathbf{x}$. Indeed, let $\mathbf{U}_i, \mathbf{U}_j \in \mathfrak{F}^\bullet_\alpha$ be different elements of the index set. By definition they are transverse. Then

$$\begin{aligned}\min\{d_{\mathbf{U}_i}(\pi_{\mathbf{U}_i}(\mathbf{x}),\rho^{\mathbf{U}_j}_{\mathbf{U}_i}), d_{\mathbf{U}_j}(\pi_{\mathbf{U}_j}(\mathbf{x}), \rho^{\mathbf{U}_i}_{\mathbf{U}})\} = \\ Do = \lim_n \min\{d_{\mathbf{U}_i}(\pi_{\mathbf{U}_i}(\mathbf{x}_n), \rho^{\mathbf{U}_j}_{\mathbf{U}_i}), d_{\mathbf{U}_j}(\pi_{\mathbf{U}_j}(\mathbf{x}_n), \rho^{\mathbf{U}_i}_{\mathbf{U}_j})\} = 0.\end{aligned}$$

Let us now see realization. Suppose that we have a set of consistent conditions C on $\mathfrak{F}^\bullet_\alpha$:

$$\min\{d_{\mathbf{U}_i}(\mathbf{t}_i, \rho^{\mathbf{U}_j}_{\mathbf{U}_i}), d_{\mathbf{U}_j}(\mathbf{t}_j, \rho^{\mathbf{U}_i}_{\mathbf{U}_j})\} = 0.$$

Let $\beta < \alpha$ and let C_β be the subset of conditions in C that make sense in T_β, that is the corresponding elements of the index set $\mathbf{U}_i$ and $\mathbf{U}_j$ belong to $\mathfrak{F}^\bullet_\beta$. Then, the conditions C_β are also consistent in $\mathfrak{F}^\bullet_\beta$ and by induction there exists $\mathbf{x}_\beta$ that realises them.

Since the conditions are consistent,

$$\sum_{\mathbf{U}_i \in \mathfrak{F}^\bullet_\alpha} d_{\mathbf{U}_i}(\mathbf{b}, \mathbf{t}_i) < \infty,$$

and from the inclusion of index sets $\mathfrak{F}^\bullet_\beta \subset \mathfrak{F}^\bullet_{\beta'}$ for all $\beta \leqslant \beta' \leqslant \alpha$ we have that

$$\sum_{\mathbf{U}_i \in \mathfrak{F}^\bullet_\beta} d_{\mathbf{U}_i}(\mathbf{b}, \mathbf{x}_\beta) \leqslant \sum_{\mathbf{U}_i \in \mathfrak{F}^\bullet_{\beta'}} d_{\mathbf{U}_i}(\mathbf{b}, \mathbf{x}_{\beta'}) \leqslant \sum_{\mathbf{U}_i \in \mathfrak{F}^\bullet_\alpha} d_{\mathbf{U}_i}(\mathbf{b}, t_i) < \infty.$$

Choose a sequence of ordinals $\{\beta_i\}_{i \in \mathbb{N}}$ so that $\cup_i \beta_i = \alpha$. Then, $\{\mathbf{x}_{\beta_i}\}$ is a convergent sequence of points in T'_α. As T_α is closed, so the limit $\mathbf{x}$ of this sequence belongs to T_α and realises the consistency conditions.

Therefore $(T_\alpha, \mathfrak{F}^\bullet_\alpha)$ is a real cubing with the required properties. The statement now follows by transfinite induction. □

Remark 7.13. Let $\mathbb{T}$ be the universal real tree described in [DP01]. Using an argument similar to the one of Lemma 7.12, but taking convex hulls of points in the boundary of $\mathbb{T}$ instead of points in the tree, one can show that $\mathbb{T}$ admits a real cubing structure with the index set having rays as associated trees.

Since any tree S isometrically embeds into $\mathbb{T}$, we conclude that S has an induced real cubing structure with associated trees consisting of rays and nontrivial segments, or, if S is trivial, a single trivial segment. $\square$

Proposition 7.14. *Let $\mathbf{X}$ be a real cubing in the sense of Definition 4.2. Then $\mathbf{X}$ is median-preservingly isometric to a real cubing in the sense of Definition 7.5.*

Proof. Let $(\mathbf{X}, \mathfrak{F}_0^\bullet)$ be a real cubing in the sense of Definition 4.2. We prove the statement in two steps. First, we show that $\mathbf{X}$ admits a real cubing structure $(\mathbf{X}, \mathfrak{F}^\bullet)$ in the sense of Definition 4.2 where all trees associated to the index set are closed segments. Then we show that $(\mathbf{X}, \mathfrak{F}^\bullet)$ is a real cubing in the sense of of Definition 7.5.

Index set. Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be obtained from $(\mathbf{X}, \mathfrak{F}_0^\bullet)$ by applying Lemma 7.12 to every real tree in $\mathbf{U} \in \mathfrak{F}_0^\bullet$.

More specifically, to each $\mathbf{Z} \in \mathfrak{F}_0^\bullet$, we have associated a family $\mathbf{U}_i^{\mathbf{Z}}$. If $\mathcal{T}\mathbf{Z}$ was a non-trivial real tree, then the real trees $\mathcal{T}\mathbf{U}_i^{\mathbf{Z}}$ are nontrivial closed segments. If $\mathcal{T}\mathbf{Z}$ is trivial, then there is a single $\mathbf{U}_i^{\mathbf{Z}}$, whose associated real tree is a single point. We define $\pi_{\mathbf{U}_i^{\mathbf{Z}}} : \mathbf{Z} \to \mathbf{U}^{\mathbf{Z}}$ to be the closest point projection.

Hence to every real tree associated to an element $\mathbf{Z} \in \mathfrak{F}_0^\bullet$ there corresponds a family of (pairwise transverse) closed segments $\mathbf{U}_i^{\mathbf{Z}}$.

Relations and ρ points. We now define $\pitchfork, \perp$ and $\sqsubseteq$ on $(\mathbf{X}, \mathfrak{F}^\bullet)$.

- Let $\mathbf{Y} \pitchfork \mathbf{Z}$ in $\mathfrak{F}_0^\bullet$. By definition, we set $\mathbf{U}_i^{\mathbf{Y}} \pitchfork \mathbf{U}_j^{\mathbf{Z}}$ for each $\mathbf{U}_i^{\mathbf{Y}}$ and $\mathbf{U}_j^{\mathbf{Z}}$. In this case, we define $\rho_{\mathbf{U}_i^{\mathbf{Y}}}^{\mathbf{U}_j^{\mathbf{Z}}} = \pi_{\mathbf{U}_i^{\mathbf{Y}}}(\rho_{\mathbf{Y}}^{\mathbf{Z}})$ and $\rho_{\mathbf{U}_i^{\mathbf{Z}}}^{\mathbf{U}_j^{\mathbf{Y}}} = \pi_{\mathbf{U}_i^{\mathbf{Z}}}(\rho_{\mathbf{Z}}^{\mathbf{Y}})$ for all i, j.
- Let $\mathbf{Y} \perp \mathbf{Z}$ in $\mathfrak{F}_0^\bullet$. By definition, we set $\mathbf{U}_i^{\mathbf{Y}} \perp \mathbf{U}_j^{\mathbf{Z}}$ for all $\mathbf{U}_i^{\mathbf{Y}}$ and $\mathbf{U}_j^{\mathbf{Z}}$.
- Suppose next that $\mathbf{Y} \sqsubseteq \mathbf{Z}$ in $\mathfrak{F}_0^\bullet$ and we have $\rho_{\mathbf{Z}}^{\mathbf{Y}} \in \mathcal{T}^\bullet\mathbf{Z}$ is a point and $\rho_{\mathbf{Y}}^{\mathbf{Z}} : \mathcal{T}^\bullet\mathbf{Z} \to \mathcal{T}^\bullet\mathbf{Y}$.

  For any $\mathbf{U}_j^{\mathbf{Z}}$ so that $\rho_{\mathbf{Z}}^{\mathbf{Y}} \in \mathbf{U}_j^{\mathbf{Z}}$ and any $\mathbf{U}_i^{\mathbf{Y}}$, we set $\mathbf{U}_i^{\mathbf{Y}} \sqsubseteq \mathbf{U}_j^{\mathbf{Z}}$. Define $\rho_{\mathbf{U}_j^{\mathbf{Z}}}^{\mathbf{U}_i^{\mathbf{Y}}} = \pi_{\mathbf{U}_j^{\mathbf{Z}}}(\rho_{\mathbf{Z}}^{\mathbf{Y}}) = \rho_{\mathbf{Z}}^{\mathbf{Y}}$ and $\rho_{\mathbf{U}_i^{\mathbf{Y}}}^{\mathbf{U}_j^{\mathbf{Z}}} = \pi_{\mathbf{U}_i^{\mathbf{Y}}} \circ \rho_{\mathbf{Y}}^{\mathbf{Z}}|_{\mathbf{U}_j^{\mathbf{Z}}}$. Otherwise, i.e. if $\rho_{\mathbf{Z}}^{\mathbf{Y}} \notin \mathbf{U}_j^{\mathbf{Z}}$, we set $\mathbf{U}_i^{\mathbf{Y}} \pitchfork \mathbf{U}_j^{\mathbf{Z}}$. In this case we define $\rho_{\mathbf{U}_j^{\mathbf{Z}}}^{\mathbf{U}_i^{\mathbf{Y}}} = \pi_{\mathbf{U}_j^{\mathbf{Z}}}(\rho_{\mathbf{Z}}^{\mathbf{Y}})$ and $\rho_{\mathbf{U}_i^{\mathbf{Y}}}^{\mathbf{U}_j^{\mathbf{Z}}} = \pi_{\mathbf{U}_i^{\mathbf{Y}}} \circ \rho_{\mathbf{Y}}^{\mathbf{Z}}|_{\mathbf{U}_j^{\mathbf{Z}}}$.

$\mathfrak{F}^\bullet$ is real cubing index set. By definition, for all $\mathbf{V}, \mathbf{W}, \mathbf{U} \in \mathfrak{F}^\bullet$ we have that if $\mathbf{V} \sqsubseteq \mathbf{W}$ and $\mathbf{W} \perp \mathbf{U}$, then $\mathbf{V} \perp \mathbf{U}$ (since the same is true in $\mathfrak{F}_0^\bullet$).

By construction, the complexity of $(\mathbf{X}, \mathfrak{F}^\bullet)$, is exactly the same as the complexity of $(\mathbf{X}, \mathfrak{F}_0^\bullet)$ (to any $\sqsubseteq$–chain in $\mathfrak{F}^\bullet$ there corresponds a $\sqsubseteq$–chain in $\mathfrak{F}_0^\bullet$). Similarly, to any subset of $\mathfrak{F}^\bullet$ whose elements are pairwise orthogonal there corresponds a subset of pairwise orthogonal elements of $\mathfrak{F}_0^\bullet$.

We next show that consistent points for $(\mathbf{X}, \mathfrak{F}_0^\bullet)$ are also consistent points for $(\mathbf{X}, \mathfrak{F}^\bullet)$. Indeed, let $\mathbf{x} \in \mathbf{X}$ is a consistent point in $(\mathbf{X}, \mathfrak{F}_0^\bullet)$. By definition if $\mathbf{Y} \pitchfork \mathbf{Z}$, then $\min\{\mathsf{d}_{\mathbf{Z}}(\pi_{\mathbf{Z}}(\mathbf{x}), \rho_{\mathbf{Z}}^{\mathbf{Y}}), \mathsf{d}_{\mathbf{Y}}(\pi_{\mathbf{Y}}(\mathbf{x}), \rho_{\mathbf{Y}}^{\mathbf{Z}})\} = 0$. Since $\pi_{\mathbf{U}_i^{\mathbf{Y}}}(\mathbf{x}) = \pi_{\mathbf{U}_i^{\mathbf{Y}}}(\pi_{\mathbf{Y}}(\mathbf{x}))$ and $\rho_{\mathbf{U}_i^{\mathbf{Y}}}^{\mathbf{U}_j^{\mathbf{Z}}} = \pi_{\mathbf{U}_i^{\mathbf{Y}}} \circ \pi_{\mathbf{Y}}(\rho_{\mathbf{Y}}^{\mathbf{Z}})$ and the same equalities hold for $\mathbf{Z}$ and $\mathbf{U}_j^{\mathbf{Z}}$, we have that

$$\min\left\{\mathsf{d}_{\mathbf{U}_j^{\mathbf{Z}}}(\pi_{\mathbf{U}_j^{\mathbf{Z}}}(\mathbf{x}), \rho_{\mathbf{U}_j^{\mathbf{Z}}}^{\mathbf{U}_i^{\mathbf{Y}}}), \mathsf{d}_{\mathbf{U}_i^{\mathbf{Y}}}(\pi_{\mathbf{U}_i^{\mathbf{Y}}}(\mathbf{x}), \rho_{\mathbf{U}_i^{\mathbf{Y}}}^{\mathbf{U}_j^{\mathbf{Z}}})\right\} = 0.$$

If $\mathbf{Y} \subsetneq \mathbf{Z}$, then we have $\min\{\mathsf{d}_{\mathbf{Z}}(\pi_{\mathbf{Z}}(\mathbf{x}), \rho_{\mathbf{Z}}^{\mathbf{Y}}), \mathsf{d}_{\mathbf{Y}}(\pi_{\mathbf{Y}}(\mathbf{x}), \rho_{\mathbf{Y}}^{\mathbf{Z}}(\pi_{\mathbf{Z}}(\mathbf{x})))\} = 0$.

If $\rho_{\mathbf{Z}}^{\mathbf{Y}} \notin \mathbf{U}_j^{\mathbf{Z}}$ and hence $\mathbf{U}_i^{\mathbf{Y}} \pitchfork \mathbf{U}_j^{\mathbf{Z}}$, the proof is the same as above.

Finally, assume that $\mathbf{U}_i^{\mathbf{Y}} \sqsubseteq \mathbf{U}_j^{\mathbf{Z}}$ and hence $\rho_{\mathbf{U}_j^{\mathbf{Z}}}^{\mathbf{U}_i^{\mathbf{Y}}} = \pi_{\mathbf{U}_j^{\mathbf{Z}}}(\rho_{\mathbf{Z}}^{\mathbf{Y}}) = \rho_{\mathbf{Z}}^{\mathbf{Y}}$.

Suppose that $\pi_{\mathbf{Z}}(\mathbf{x}) = \rho^{\mathbf{Y}}_{\mathbf{Z}}$. Since $\rho^{\mathbf{Y}}_{\mathbf{Z}} \in \mathbf{U}^{\mathbf{Z}}_j$, then $\pi_{\mathbf{Z}}(\mathbf{x}) = \pi_{\mathbf{U}^{\mathbf{Z}}_j}(\mathbf{x})$ and so $\pi_{\mathbf{U}^{\mathbf{Z}}_j}(\mathbf{x}) = \rho^{\mathbf{Y}}_{\mathbf{Z}} = \rho^{\mathbf{U}^{\mathbf{Y}}_i}_{\mathbf{U}^{\mathbf{Z}}_j}$ and the last equality is by definition.

Suppose now that $\pi_{\mathbf{Z}}(\mathbf{x}) \neq \rho^{\mathbf{Y}}_{\mathbf{Z}}$. Then $\pi_{\mathbf{Y}}(\mathbf{x}) = \rho^{\mathbf{Z}}_{\mathbf{Y}}(\pi_{\mathbf{Z}}(\mathbf{x}))$. It follows that the same equality holds after applying $\pi_{\mathbf{U}^{\mathbf{Y}}_i}$. Therefore, from the definition, we have that $\pi_{\mathbf{U}^{\mathbf{Y}}_i}(\mathbf{x}) = \rho^{\mathbf{U}^{\mathbf{Z}}_j}_{\mathbf{U}^{\mathbf{Y}}_i}(\pi_{\mathbf{U}^{\mathbf{Z}}_j}(\mathbf{x}))$. This concludes the proof that consistent points in $(\mathbf{X}, \mathfrak{F}^{\bullet}_0)$ are consistent in $(\mathbf{X}, \mathfrak{F}^{\bullet})$.

We are left to show that consistent points in $(\mathbf{X}, \mathfrak{F}^{\bullet})$ are also consistency in $(\mathbf{X}, \mathfrak{F}^{\bullet}_0)$. Let $\mathbf{x}$ be a consistent point in $(\mathbf{X}, \mathfrak{F}^{\bullet})$. Let $\mathbf{Y}, \mathbf{Z} \in \mathfrak{F}^{\bullet}_0$ be so that $\mathbf{Y} \pitchfork \mathbf{Z}$. We show that either $\pi_{\mathbf{Y}}(\mathbf{x}) = \rho^{\mathbf{Z}}_{\mathbf{Y}}$ or $\pi_{\mathbf{Z}}(\mathbf{x}) = \rho^{\mathbf{Y}}_{\mathbf{Z}}$.

Suppose towards contradiction that $\pi_{\mathbf{Y}}(\mathbf{x}) \neq \rho^{\mathbf{Z}}_{\mathbf{Y}}$ and $\pi_{\mathbf{Z}}(\mathbf{x}) \neq \rho^{\mathbf{Y}}_{\mathbf{Z}}$, then there exist i and j so that $\pi_{\mathbf{U}^{\mathbf{Y}}_i}(\pi_{\mathbf{Y}}(\mathbf{x})) \neq \pi_{\mathbf{U}^{\mathbf{Y}}_i}(\rho^{\mathbf{Z}}_{\mathbf{Y}})$ and $\pi_{\mathbf{U}^{\mathbf{Z}}_j}(\pi_{\mathbf{Z}}(\mathbf{x})) \neq \pi_{\mathbf{U}^{\mathbf{Z}}_j}(\rho^{\mathbf{Y}}_{\mathbf{Z}})$. Since $\mathbf{Y}$ and $\mathbf{Z}$ are trees, $\mathbf{U}^{\mathbf{Y}}_i$ and $\mathbf{U}^{\mathbf{Z}}_j$ are segments, and from Lemma 7.12, it follows that $\pi_{\mathbf{U}^{\mathbf{Y}}_i}(\pi_{\mathbf{Y}}(\mathbf{x})) = \pi_{\mathbf{U}^{\mathbf{Y}}_i}(\mathbf{x})$ and $\pi_{\mathbf{U}^{\mathbf{Z}}_j}(\pi_{\mathbf{Z}}(\mathbf{x})) = \pi_{\mathbf{U}^{\mathbf{Z}}_j}(\mathbf{x})$, and $\pi_{\mathbf{U}^{\mathbf{Y}}_i}(\rho^{\mathbf{Z}}_{\mathbf{Y}}) = \rho^{\mathbf{U}^{\mathbf{Z}}_j}_{\mathbf{U}^{\mathbf{Y}}_i}$ and $\pi_{\mathbf{U}^{\mathbf{Z}}_j}(\rho^{\mathbf{Y}}_{\mathbf{Z}}) = \rho^{\mathbf{U}^{\mathbf{Y}}_i}_{\mathbf{U}^{\mathbf{Z}}_j}$. Hence, $\pi_{\mathbf{U}^{\mathbf{Z}}_j}(\mathbf{x}) \neq \rho^{\mathbf{U}^{\mathbf{Y}}_i}_{\mathbf{U}^{\mathbf{Z}}_j}$ and $\pi_{\mathbf{U}^{\mathbf{Y}}_i}(\mathbf{x}) \neq \rho^{\mathbf{U}^{\mathbf{Z}}_j}_{\mathbf{U}^{\mathbf{Y}}_i}$. Since $\mathbf{U}^{\mathbf{Y}}_i \pitchfork \mathbf{U}^{\mathbf{Z}}_j$, we obtain a contradiction with consistency of $\mathfrak{F}^{\bullet}$. This proves that $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is a real cubing in the sense of Definition 4.2.

We would like to show that the real cubing is also a real cubing in the sense of Definition 7.5. In order to do that, one would like to interpret the relations in a natural way: nesting is just domination, transversality corresponds to quadratic relation and independence to orthogonality.

However, by definition of domination, if $\mathbf{V}$ dominates $\mathbf{U}$ we must have $b \neq c$, see Definition 7.1. (This is due to the fact that in a cubical semialgebraic set, we need to have that if $\mathbf{V}$ dominates $\mathbf{U}$, then $\mathbf{U}$ does not dominate $\mathbf{V}$ in order to proof that domination is a partial order.) The latter translates to the following statement about the real cubing $(\mathbf{X}, \mathfrak{F}^{\bullet})$: if $\mathbf{U} \subsetneq \mathbf{V}$, then $\rho^{\mathbf{V}}_{\mathbf{U}}$ is not a single point. A priori, this does not have to hold in real cubings $(\mathbf{X}, \mathfrak{F}^{\bullet}_0)$ and $(\mathbf{X}, \mathfrak{F}^{\bullet})$.

To solve this issue, we modify the index set $\mathfrak{F}^{\bullet}$ as follows: for any $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$, so that $\mathbf{U} \subsetneq \mathbf{V}$ and $\rho^{\mathbf{V}}_{\mathbf{U}}$ is just a point we declare $\mathbf{U} \pitchfork \mathbf{V}$. It is immediate to check that $\mathfrak{F}^{\bullet}$ is a real cubing index set and that $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is a real cubing. Abusing the notation, we denote the obtained index set by $\mathfrak{F}^{\bullet}$. Using this index set, we now can interpret nesting as domination (independence, as orthogonality and transversality, by quadratic relation).

We are now ready to show that with this index set, the real cubing is also a real cubing in the sense of Definition 7.5.

We note that $(\mathbf{X}, \mathfrak{F}^{\bullet})$ has finite depth and finite dimension, in fact it has the same depth and dimension as $(\mathbf{X}, \mathfrak{F}^{\bullet}_0)$.

Semialgebraic projections. We show that if $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$, then $P_{\mathbf{U},\mathbf{V}}(\mathbf{X})$ is a semialgebraic set defined by a $(\mathbf{U}, \mathbf{V})$-cubical system. Let now $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\bullet}$ be distinct. Suppose first that $\mathbf{U} \perp \mathbf{V}$. We show that $\pi_{\mathbf{U}} \times \pi_{\mathbf{V}} : \mathbf{X} \to \mathcal{T}^{\bullet}\mathbf{U} \times \mathcal{T}^{\bullet}\mathbf{V}$ is surjective. Since $\mathbf{U} \perp \mathbf{V}$, then the nonempty products hypothesis and Proposition 4.37 provide a convex subspace $\mathbf{F}_{\mathbf{U}} \times \mathbf{F}_{\mathbf{V}}$. Since $\pi_{\mathbf{U}}, \pi_{\mathbf{V}}$ are surjective, and the gate map to the product region does not change $\mathbf{U}, \mathbf{V}$–coordinates of any point, and $\pi_{\mathbf{U}}$ is constant on each parallel copy of $\mathbf{F}_{\mathbf{V}}$ and vice versa, we get that $\pi_{\mathbf{U}} \times \pi_{\mathbf{V}}$ is surjective, as required. We note that this argument is general and does not require that $\mathcal{T}^{\bullet}\mathbf{U}$ and $\mathcal{T}^{\bullet}\mathbf{V}$ be subspaces of $\mathbb{R}$.

Suppose next that $\mathbf{U} \pitchfork \mathbf{V}$. In this case, we have that $\pi_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}}) = \rho^{\mathbf{V}}_{\mathbf{U}}$ and $\pi_{\mathbf{V}}(\mathbf{F}_{\mathbf{U}}) = \rho^{\mathbf{U}}_{\mathbf{V}}$. It follows that the preimage of $(\pi_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}}))$ in $\mathcal{T}^{\bullet}\mathbf{U} \times \mathcal{T}^{\bullet}\mathbf{V}$ is a copy of $\mathbf{F}_{\mathbf{V}}$ and similarly the preimage of $\pi_{\mathbf{V}}(\mathbf{F_U})$ in $\mathcal{T}^\bullet\mathbf{U} \times \mathcal{T}^\bullet\mathbf{V}$ is a parallel copy of $\mathbf{F_U}$. Both of them are closed connected subspaces of $\mathbb{R}$ and the statement follows in this case.

Finally, assume that $\mathbf{U} \sqsubseteq \mathbf{V}$. Arguing as above, since $\pi_{\mathbf{V}}(\mathbf{F_U}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ is a point it follows that the preimage of $\pi_{\mathbf{V}}(\mathbf{F_U})$ in $\mathcal{T}^\bullet\mathbf{U} \times \mathcal{T}^\bullet\mathbf{V}$ is a parallel copy of $\mathbf{F_U}$, a closed connected subset of $\mathbb{R}$. Since $\mathbf{F_V}$ is a closed connected subset of $\mathbb{R}$, it follows that $\mathbf{F_V} \smallsetminus \rho^{\mathbf{U}}_{\mathbf{V}}$ has at most 2 connected components. Therefore, $P_{\mathbf{U},\mathbf{V}}(\mathbf{X})$ is a subset of the set pictured on Figure 17.

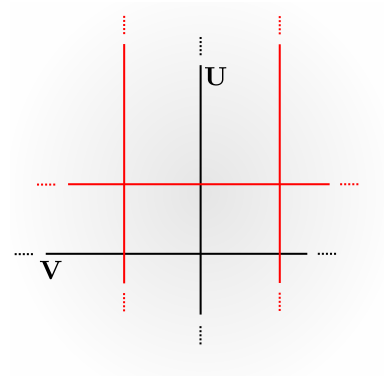

FIGURE 17. $P_{\mathbf{U},\mathbf{V}}(\mathbf{X})$ is a subset of the set marked red.

Finally, by Definition 4.2(5), $\mathbf{X}$ isometrically embeds into $\ell_1(\mathfrak{F}^\bullet)$ and so $P_{\mathbf{U},\mathbf{V}}(\mathbf{X})$ also *isometrically* embeds into $\ell_1(\{\mathbf{U},\mathbf{V}\})$. It follows that $P_{\mathbf{U},\mathbf{V}}(\mathbf{X})$ corresponds to domination (see Definition 7.1).

We are left to observe that the equality $\mathbf{X} = \bigcap_{\mathbf{U}\neq\mathbf{V}} P^{-1}_{\mathbf{U},\mathbf{V}}(\mathbf{X}(\mathbf{U},\mathbf{V}))$ holds, since we established that $\mathbf{X}(\mathbf{U},\mathbf{V}) = P_{\mathbf{U},\mathbf{V}}(\mathbf{X})$ for all $\mathbf{U},\mathbf{V}$. □

8. ☺ MISCELLANEOUS REMARKS

We collect here some miscellaneous remarks about real cubings that may be of interest.

Remark 8.1 (Tree-graded structures)**.** The class of real cubings is closed under taking finitely many direct products and tree-graded structures. More precisely, if $\mathbf{X}_1, \ldots, \mathbf{X}_k$ are real cubings, then so is $\mathbf{X}_1 \times \cdots \times \mathbf{X}_k$. The real trees for $\mathbf{X}_i$ are declared to be orthogonal to those for $\mathbf{X}_j$ whenever $i \neq j$.

Let $\mathbf{X}$ be a tree-graded space, in the sense of [DS05]. Suppose the *pieces* in the tree-graded structure are $\mathbb{R}$–cubings with globally bounded rank N and a global bound D on the length of $\sqsubseteq$–chains. Then $\mathbf{X}$ is isometric to an $\mathbb{R}$–cubing of rank N, with $\sqsubseteq$–chains of length at most $D+1$. Indeed, let $\mathfrak{F}^\bullet$ be the union of the index sets $\{\mathfrak{F}^\bullet_i\}$ corresponding to the pieces and a maximal element $\mathbf{S}$. Declare the following relations:

- for each $\mathbf{U}_i \in \mathfrak{F}^\bullet_i$, we have $\mathbf{U} \sqsubseteq \mathbf{S}$;
- if $\mathbf{U}_i, \mathbf{U}'_i \in \mathfrak{F}^\bullet_i$, then the relation between them is the same as in $\mathfrak{F}^\bullet_i$;
- for all $i \neq j$, for all $\mathbf{U}_i \in \mathfrak{F}^\bullet_i$ and for all $\mathbf{U}_j \in \mathfrak{F}^\bullet_j$, $\mathbf{U}_i \pitchfork \mathbf{U}_j$;

To the maximal element we associate the transverse tree obtained by collapsing all the pieces. The ρ maps are defined naturally by the composition of the closest point projection of the pieces in the space and the ρ maps in the pieces. One can check that with this data, $\mathbf{X}$ is a real cubing.

Alternatively, suppose that the pieces are real cubings where the index set has clean containers. From Corollary 6.9 we have that the pieces are real cubings with orthogonal poset-colouring of finite depth. Assume further that the pieces have a global bound D on the depth of the orthogonal poset-colouring and a global bound N on the rank.

A tree-graded space with geodesic median spaces as pieces is again median. We leave this part as an exercise; it can also be assembled from results in [Dru09] and [DS08, Section 2.2], in particular Lemma 2.7 of the latter paper.

The set of walls of the tree-graded space $\mathbf{X}$ is the union of the walls in the pieces together with walls arising from the transverse tree associated to $\mathbf{X}$. The orthogonal poset-colouring associated to the median space $\mathbf{X}$ is the union of the orthogonal poset-colourings of the pieces and a maximal element $\mathbf{S}$ the maximal elements of the index sets of the piece have been identified with $\mathbf{S}$. Therefore, $\mathbf{X}$ is a median space of rank N and with orthogonal poset-colouring of depth $D + 1$ and so by Corollary 6.9 it is a real cubing with clean containers.

The motivating example of an $\mathbb{R}$–cubing is an asymptotic cone of a hierarchically hyperbolic space (Theorem 26.3), and a motivating example of a tree-graded space (with additional properties of the pieces) is an asymptotic cone of a relatively hyperbolic group [DS05, DS08]. In this situation, since hierarchical hyperbolicity of parabolic subgroups implies hierarchical hyperbolicity of the whole group [BHS19, Theorem 9.1], it is true that tree-graded spaces arising as asymptotic cones of hierarchically hyperbolic groups are $\mathbb{R}$–cubings. □

Remark 8.2 (Relative real cubings)**.** Just as there is a notion of a *relatively hierarchically hyperbolic space* [BHS17a], one can imagine a notion of a *relative* $\mathbb{R}$*–cubing*. We imagine that the definition would be the same as Definition 4.2, except that for $\mathbf{W} \in \mathfrak{F}^{\bullet}$ a $\sqsubseteq$–minimal element, we would require only that $\mathcal{T}^{\bullet}\mathbf{W}$ is a geodesic space, not necessarily an $\mathbb{R}$–tree. (There might be some in-between notion, where we ask that such $\mathcal{T}^{\bullet}\mathbf{W}$ is a geodesic median space of rank possibly larger than 1.) We wonder if such a notion would be useful for studying asymptotic cones of, say, graph products, since graph products admit relative HHS structures [BR20b] (we are grateful to Anthony Genevois for a question related to this). □

Remark 8.3 (Injective metrics)**.** Let $(\mathbf{X}, \mathfrak{F}^{\bullet})$ be an $\mathbb{R}$–cubing. Replace the ℓ_1 metric by the ℓ_∞ metric. Presumably, as in the more general theorem of Bowditch about finite rank median spaces from [Bow20], this changes $\mathbf{X}$ within its bilipschitz equivalence class to become an *injective space*. Is this useful for anything? For example, is it useful to know that asymptotic cones of hierarchically hyperbolic groups are bilipschitz equivalent to injective spaces arising by restricting the ℓ_∞ metric on a product of real trees? (Note that Haettel-Hoda-Petyt have already shown that any hierarchically hyperbolic group is quasi-isometric to an injective space [HHP20], but it does not seem clear that asymptotic cones of injective spaces should be injective.) □

Remark 8.4 (Approximating median spaces by real cubings)**.** We have proven in Proposition 3.25 that any median metric space admits an orthogonal poset-colouring. Furthermore, if this poset-colouring has finite depth, then the space can be given a real cubing structure. Informally speaking, this hints that truncating the depth in a metric median space, one would get a real cubing and this process could give a way to approximate median metric spaces of finite rank by real cubings. Can **any** complete median metric space of finite rank be **approximated** by real cubings (as direct or inverse limit of real cubings)? □

Remark 8.5 (Eliminating isoorthogonality more generally)**.** It is possible that there is a version of Proposition 6.6 not using clean containers. Specifically, let $(\mathbf{X}, \mathfrak{F}^{\bullet})$ be a real cubing. Can we replace $\mathfrak{F}^{\bullet}$ by a new real cubing structure $\mathfrak{F}_1^{\bullet}$ in which no two distinct elements are isoorthogonal?

The idea is the following. If $\{\mathbf{U}_i\}_i$ is a set of isoorthogonal (and hence pairwise non-orthogonal elements), then using consistency one can show that the image $\mathcal{T}^{\bullet}\{\mathbf{U}_i\}$ of $\mathbf{X}$ in $\prod_i \mathcal{T}^{\bullet}\mathbf{U}_i$ is a real tree, and moreover if $\mathbf{W}$ is such that $\rho^{\mathbf{W}}_{\mathbf{U}_i}$ is defined for each $\mathbf{U}_i$, the point $(\rho^{\mathbf{W}}_{\mathbf{U}_i})_i$ in the product lies in $\mathcal{T}^{\bullet}\{\mathbf{U}_i\}$. (Similar statements hold for nesting.) This seems to us to be quite elegant, and suggests the idea that one should form $\mathfrak{F}_1^\bullet$ as the set of isoorthogonality classes, using the various $\mathcal{T}^\bullet\{\mathbf{U}_i\}$ as the associated real trees.

What is less elegant is what one has to do to arrange for the required combinatorial properties of the index set from Definition 4.2 to hold for $\mathfrak{F}_1^\bullet$. Specifically, defining the new nesting relation so that it is transitive seems to require a bit of a hack. We did not need this construction above, because in our situation clean containers allowed us to, essentially, replace each isoorthogonality class with an element of $\mathfrak{F}^\bullet$, and simply restrict the relations. Without clean containers, it is more complicated, but we think this possibility is worth pointing out. □

Remark 8.6 (More flexible sets of walls)**.** In this part we have considered the set of all walls $\mathcal{W}$ in a median space. Any measurable subset $\mathcal{W}' \subset \mathcal{W}$ of walls in a metric median space defines a pseudo-metric on it. We say that a subset of walls is *complete* if it defines the original metric of the space, that is for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\text{fio}(\mathcal{H}(\mathbf{x},\mathbf{y})\backslash(\mathcal{H}(\mathbf{x},\mathbf{y}) \cap \mathcal{H}')) = 0$ where $\mathcal{H}'$ is the set of halfspaces associated to walls in $\mathcal{W}'$. In this part we could have consider complete set of walls instead of the set of all walls (and adjust correspondingly the definitions).

In particular, the domain of poset-colouring map is $\mathcal{W}$. In order to obtain extra properties on a real cubing, for instance wedges and clean containers, we modify the poset-colouring and remove the redundant colours, see the proof of Corollary 6.9.

One could have proceeded in a different way as follows. We could define poset-colouring on complete subsets of walls. This would allow us to remove set of walls of measure 0 and obtain poset-colouring with the extra properties, for instance, without redundant colours. In other words, there are two equivalent approaches to obtain better structures on a real cubing - one can either modify the domain of the poset-colouring (from the set of walls to a complete set of walls) or the co-domain (the set of colours). □

9. Summary

Thus far we have defined what it means for a space $(\mathbf{X}, \mathsf{d}_\mathbf{X})$ to be an ℝ–cubing, described the structure of an ℝ–cubing as a complete, finite-rank median metric space, and discussed convexity. We have also discussed automorphisms of ℝ–cubings, the local structure of an ℝ–cubing (via the notion of a grove), and how these two notions interact.

We have also characterised real cubings as semialgebraic sets, and related real cubings with some additional properties for their index sets — wedges and clean containers — to properties of walls (the orthogonal poset-colouring).

We have seen some examples of ℝ–cubings from "nature", like finite products of ℝ–trees, and finite-dimensional CAT(0) cube complexes, and we have promised that more examples are provided by asymptotic cones of hierarchically hyperbolic spaces and groups.

Part 2. Hierarchically hyperbolic spaces and groups

In this part, we discuss hierarchically hyperbolic spaces and groups (respectively abbreviated as HHS and HHG). We aim to keep things somewhat self-contained, stating many of the results from the literature that we shall use later, and including some proofs (in particular, we give a more detailed account of *standard product regions* than appeared in [BHS17b, BHS19], and correct two misstatements from [BHS19, Section 5]).

At the end of this part, we revisit the construction of hierarchically hyperbolic structures on cube complexes from [BHS17b] in light of the orthogonal colouring, and in the following section, speculate on some related questions; Sections 20 and Section 21 are therefore not necessary for our applications to asymptotic cones.

A gentle but detailed explanation of hierarchically hyperbolic spaces, clearly explaining the geometric intuition behind each part of the definition, was recently written by Sisto [Sis19]. The presentation here is somewhat more technical, and aimed at our specific applications.

10. The definition of an HHS

We now recall from Definition 1.1 of [BHS19] the notion of a *hierarchically hyperbolic space* and the associated *hierarchically hyperbolic structure*.

In the remainder of the paper, given a metric space M, a subspace A, and $r \geqslant 0$, we let $\mathcal{N}_r(A)$ denote the (closed) r–neighbourhood of A in M.

Definition 10.1 (Hierarchically hyperbolic space)**.** The (q,q)–quasigeodesic space $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ is a *hierarchically hyperbolic space (abbreviated as HHS)* if there exist

- an index set $\mathfrak{F}$,
- a constant $\delta \geqslant 0$, and
- a set $\{\mathcal{C}W : W \in \mathfrak{F}\}$ of δ–hyperbolic geodesic metric spaces,

such that the following conditions are satisfied:

(1) **(Projections.)** There exist $K, \xi \geqslant 0$ and, for each $W \in \mathfrak{F}$, a *projection* $\pi_W : \mathcal{X} \to 2^{\mathcal{C}W}$ such that
  - for each $x \in \mathcal{X}$, the set $\pi_W(x)$ is nonempty and of diameter at most ξ;
  - π_W is (K,K)–coarsely lipschitz[7];
  - the image of π_W is K–quasiconvex in $\mathcal{C}W$.[8]

(2) **(Nesting.)** $\mathfrak{F}$ is equipped with a partial order $\sqsubseteq$ with the following properties:
  - if $\mathfrak{F} \neq \varnothing$, then $\mathfrak{F}$ has a unique $\sqsubseteq$–maximal element, denoted S;
  - $W \sqsubseteq W$ for all $W \in \mathfrak{F}$ (i.e. our partial order is reflexive).

  When $V \sqsubseteq W$, we say that V is *nested* in W, and when $V \sqsubsetneq W$, we say that this nesting is *proper*.

  For each $W \in \mathfrak{F}$, we denote by $\mathfrak{F}_W$ the set of $V \in \mathfrak{F}$ such that $V \sqsubseteq W$. For all $W \in \mathfrak{F}$ and all $V \in \mathfrak{F} - \{W\}$, there is a nonempty subset $\rho^V_W \subset \mathcal{C}W$ with $\mathsf{diam}_{\mathcal{C}W}(\rho^V_W) \leqslant \xi$. There is also a *projection* $\rho^W_V : \mathcal{C}W \to 2^{\mathcal{C}V} - \{\varnothing\}$.

(3) **(Orthogonality.)** $\mathfrak{F}$ has a relation $\perp$, called *orthogonality*, satisfying:
  - $\perp$ is symmetric;
  - $\perp$ is anti-reflexive;
  - $V \sqsubseteq W$ and $W \perp U$ imply $V \perp U$ for all $U, V, W \in \mathfrak{F}$;
  - if $V \perp W$, then V, W are not $\sqsubseteq$–comparable.

  Furthermore, the following holds for all $T \in \mathfrak{F}$. Suppose that $U \in \mathfrak{F}_T$ and $\{V \in \mathfrak{F}_T : V \perp U\} \neq \varnothing$. Then there exists $W \in \mathfrak{F}_T - \{T\}$ with the property that $V \perp U$ and $V \sqsubseteq T$ together imply $V \sqsubseteq W$.

  This V is called a *container for the orthogonal complement* of U in T. It need not be unique, or orthogonal to U, although in practice the latter often holds and simplifies various things.

(4) **(Transversality and consistency.)** If $V, W \in \mathfrak{F}$ are not orthogonal and neither is nested in the other, then we say V, W are *transverse*, denoted $V \pitchfork W$.

  There exists $\kappa_0 \geqslant 0$ such that if $V \pitchfork W$, then there are nonempty sets $\rho^V_W \subseteq \mathcal{C}W$ and $\rho^W_V \subseteq \mathcal{C}V$ each of diameter at most ξ and satisfying:

$$\min\left\{\mathsf{d}_W(\pi_W(x), \rho^V_W), \mathsf{d}_V(\pi_V(x), \rho^W_V)\right\} \leqslant \kappa_0$$

[7] Where it will not cause confusion, we will sometimes treat coarse maps like maps, e.g. this condition means that for all $x, y \in \mathcal{X}$, and all $U \in \mathfrak{F}$, we have $\mathsf{d}_{\mathcal{C}U}(\pi_U(x), \pi_U(y)) \leqslant K\mathsf{d}_{\mathcal{X}}(x,y) + K$.

[8] This means the set $\bigcup_{x \in \mathcal{X}} \pi_U(x)$ is K–quasiconvex in the hyperbolic space $\mathcal{C}U$.

for all $x \in \mathcal{X}$.

For $V, W \in \mathfrak{F}$ satisfying $V \subsetneq W$ and for all $x \in \mathcal{X}$, we have:

$$\min\left\{\mathsf{d}_W(\pi_W(x), \rho^V_W), \mathsf{diam}_{\mathcal{C}V}(\pi_V(x) \cup \rho^W_V(\pi_W(x)))\right\} \leqslant \kappa_0.$$

The preceding two inequalities are the *consistency inequalities* for points in $\mathcal{X}$.

Similarly, if $U \sqsubseteq V$, then $\mathsf{d}_W(\rho^U_W, \rho^V_W) \leqslant \kappa_0$ whenever $W \in \mathfrak{F}$ satisfies either $V \subsetneq W$ or $V \pitchfork W$ and $W \not\perp U$.

(5) **(Finite complexity.)** There exists $\chi \geqslant 0$, the *complexity* of $(\mathcal{X}, \mathfrak{F})$, so that any set of pairwise $\sqsubseteq$–comparable elements has cardinality at most χ.

(6) **(Large links.)** There exist $\lambda \geqslant 1$ and $E \geqslant \max\{\xi, \kappa_0\}$ such that the following holds. Let $W \in \mathfrak{F}$ and let $x, x' \in \mathcal{X}$. Let $N = \lambda \mathsf{d}_W(\pi_W(x), \pi_W(x')) + \lambda$.

Then there exists $\{T_i\}_{i=1}^{\lfloor N \rfloor} \subseteq \mathfrak{F}_W - \{W\}$ such that for all $T \in \mathfrak{F}_W - \{W\}$, either $T \in \mathfrak{F}_{T_i}$ for some i, or $\mathsf{d}_T(\pi_T(x), \pi_T(x')) < E$. Also, $\mathsf{d}_W(\pi_W(x), \rho^{T_i}_W) \leqslant N$ for each i.

(7) **(Bounded geodesic image.)** There exists $B \geqslant 0$ such that the following holds.

For all $W \in \mathfrak{F}$, all $V \in \mathfrak{F}_W - \{W\}$, and all geodesics γ of $\mathcal{C}W$, either $\mathsf{diam}_{\mathcal{C}V}(\rho^W_V(\gamma)) \leqslant B$ or $\gamma \cap \mathcal{N}_B(\rho^V_W) \neq \emptyset$.

(8) **(Partial Realization.)** There exists a constant α with the following property. Let $\{V_j\}$ be a family of pairwise orthogonal elements of $\mathfrak{F}$, and let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq \mathcal{C}V_j$. Then there exists $x \in \mathcal{X}$ so that:
   - $\mathsf{d}_{V_j}(x, p_j) \leqslant \alpha$ for all j,
   - for each j and each $V \in \mathfrak{F}$ with $V_j \sqsubseteq V$, we have $\mathsf{d}_V(x, \rho^{V_j}_V) \leqslant \alpha$, and
   - for each j and each W with $W \pitchfork V_j$, we have $\mathsf{d}_W(x, \rho^{V_j}_W) \leqslant \alpha$.

(9) **(Uniqueness.)** For each $\kappa \geqslant 0$, there exists $\theta_u = \theta_u(\kappa)$ such that if $x, y \in \mathcal{X}$ and $\mathsf{d}(x, y) \geqslant \theta_u$, then there exists $V \in \mathfrak{F}$ such that $\mathsf{d}_V(x, y) \geqslant \kappa$.

We refer to $\mathfrak{F}$, together with the nesting and orthogonality relations, the projections, and the hierarchy paths, as a *hierarchically hyperbolic structure* for $\mathcal{X}$. Given $A \subset \mathcal{X}$ and $U \in \mathfrak{F}$ we let $\pi_U(A)$ denote $\cup_{a \in A} \pi_U(a)$. □

Notation 10.2. Where it will not cause confusion, we will often suppress the symbols π and $\mathcal{C}$ in the following way: we write, e.g., $\mathsf{d}_U(x, y)$ to mean $\mathsf{d}_{\mathcal{C}U}(\pi_U(x), \pi_U(y))$ for $U \in \mathfrak{F}$ and $x, y \in \mathcal{X}$. □

Remark 10.3. One can replace each $\mathcal{C}U$ with a suitable geodesic thickening of $\pi_U(\mathcal{X})$ to make each π_U coarsely surjective. Hence we can and shall assume that π_U is uniformly coarsely surjective for each $U \in \mathfrak{F}$.

An HHS where the projections are uniformly coarsely surjective is *normalised*, and this procedure is discussed in [BHS19, Section 1] and [DHS17, Section 1].

The reason that coarse surjectivity is not imposed in the definition is that the more flexible definition is useful for verifying that certain subsets of HHSes are again HHSes.

So, when introducing an HHS, we will assume that it is normalised. If we encounter an HHS (arising as a subspace) that is not normalised, we will warn the reader. □

Remark 10.4 (Summary of constants)**.** As in [BHS19, Remark 1.5], we choose our constant $E < \infty$ to exceed each of the constants $\delta, K, \xi, \kappa_0, B, \alpha$. So, the reader can usually forget those constants and just use E.

The constant E, the complexity χ, the large link constant λ, and the function $\kappa \mapsto \theta_u(\kappa)$ from the uniqueness axiom will often be referred to as *the HHS constants*. For example, when we say that some other constant *depends on the HHS constants* (or *depends on the HHS structure*), we are saying that it depends on the space $\mathcal{X}$ and the particular choice of HHS structure, but not on any other feature of the situation at hand. □

10.1. **Realisation and the distance formula.** The two main technical theorems about hierarchically hyperbolic spaces are the *realisation theorem* and the *distance formula*, stated immediately below. If they wish, the reader can take these to be part of the definition of a hierarchically hyperbolic space.

Theorem 10.5 (Realisation (with linear bound))**.** *Let $(\mathcal{X},\mathfrak{F})$ be a hierarchically hyperbolic space with each π_U an E–coarsely surjective coarse map. Then there exists r_0, depending only on the constant E, such that the following holds. Let $\kappa \geqslant 1$ and let $(b_V)_{V\in\mathfrak{F}} \in \prod_{V\in\mathfrak{F}} \mathcal{C}V$ be κ–*consistent*, i.e.*

- *if $U,V \in \mathfrak{F}$ satisfy $U \pitchfork V$, then $\min\{\mathsf{d}_U(\rho_U^V, b_U), \mathsf{d}_V(\rho_V^U, b_V\} \leqslant \kappa$;*
- *if $U \subsetneq V$, then $\min\{\mathsf{d}_V(\rho_V^U, b_V), \mathsf{diam}(b_U \cup \rho_U^V(b_V))\} \leqslant \kappa$.*

Then there exists $x \in \mathcal{X}$ such that, for all $V \in \mathfrak{F}$, we have $\mathsf{d}_V(x, b_V) \leqslant r_0\kappa$.

Proof. This is almost the content of Theorem 3.1 and Remark 3.2 of [BHS19]. The only difference is that the statement of [BHS19, Theorem 3.1] does not make the bound on $\mathsf{d}_V(x, b_V)$, in terms of κ, explicit. However, inspecting the proof of that theorem shows that we can take $r_0 = 10^{10\chi}E^2$, where χ (the complexity) bounds the cardinalities of pairwise orthogonal subsets of $\mathfrak{F}$ by Lemma 2.1 of [BHS19] and E is the constant from Remark 10.3. $\square$

Remark 10.6 (Admissibility)**.** In the realisation theorem, we are assuming uniform coarse surjectivity of the π_U maps, as mentioned earlier. If we didn't assume this, then the same conclusion would hold, except one would have to add the hypothesis that $\mathsf{d}_U(b_U, \pi_U(\mathcal{X})) \leqslant \kappa$ for all $U \in \mathfrak{F}$. This blanket coarse surjectivity assumption is why we have omitted the notion of an *admissible tuple* used in [BHS19]. $\square$

The realisation theorem says roughly that the consistency inequalities coarsely characterise $\mathcal{X}$ as a subspace of $\prod_{V\in\mathfrak{F}} \mathcal{C}V$. The next theorem, the distance formula, says that the projections completely control the coarse geometry of $\mathcal{X}$.

Theorem 10.7 (Distance formula and existence of hierarchy paths for HHS)**.** *Let $(\mathcal{X},\mathfrak{F})$ be a hierarchically hyperbolic space. Then there exists a constant s_0 such that for all $s \geqslant s_0$ there exists K such that for all $x,y \in \mathcal{X}$, we have*

$$K^{-1}\mathsf{d}_{\mathcal{X}}(x,y) - K \leqslant \sum_{\{U\in\mathfrak{F}:\mathsf{d}_U(x,y)\geqslant s\}} \mathsf{d}_U(x,y) \leqslant K\mathsf{d}_{\mathcal{X}}(x,y) + K.$$

Moreover, there exists a constant $D = D(\mathcal{X},\mathfrak{F})$ such that the following holds: for all $x,y \in \mathcal{X}$, there is a (D,D)–quasigeodesic $\gamma : [0,L] \to \mathcal{X}$, with $\gamma(0) = x, \gamma(L) = y$, such that, for all $U \in \mathfrak{F}$, the composition $\pi_U \circ \gamma$ is an unparameterised (D,D)–quasigeodesic in the E–hyperbolic space $\mathcal{C}U$, lying D–close to any $\mathcal{C}U$–geodesic joining its endpoints.

A path as in the statement is a (D,D)*–hierarchy path*.

Theorem 10.7 follows from Theorem 4.4 and Theorem 4.5 in [BHS19]. An alternative proof of the above theorem, in a slightly more general setting, was recently given by Bowditch [Bow18a].

Remark 10.8 (Distances and realisation in the main examples)**.** In this paper, we are mainly interested in the case where $\mathcal{X}$ is either a Cayley graph of a mapping class group of an orientable surface of finite type, or the Cayley graph of a group that is compact special in the sense of [HW08]. In the former case, the realisation theorem was also proved in [BKMM12] and the distance formula in [MM00]. In the latter case, realisation and the distance formula were established in [BHS17b]. $\square$

10.2. ☺ **Differences between hierarchically hyperbolic spaces and real cubings.** Definition 10.1 and Definition 4.2 are morally very similar — one might say that the definitions are roughly the same, except in a real cubing, hyperbolic spaces are replaced by real trees, various bounded sets are replaced by points, coarse maps are replaced by maps, etc. However, there are a few more important differences that we now highlight:

- In a real cubing, we do not require $\mathfrak{F}^{\bullet}$ to have a unique $\sqsubseteq$–maximal element, as we do for $\mathfrak{F}$ in a hierarchically hyperbolic space;
- The part of Definition 10.1.(3) asking that every element orthogonal to a given U is nested in a single, non-maximal element has no analogue for real cubings.
- There is no real cubing analogue of the large link axiom, and in particular in a real cubing, the set of $\mathbf{U} \in \mathfrak{F}^{\bullet}$ such that $\pi_{\mathbf{U}}(\mathbf{x}) \neq \pi_{\mathbf{U}}(\mathbf{y})$ for a given $\mathbf{x}, \mathbf{y}$ may be infinite (although the ℓ_1 condition means it must be countable); in an HHS, this is ruled out by Lemma 11.4, which relies on the large link axiom. Similarly, the "passing up lemma", so important for HHSes, Lemma 11.1, has no real cubing analogue.
- The distance formula, Theorem 10.7, does *not* say that the projections π_U give a quasi-isometric embedding $\mathcal{X} \to \prod_{U\in\mathfrak{F}} \mathcal{C}U$, because of the thresholding in the sum in the theorem. On the other hand, just by definition, in a real cubing the projections give an isometric embedding $\mathbf{X} \to \prod_{\mathbf{U}\in\mathfrak{F}^{\bullet}} \mathcal{T}^{\bullet}\mathbf{U}$.
- Real cubings have no analogue of the uniqueness axiom, Definition 10.1.(9). For hierarchically hyperbolic spaces, this axiom implies that if $\mathcal{X}$ is unbounded, then there are associated hyperbolic spaces $\mathcal{C}U$ of arbitrarily large diameter. We have seen an example of an (unbounded) real cubing where all of the associated real trees are unit intervals, Example 4.25.

The real cubings considered later, that arise as asymptotic cones of hierarchically hyperbolic spaces, will retain some features of hierarchically hyperbolic spaces that do not hold for general real cubings (e.g. $\mathfrak{F}^{\bullet}$ will have a unique $\sqsubseteq$–maximal element), but some of the differences will persist (e.g. there will certainly be many pairs of points whose projections to infinitely many real trees differ).

10.3. **Group actions on hierarchically hyperbolic spaces.** We now discuss group actions on hierarchically hyperbolic spaces. The formulation is a bit more modern than in [BHS17b], following [DHS20]. The present formulation of the definition of a hierarchically hyperbolic group was introduced explicitly, to our knowledge, in [PS20].

Definition 10.9 (HHS automorphisms, hierarchically hyperbolic group)**.** Let $(\mathcal{X}, \mathfrak{F})$ be an HHS. Let G be a group acting on $\mathcal{X}$, and suppose that all of the following hold:

- the action of G on $\mathcal{X}$ is by uniform quasi-isometries (i.e. the quasi-isometry constants can be taken independently of the group element);
- G acts on $\mathfrak{F}$, preserving the relations $\sqsubseteq, \perp, \pitchfork$;
- for each $g, h \in G$ and each $U \in \mathfrak{F}$, we have isometries $g : \mathcal{C}U \to \mathcal{C}gU$ and $h : \mathcal{C}U \to \mathcal{C}hU$, such that $gh : \mathcal{C}U \to \mathcal{C}ghU$ is the composition of the isometries g and h;
- for all $x \in \mathcal{X}, g \in G, U \in \mathfrak{F}$, we have $\pi_{gU}(gx) = g(\pi_U(x))$;
- for all $g \in G$ and all $U, V \in \mathfrak{F}$ with either $U \subsetneq V$ or $U \pitchfork V$, we have

$$g(\rho^U_V) = \rho^{gU}_{gV}.$$

Then each element of g is an *HHS automorphism* of $(\mathcal{X}, \mathfrak{F})$.

If, in addition, the action of G on $\mathfrak{F}$ is cofinite, and the action of G on $\mathcal{X}$ is (metrically) proper and cobounded, then $(G, \mathfrak{F})$ is a *hierarchically hyperbolic group*, abbreviated *HHG*. □

Remark 10.10. Suppose that $(\mathcal{X}, \mathfrak{F})$ is an HHS on which G acts (metrically) properly and coboundedly by HHS automorphisms, acting cofinitely on $\mathfrak{F}$.

Fix a basepoint $x_0 \in \mathcal{X}$. Let $\psi : G \to \mathcal{X}$ be the orbit map $\psi(g) = gx_0$. For each $U \in \mathfrak{F}$, define $\pi'_U : G \to \mathcal{C}U$ by $\pi'_U(g) = \pi_U(\psi(g))$. Note that

$$g\pi'_U(h) = g\pi_U(hx_0) = \pi_{gU}(ghx_0) = \pi'_{gU}(gh).$$

So, to check that the left-multiplication action of G on $(G, \mathfrak{F})$ is an action by HHS automorphisms, we just need to check that G is finitely generated and $(G, \mathfrak{F})$, with the above projections, is an HHS. By the definition of π'_U, the HHS axioms will hold (after uniform enlargement of constants) once we show that G is finitely-generated and quasi-isometric to $\mathcal{X}$. (Indeed, by [BHS19, Proposition 1.10], if G is quasi-isometric to $\mathcal{X}$, then composing projections $\pi_U : \mathcal{X} \to \mathcal{C}U$ with the quasi-isometry makes $(G, \mathfrak{F})$ an HHS.)

Now, since $\mathcal{X}$ is a quasigeodesic space, it is quasi-isometric to a geodesic metric space [CdlH16, Lemma 3.B.6]. So, G quasi-acts on $\mathcal{X}'$ by uniform quasi-isometries. Applying Lemma 1.4 of [FLS15], we see that G is finitely generated and ψ is a quasi-isometry. Hence $(G, \mathfrak{F})$ is an HHS where G acts properly and coboundedly by automorphisms, and cofinitely on $\mathfrak{F}$. □

Accordingly, we will always use the following simpler definition of an HHG:

Definition 10.11 (Hierarchically hyperbolic group)**.** A finitely generated group G is a *hierarchically hyperbolic group* (HHG) if, fixing a word-metric on G associated to a finite generating set, we have an HHS $(G, \mathfrak{F})$ on which G acts by HHS automorphisms in such a way that the action of G on G is left-multiplication, and the action of G on $\mathfrak{F}$ is cofinite. □

Remark 10.12. In the preceding definition, we could just as well have used the Cayley graph of G as our HHS. Typically, one does not bother with this because in the HHS setting, one works coarsely, so there's nothing gained by having the space be geodesic; quasigeodesic is enough. Moreover, even if one did use a geodesic space like the full Cayley graph, one must take care since geodesics may not interact well with the rest of the HHS structure (they may not be hierarchy paths). The mantra is that the geodesics in an HHS are a mystery, but we have special quasigeodesics — the hierarchy paths — instead. □

In particular, asking that G be an HHG is strictly stronger than simply requiring every Cayley graph of G to be an HHS: in fact, every Cayley graph of the $(3, 3, 3)$ triangle group is an HHS, but this group admits no HHG structure [PS20].

Although many of the results of this paper are for general HHGs (or with mild combinatorial hypotheses on the index set) the uniqueness of the asymptotic cone will be established for a more restrictive class of groups, *algebraic HHG*, which are introduced in Section 35. Thus far, hierarchical hyperbolicity is just an abstract geometric property of a group; later, we will ask that the HHG structure arise in a particular way from algebraic features of G.

11. Counting, ordering, and colouring in the index set

When working with an HHS $(X, \mathfrak{F})$, one frequently uses directly the consistency and bounded geodesic image axioms. It is less typical to use the large link axiom as stated; one more frequently uses it in conjunction with finiteness of complexity, in the following form, which is Lemma 2.5 in [BHS19]:

Lemma 11.1 (Passing up large projections)**.** *Let $(\mathcal{X}, \mathfrak{F})$ be an HHS. For all $C \geqslant 0$ there exists $N \in \mathbb{N}$ such that the following holds. Let $x, y \in \mathcal{X}$ and $V \in \mathfrak{F}$. Let $\{U_i\}_{i=1}^n \subset \mathfrak{F}$ be a subset of the index set with the following properties:*

- *$n \geqslant N$;*
- *$U_i \sqsubseteq V$ for all i;*
- *$\mathsf{d}_{U_i}(x, y) \geqslant E$ for all i.*

Then there exists $U \in \mathfrak{F}_V$ such that $U_i \subsetneq U$ for some i, and $\mathsf{d}_U(x,y) > C$.

The preceding lemma plays an important role in several places in the literature on HHSes, and in the present paper.

We also record that, while the finite complexity axiom is about nesting, it combines with the orthogonality axiom to yield the following, which is Lemma 2.1 in [BHS19]:

Lemma 11.2 (Finite dimension)**.** *Let $(\mathcal{X},\mathfrak{F})$ be an HHS. Let $\mathcal{O} \subset \mathfrak{F}$ be a subset of the index set whose elements are pairwise orthogonal. Then $|\mathcal{O}| \leqslant \chi$.*

Given $x, y \in \mathcal{X}$, it is often useful to consider the set of $\mathcal{C}U$ on which x, y project "far apart", and get some combinatorial control on that set; Lemma 11.1 does that using the large link axiom and finite complexity. Here's an analogous statement using finite complexity and consistency, which also appears in [DMS20, Section 2]:

Lemma 11.3 (Covering)**.** *Let $(\mathcal{X},\mathfrak{F})$ be an HHS. Then there exists $N \in \mathbb{N}$, depending only on the HHS constants, such that the following holds. Let $x, y \in \mathcal{X}$. Let $C \geqslant 100E$, and consider the set $\mathrm{Rel}_C(x,y)$ of $V \in \mathfrak{F}$ such that $\mathsf{d}_V(x,y) \geqslant C$. Then for all $U \in \mathrm{Rel}_C(x,y)$, there are at most N elements $V \in \mathrm{Rel}_C(x,y)$ such that $U \sqsubseteq V$.*

Proof. Fix U as in the statement and let $\mathcal{V}$ be the set of $V \in \mathrm{Rel}_C(x,y)$ such that $U \subsetneq V$. We want to bound $|\mathcal{V}|$ uniformly. First, observe that any $\sqsubseteq$–chain in $\mathcal{V}$ has length at most χ. In other words, if $\mathcal{V}'$ is a set of pairwise $\sqsubseteq$–related distinct elements of $\mathcal{V}$, then $|\mathcal{V}'| \leqslant \chi$. Any set of pairwise orthogonal elements has cardinality at most χ by Lemma 11.2.

Now suppose that $V_1, V_2, V_3 \in \mathcal{V}$ and suppose that $V_i \pitchfork V_j$ for all $i \neq j$ with $i, j \in \{1,2,3\}$. Since $C > E$, the consistency axiom implies that, up to relabelling, we have $\mathsf{d}_{V_2}(\rho^{V_1}_{V_2}, x) \leqslant E$ and $\mathsf{d}_{V_2}(\rho^{V_3}_{V_2}) \leqslant E$. Hence $\mathsf{d}_{V_2}(\rho^{V_1}_{V_2}, \rho^{V_3}_{V_2}) \geqslant C - 4E > E$, which contradicts Definition 10.1.(4) since $U \subsetneq V_1, U \subsetneq V_2, U \subsetneq V_3$. Hence subsets of $\mathcal{V}$ whose elements are pairwise transverse have cardinality at most 2.

By Ramsey's theorem [Ram29], with $N = \mathrm{Ram}(\chi+1, \chi+1, 3) - 1$ (where Ram denotes the Ramsey number), we have $|\mathcal{V}| \leqslant N$, since any two elements of $\mathcal{V}$ are either $\sqsubseteq$–related, orthogonal, or transverse. $\square$

We also have the following lemma. It follows from the distance formula, but one does not need the distance formula to prove it (and in fact it plays a role in the proof of the distance formula). So, we include a proof for illustrative purposes:

Lemma 11.4. *Let $C \geqslant E$. Then for all $x, y \in \mathcal{X}$, the set $\mathrm{Rel}_C(x,y)$ is finite.*

Proof. Given $U \in \mathfrak{F}$, let $\mathcal{R}^U$ be the set of $V \in \mathrm{Rel}_C(x,y)$ with $V \subsetneq U$.

Letting S be the unique $\sqsubseteq$–maximal element of $\mathfrak{F}$, we have by the large link axiom that there exist $T_1, \ldots, T_k \subsetneq S$ such

$$\mathrm{Rel}_C(x,y) \subset \{S\} \cup \bigcup_{i=1}^{k} \mathcal{R}^{T_i}.$$

By induction on the maximum length of a $\sqsubseteq$–chain terminating at T_i (which is strictly less than the corresponding number for S, namely χ), each $\mathcal{R}^{T_i}$ is finite, and we are done. (In the base case, T_i is $\sqsubseteq$–minimal, and $|\mathcal{R}^{T_i}| \leqslant 1$.) $\square$

We record two more important combinatorial facts about the set $\mathrm{Rel}_C(x,y)$ for C sufficiently large. The first is used later in the paper, where it is instrumental in the construction of the *diagonal decomposition* of a sequence of elements. The second, related, fact is not used directly, but we include it because it is instrumental in the original proof of the distance formula (Theorem 10.7), and it is helpful for understanding the *cubical approximation theorem*, discussed below.

Definition 11.5 (Level)**.** The *level* of $U \in \mathfrak{F}$ is the length of a longest $\sqsubseteq$–chain having U as the maximal element, and is denoted $\mathrm{Level}(U)$. Note that $\mathrm{Level}(U) \leqslant \chi$ for all U. □

Lemma 11.6 (Partial order on relevant elements)**.** *For all $C \geqslant 100E$, the following holds. Let $x, y \in \mathcal{X}$ and let $\mathrm{Rel}_C(x,y)$ be the set of $V \in \mathfrak{F}$ with $\mathsf{d}_V(x,y) \geqslant C$. Let $\mathcal{V} \subset \mathrm{Rel}_C(x,y)$. Define a relation $\prec$ on $\mathcal{V}$ by $U \preceq V$ if $V = U$, or $V \pitchfork U$ and $\mathsf{d}_V(x, \rho^U_V) \leqslant E$. Then $\preceq$ is a partial order on $\mathcal{V}$. Moreover, $U, V \in \mathrm{Rel}_C(x,y)$ are either orthogonal, $\sqsubsetneq$–related, or $\preceq$–comparable.*

Proof. This is Proposition 2.8 in [BHS19], and is illustrated in Figure 11. □

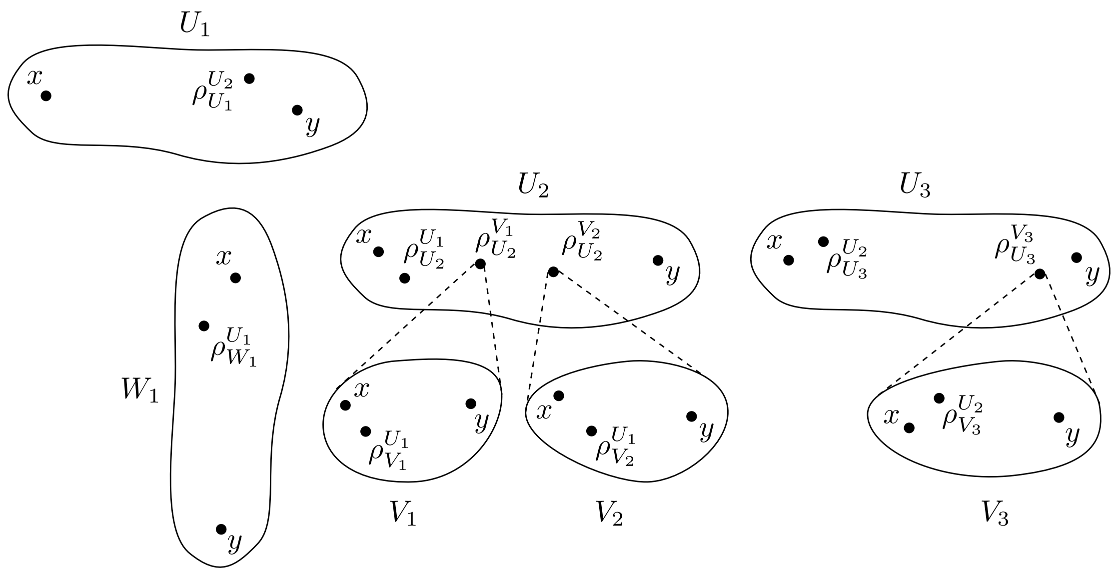

FIGURE 18. A $\preceq$–example. Here, $\mathrm{Rel}_C(x,y) = \{U_1, U_2, U_3, V_1, V_2, V_3, W_1\}$. We have that $W_1 \perp U_2, U_3$ and $V_1, V_2 \sqsubseteq U_2$ and $V_3 \sqsubseteq U_3$. Hence $W_1 \perp V_i$ for all i. The other pairs are transverse, and some $\rho^{\bullet}_{\bullet}$ points are shown. We have, for example, $U_1 \prec U_2 \prec U_3$, and $U_1 \prec W_1$, and $V_1 \prec V_2 \prec U_3$. While $V_2 \prec U_3$ and $V_2 \prec V_3$, the elements U_3, V_3 are not transverse and hence $\prec$–incomparable.

We could have equivalently defined $\prec$ by asking $\mathsf{d}_U(y, \rho^V_U) \leqslant E$, because of consistency. The partial order intuitively means that, when moving along any hierarchy path from x to y, one must first change the U coordinate before one can change the V coordinate.

For each $\ell \leqslant \chi$, and any $C \geqslant 100E$, we can consider the set of $V \in \mathrm{Rel}_C(x,y)$ with $\mathrm{Level}(V) = \ell$, denoted $\mathrm{Rel}^\ell_C(x,y)$.

The following lemma is proved in [BHS19, Section 2], but we reproduce the proof here, because it is a nice quick application of Dilworth's theorem.

Lemma 11.7 (Colouring of relevant elements)**.** *Let $C \geqslant 100E$, let $\ell \leqslant \chi$, and let $x, y \in \mathcal{X}$. Then there exists $n \leqslant \chi$ and disjoint subsets $\mathcal{V}_1, \dots, \mathcal{V}_n$ of $\mathrm{Rel}^\ell_C(x,y)$ such that*

- $\bigsqcup_i \mathcal{V}_i = \mathrm{Rel}^\ell_C(x,y)$;
- *the elements of $\mathcal{V}_i$ are pairwise transverse, and in particular form a $\prec$–chain, for each $i \leqslant n$.*

Hence $\mathrm{Rel}_C(x,y)$ can be partitioned into at most χ^2 $\prec$–chains.

Proof. The "hence" part is immediate from the first part of the lemma. So, it suffices to do the colouring for a fixed $\ell \leqslant \chi$.

But $\mathrm{Rel}^{\ell}_{C}(x,y)$ is partially ordered by $\preceq$ in such a way that transverse elements are comparable, and incomparable elements are orthogonal. Hence, by Lemma 11.2, antichains have cardinality bounded by χ, so the claim follows from Dilworth's theorem [Dil50]. $\square$

The partial order and the colouring will play a role later in this section, in our discussion of the cubical approximation theorem, which essentially transforms the partial order $\prec$ on the set of relevant projections for a pair of points into the usual partial order on halfspaces in an appropriately chosen finite CAT(0) cube complex.

12. Coarse median operator on an HHS

Let $(\mathcal{X},\mathfrak{F})$ be a hierarchically hyperbolic space. Here we recall the *coarse median* structure on $\mathcal{X}$. Coarse median spaces are a common generalisation of hyperbolic spaces and CAT(0) cube complexes introduced by Bowditch in [Bow13] and play a fundamental role in hierarchically hyperbolic spaces.

We do not need the general definition of a coarse median space here, just the definition of the coarse median in the HHS context. This closely parallels the construction of the median operator on an ℝ–cubing.

Definition 12.1 (Coarse median coordinates)**.** Let $x,y,z\in\mathcal{X}$ and let $W\in\mathfrak{F}$. Consider any geodesic triangle in the E–hyperbolic space $\mathcal{C}W$ formed by geodesics between the points $\pi_W(x),\pi_W(y),\pi_W(z)$. Let $\mu_W(x,y,z)\in\mathcal{C}W$ be any point lying E–close to each of the three geodesics. $\square$

Now consider the tuple $(\mu_W(x,y,z))_{W\in\mathfrak{F}}$. Lemma 2.6 of [BHS19] shows that this tuple is $100E$–consistent. So the realisation theorem, Theorem 10.5, provides a point $\mu=\mu(x,y,z)\in\mathcal{X}$ such that $\mathsf{d}_W(\mu,\mu_W)\leqslant 100r_0E$ for all $W\in\mathfrak{F}$.

Define $\mu:\mathcal{X}^3\to\mathcal{X}$ by $(x,y,z)\mapsto\mu(x,y,z)$. The choice of $\mu(x,y,z)$ relied on two sets of choices, namely the choice inherent in the various μ_W, and the choice in output of the realisation theorem. However, the choice of μ_W is well-defined up to distance bounded in terms of E, and hence, by the uniqueness axiom, μ is well-defined up to uniformly bounded error.

Moreover, if $x,y,z\in\mathcal{X}$ and σ is any permutation of $\{x,y,z\}$, then the points $\mu(x,y,z)$ and $\mu(\sigma(x),\sigma(y),\sigma(z))$ uniformly coarsely coincide (i.e. lie at distance bounded in terms of the HHS structure only). So we will not make a big deal of the order of x,y,z in the coarse median operation.

Theorem 7.3 in [BHS19] says that the *coarse median operator* μ makes $\mathcal{X}$ into a coarse median space.

If a group G acts on $(\mathcal{X},\mathfrak{F})$ freely by coarse median automorphisms, and $\mu:\mathcal{X}^3\to X$ is the coarse median operator defined above, we can always assume that $\mu(gx,gy,gz)=g\mu(x,y,z)$ for all $g\in G, x,y,z\in\mathcal{X}$.

Indeed, fix $x,y,z\in\mathcal{X}$ and let $W\in\mathfrak{F}$. Then

$$\mathsf{d}_{gW}(\mu_W(gx,gy,gz),g\mu_W(x,y,z))\leqslant 100E,$$

because, by Definition 10.9, the isometry $g:\mathcal{C}W\to\mathcal{C}gW$ takes any geodesic triangle with vertices $\pi_W(x),\pi_W(y),\pi_W(z)$ to a geodesic triangle with vertices $\pi_{gW}(gx),\pi_{gW}(gy),\pi_{gW}(gz)$, so computing the coarse median in $\mathcal{C}W$ and applying g is uniformly coarsely the same as computing the coarse median in $\mathcal{C}gW$. Hence, by the uniqueness axiom, there exists $C=C(E)$ such that $\mathsf{d}_{\mathcal{X}}(g\mu(x,y,z),\mu(gx,gy,gz))\leqslant C$.

We now choose one ordered triple (x,y,z) in each G–orbit, and let $\mu(x,y,z)$ be as defined above, and then define $\mu(gx,gy,gz)=g\mu(x,y,z)$ for all $g\in G$. Since this is only a bounded perturbation of the original coarse median, it is still a coarse median operator.

In particular, given an HHG $(G, \mathfrak{F})$, we will denote by $\mu : G^3 \to G$ the coarse median operator, which has the following two properties:

- $g\mu(x, y, z) = \mu(gx, gy, gz)$ for all $g, x, y, z \in G$;
- for all $W \in \mathfrak{F}$ and $x, y, z, \in G$, the point $\pi_W(\mu(x, y, z))$ lies C_0–close to any geodesic joining $\pi_W(a), \pi_W(b)$ whenever $a, b \in \{x, y, z\}$ are distinct; the constant C_0 depends only on E.

Beyond that, we will only use a few additional properties of coarse media from the literature, which we will mention as they are used.

Remark 12.2 (Equivariance and permutation-invariance simultaneously)**.** At the expense of uniformly increasing constants, we can perturb $\mu : \mathcal{X}^3 \to \mathcal{X}$ so that $\mu(x, y, z) = \mu(\sigma(x), \sigma(y), \sigma(z))$ and $\mu(x, x, z) = x$ whenever σ is a permutation of $\{x, y, z\}$. In practice, we do not ever need exact permutation-invariance of this type.

When $(\mathcal{X}, \mathfrak{F})$ has a free action by a group G of HHS automorphisms, we *are* often interested in μ being exactly (not just coarsely) equivariant, as arranged above.

In general, one cannot arrange this simultaneously with the perturbation needed to make μ invariant under permutation of its three arguments. However, one *can* do this provided the action of G on the set of *unordered* triples in $\mathcal{X}$ is free, i.e. if G doesn't have subgroups embedding in S_3. The reader really wishing for a G–equivariant *and* permutation-invariant coarse median is invited, in all later sections, to assume our HHGs are torsion-free and to replace mapping class groups everywhere with a fixed torsion-free finite-index subgroup. □

13. Hierarchical quasiconvexity and hulls

We now introduce the notion of (quasi)convexity appropriate to HHSes, following [BHS19, Section 5]. The reader may notice that it is a "coarse-ification" of the notion of convexity in an $\mathbb{R}$–cubing. Fix an HHS $(\mathcal{X}, \mathfrak{F})$.

Definition 13.1 (Hierarchical quasiconvexity)**.** Let $\kappa : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be a function. Let $\mathcal{Y} \subset \mathcal{X}$. Then $\mathcal{Y}$ is *κ–hierarchically quasiconvex* if both of the following hold:

- for all $U \in \mathfrak{F}$, the subset $\pi_U(\mathcal{Y})$ is $\kappa(0)$–quasiconvex in $\mathcal{C}U$;
- for all $t \in \mathbb{N}$ there exists $s = \kappa(t) \in \mathbb{N}$ such that the following holds for all $x \in \mathcal{X}$: if $\mathsf{d}_U(x, \mathcal{Y}) \leqslant t$ for all $U \in \mathfrak{F}$, then $\mathsf{d}_{\mathcal{X}}(x, \mathcal{Y}) \leqslant s$.

Occasionally, when κ is not important, we will say that $\mathcal{Y}$ is *hierarchically quasiconvex* if the above two properties hold for some function κ. □

Just as convexity in an $\mathbb{R}$–cubing is closely related to median-convexity, a theorem of Russell-Spriano-Tran relates hierarchical quasiconvexity to quasiconvexity in the coarse median sense. Recall that $\mu : \mathcal{X}^3 \to \mathcal{X}$ denotes the coarse median from Section 12.

Definition 13.2 (Coarse median quasiconvexity)**.** Let $Q \geqslant 0$. A subset $\mathcal{Y} \subset \mathcal{X}$ is *Q–median quasiconvex* if for all $y, y' \in \mathcal{Y}$ and $x \in \mathcal{X}$, we have $\mathsf{d}_{\mathcal{X}}(\mu(x, y, y'), \mathcal{Y}) \leqslant Q$. □

From [RST18, Proposition 5.11], we obtain:

Proposition 13.3. *Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space and let $\mathcal{Y} \subset \mathcal{X}$. Then the following are equivalent:*

- *there exists Q such that $\mathcal{Y}$ is Q–median convex;*
- *there exists κ such that $\mathcal{Y}$ is κ–hierarchically quasiconvex.*

Moreover, Q depends only on κ and the HHS structure, and κ depends only on Q and the HHS structure.

The preceding proposition is extremely useful, because, depending on the situation, either the definition of hierarchical quasiconvexity using the HHS structure or the coarse median characterisation may be much easier to work with than the other.

Given a notion of "convexity", one should have a notion of "convex hull". This is achieved by the following, which is Definition 6.1 of [BHS19]:

Definition 13.4 (Hull)**.** Let $\theta \geqslant 0$. Given $A \subset \mathcal{X}$, let $H_\theta(A)$ be the set of all $x \in \mathcal{X}$ such that, for all $W \in \mathfrak{F}$, the point $\pi_W(x)$ lies θ–close to a geodesic in $\mathcal{C}W$ joining two points in $\pi_W(A)$. □

Lemma 6.3 of [BHS19] provides a **constant** θ_0, depending only on the HHS constants, such that for all $\theta \geqslant \theta_0$, there is a function κ_0 such that $H_\theta(A)$ is κ_0–hierarchically quasiconvex for any A. We will *frequently* refer to the constant θ_0 and the associated function κ_0.

We direct the reader to [RST18] for alternate characterisations of the hull in terms of hierarchy paths.

14. Gates

Recall that median convex subsets of a median space admit gate maps, which are 1–lipschitz retractions, and indeed coincide with closest-point projection. Here we discuss the appropriate coarse-ification of this notion in the HHS context.

Let $(\mathcal{X}, \mathfrak{F})$ be an HHS and let $\mathcal{Y} \subset \mathcal{X}$ be κ–hierarchically quasiconvex, where κ is some arbitrary hierarchical quasiconvexity function.

Definition 14.1 (Gate tuple)**.** Let $x \in \mathcal{X}$. Fix $W \in \mathfrak{F}$. Recall that $\pi_W(\mathcal{Y})$ is $\kappa(0)$–quasiconvex in $\mathcal{C}W$. Accordingly, we have a uniformly coarsely lipschitz coarse projection $p_W : \mathcal{C}W \to \pi_W(\mathcal{Y})$ given by

$$p_W(a) = \{b \in \pi_W(\mathcal{Y}) : \mathsf{d}_W(a, b) \leqslant \mathsf{d}_W(a, \pi_W(\mathcal{Y})) + 1\},$$

which sends points in $\mathcal{C}W$ to nonempty sets of diameter bounded in terms of $\kappa(0)$ and E. Let $b_W(x)$ be an arbitrary point in $p_W(\pi_W(x))$, for each $W \in \mathfrak{F}$. □

Lemma 5.5 of [BHS19] produces a constant C_1, depending on E and κ, such that the tuple $(b_W(x))_{W \in \mathfrak{F}}$ is C_1–consistent. Applying realisation (Theorem 10.5) and hierarchical quasiconvexity of $\mathcal{Y}$ yields a point $\mathfrak{g}_{\mathcal{Y}}(x) \in \mathcal{Y}$ such that $\mathsf{d}_W(b_W(x), \mathfrak{g}_{\mathcal{Y}}(x)) \leqslant C_2$ for all W, where C_2 depends on κ, E, and C_1.

Accordingly, we have a *gate map* $\mathfrak{g}_{\mathcal{Y}} : \mathcal{X} \to \mathcal{Y}$ with the following properties:

- $\mathfrak{g}_{\mathcal{Y}}$ is an (L_1, L_1)–coarsely lipschitz L_1–coarse retraction, with L_1 depending only on the HHS constants and the function κ.
- For all $U \in \mathfrak{F}$ and all $x \in \mathcal{X}$, the set $\pi_U(x)$ uniformly coarsely coincides with the image of $\pi_U(x)$ under coarse closest-point projection to the quasiconvex subspace $\pi_U(\mathcal{Y})$. (Here, "uniformly" means "dependent on $(\mathcal{X}, \mathfrak{F})$ but independent of U and x".)

Remark 14.2 (Iterated gates)**.** If $\mathcal{Y} \subset \mathcal{Z}$ are hierarchically quasiconvex subsets, then $\mathfrak{g}_{\mathcal{Y}}(x)$ lies at bounded distance from $\mathfrak{g}_{\mathcal{Y}}(\mathfrak{g}_{\mathcal{Z}}(x))$, where the bound depends only on the hierarchical quasiconvexity functions of $\mathcal{Y}, \mathcal{Z}$ and the HHS constants. □

A great deal of further discussion of the gate map can be found in [BHS17c, Section 1]. We will introduce various other properties of gates as needed. For example, later we will see that in an HHG $(G, \mathfrak{F})$, given a hierarchically quasiconvex subset $\mathcal{Y}$, it is possible to make $\mathfrak{g}_{\mathcal{Y}}$ a $\mathrm{Stab}_G(\mathcal{Y})$–equivariant map, using a bounded perturbation.

15. Standard product regions

One of the most important notions in the HHS world is that of *standard product regions*. One intuitive idea the reader should have in mind is: in an HHS, there are various hierarchically quasiconvex subspaces, each of which is coarse-median-preservingly quasi-isometric to the product of "simpler" hierarchically hyperbolic spaces. These are *standard product regions*, and the ambient HHS is weakly hyperbolic relative to them — indeed, coning them off yields a space quasi-isometric to $\mathcal{C}S$, where $S \in \mathfrak{F}$ is the unique $\sqsubseteq$–maximal element.

We recall from [BHS19, Section 5] the notion of a standard product region. We will formulate things somewhat more explicitly than in [BHS19], give a bit more detail, and correct two misstatements.

Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space. Fix $U \in \mathfrak{F}$. Recall that $\mathfrak{F}_U$ is the set of $V \in \mathfrak{F}$ with $V \sqsubseteq U$, and let $\mathfrak{F}_U^\perp$ be the set of V with $V \perp U$.

Let $\kappa \geqslant E$, and let

- $F_U{}^\kappa$ be the set of tuples $(p_V)_{V \in \mathfrak{F}_U} \in \prod_{V \in \mathfrak{F}_U} \mathcal{C}V$ that are κ–consistent, and let
- $E_U{}^\kappa$ be the corresponding set of κ–consistent tuples, with $\mathfrak{F}_U$ replaced by $\mathfrak{F}_U^\perp$.

For each $(\vec{p}, \vec{q}) \in F_U^\kappa \times E_U^\kappa$, we obtain a $(\kappa + E)$–consistent tuple $f(\vec{p}, \vec{q}) \in \prod_{V \in \mathfrak{F}} \mathcal{C}V$ whose V–coordinate is p_V if $V \sqsubseteq U$, and q_V if $V \perp U$, and ρ_V^U otherwise.

(The consistency of the tuple $f(\vec{p}, \vec{q})$ is verified in [BHS19, Construction 5.10]. This relies on [BHS19, Proposition 1.8] (the source of the $\kappa + E$), which is a coarse analogue of Lemma 4.12 about real cubings, and which essentially depends on the partial realisation axiom. Moreover, nonemptiness of F_U^κ and E_U^κ also relies on the partial realisation and consistency axioms.)

So, Theorem 10.5 provides $x \in \mathcal{X}$ such that for all $V \in \mathfrak{F}$, we have $\mathsf{d}_V(x, f_V) \leqslant r_0(\kappa + E)$, where f_V is the V–coordinate of $f(\vec{p}, \vec{q})$.

This gives a map $f : F_U^\kappa \times E_U^\kappa \to \mathcal{X}$ whose image we denote by $P_U{}^\kappa$. We recall the following from [BHS19, Section 5]:

Lemma 15.1. *P_U^κ is hierarchically quasiconvex, with hierarchical quasiconvexity function depending only on κ and $(\mathcal{X}, \mathfrak{F})$ and the associated constant E.*

Moreover, if $V \pitchfork U$ or $U \subsetneq V$, then $\mathsf{diam}(\pi_V(P_U^\kappa) \cup \rho_V^U)$ is bounded above by a constant depending only on κ and the HHS structure (but independent of U).

Proof. Let $x, y \in P_U^\kappa$. Then by construction, for all V as in the "moreover" part of the statement, we have that $\pi_V(x)$ and $\pi_V(y)$ are both $r_0(\kappa + E)$–close to ρ_V^U. This proves the "moreover" assertion.

We now show that for all $V \sqsubseteq U$ or $V \perp U$, we have that $\pi_V(P_U^\kappa)$ uniformly coarsely coincides with the uniformly quasiconvex set $\pi_V(\mathcal{X})$, and is therefore uniformly quasiconvex.

Indeed, fix such a V and consider $\pi_V(a)$ for some $a \in \mathcal{X}$. Then $\{\pi_W(a) : W \in \mathfrak{F}\}$ is an E–consistent tuple. Form a tuple $\vec{a}$ by changing the $\mathcal{C}W$–coordinate of the preceding tuple to ρ_W^U whenever $U \subsetneq W$ or $U \pitchfork W$. Restricting $\vec{a}$ to $\mathfrak{F}_U$ and $\mathfrak{F}_U^\perp$ gives κ–consistent tuples $\vec{a}_1 \in F_U^\kappa$ and $\vec{a}_2 \in E_U^\kappa$. By construction, $\pi_V(f(\vec{a}_1, \vec{a}_2))$ is uniformly close to $\pi_V(a)$, as required.

Now fix a constant L and suppose that $a \in \mathcal{X}$ satisfies $\mathsf{d}_V(a, P_U^\kappa) \leqslant L$ for all $V \in \mathfrak{F}$. Then $(\pi_V(a))_{V \in \mathfrak{F}}$ is an E–consistent tuple. Let $\vec{b}$ be the tuple obtained by changing the V–coordinate to ρ_V^U for $V \pitchfork U$ or $U \subsetneq V$. Then $\vec{b}$ is a $(\kappa + E)$–consistent tuple, and there exists $x \in P_U^\kappa$ such that $\pi_V(x)$ is $r_0(\kappa + E)$–close to the V–coordinate of $\vec{b}$ for all V. On the other hand, $\mathsf{d}_V(a, x) \leqslant L + r_0(\kappa + E)$ for all V, so by Theorem 10.7, $\mathsf{d}_{\mathcal{X}}(a, x)$ is bounded uniformly in terms of L, κ, i.e. a is close to P_U^κ. Thus P_U^κ is hierarchically quasiconvex. $\square$

Fix once and for all a constant κ as above (we can take $\kappa = E$), and let $P_U = P_U^\kappa$. Fix an arbitrary basepoint $x_0 = f(\vec{p}_0, \vec{q}_0)$. Consider the set $F_U^\kappa \times \{\vec{q}_0\}$, and let $F_U = f(F_U^\kappa \times \{\vec{q}_0\})$.

Similarly, let $E_U = f(\{\vec{p}_0\} \times E_U^\kappa)$. Arguing exactly as in the proof of Lemma 15.1 shows that E_U and F_U are hierarchically quasiconvex, with hierarchical quasiconvexity function depending only on κ and E. In particular, this function is independent of the basepoint $f(\vec{p}_0, \vec{q}_0)$.

We metrise F_U, E_U by equipping them with the subspace metric coming from $\mathcal{X}$.

Remark 15.2 (Gates in product regions)**.** We denote by $\kappa^\times$ a function, depending only on the HHS constants, such that each P_U, E_U, F_U is $\kappa^\times$–hierarchically quasiconvex.

Let $\mathfrak{g}_U : \mathcal{X} \to P_U$ denote the gate map. Then (up to uniform enlargement of $\kappa^\times(0)$), we have that, for all $V \in \mathfrak{F}$ and all $x \in \mathcal{X}$, the point $\pi_V(\mathfrak{g}_U(x))$ is $\kappa^\times(0)$–close to:

- ρ_V^U if $U \sqsubsetneq V$ or $U \pitchfork V$;
- $\pi_V(x)$ if $U \perp V$ or $U \sqsubseteq V$.

Similarly, for any choice of $\vec{q}_0 \in F_U^\kappa$, letting $\mathfrak{g}_{F_U} : \mathcal{X} \to F_U$, we have for all $x \in \mathcal{X}, V \in \mathfrak{F}$ that $\pi_V(\mathfrak{g}_{F_U}(x))$ is $\kappa^\times(0)$–close to:

- ρ_V^U if $U \sqsubsetneq V$ or $U \pitchfork V$;
- $\pi_V(x)$ if $V \sqsubseteq U$;
- $\pi_V(\vec{q}_0)$ if $V \perp U$.

A similar statement about the gate map to $E_U = f(\vec{p}_0 \times E_U^\kappa)$ holds, except in the second bullet point, $\perp$ replaces $\sqsubseteq$, and in the third bullet point, $\sqsubseteq$ replaces $\perp$ and $\pi_V(\vec{p}_0)$ replaces $\pi_V(\vec{q}_0)$. □

As in Proposition 5.11 of [BHS19], F_U and E_U are hierarchically hyperbolic spaces. In general, the statement of Proposition 5.11 is not quite accurate in the case of E_U; here is the corrected version:

Proposition 15.3. *Let $\mathfrak{F}_U$ be the set of $V \in \mathfrak{F}$ with $V \sqsubseteq U$. Let $A \in \mathfrak{F}$ be such that every $V \in \mathfrak{F}$ with $V \perp U$ satisfies $V \sqsubseteq A$ and A is not the unique $\sqsubseteq$–maximal element of $\mathfrak{F}$. Then:*

1. *$(F_U, \mathfrak{F}_U)$ is an HHS, with constants depending only on E.*
2. *$(F_U, \mathfrak{F})$ is an HHS, with constants depending only on E.*
3. *$(E_U, \mathfrak{F})$ is an HHS, with constants depending only on E. Moreover, for all $V \in \mathfrak{F}$ with $V \not\perp U$, the diameter of $\pi_V(E_U)$ is bounded in terms of E.*
4. *$(E_U, \mathfrak{F}_A)$ is an HHS, with constants depending only on E. Moreover, the complexity is strictly lower than that of $(\mathcal{X}, \mathfrak{F})$.*

Proof. The assertions about $(F_U, \mathfrak{F})$ and $(E_U, \mathfrak{F})$ follow from hierarchical quasiconvexity and [BHS19, Proposition 5.6]. The assertion about $(F_U, \mathfrak{F}_U)$ follows exactly as in the proof of Proposition 5.11 of [BHS19]. In particular, $(F_A, \mathfrak{F}_A)$ is an HHS, and E_U is uniformly quasi-isometric to a hierarchically quasiconvex subset of F_A. Hence, by [BHS19, Proposition 5.6], $(E_U, \mathfrak{F}_A)$ is an HHS. The statement about complexity follows since A is not $\sqsubseteq$–maximal in $\mathfrak{F}$. □

Remark 15.4 (Visitor from the past and future: clean containers)**.** In most of the natural examples (e.g. mapping class groups and fundamental groups of compact special cube complexes), the HHS structure satisfies a stronger version of the orthogonality axiom called *clean containers.*

This says that for all $V \in \mathfrak{F}$, and all $U \sqsubseteq V$, if there exists $W \sqsubseteq V$ such that $W \perp U$, then there exists $T \sqsubsetneq V$ such that $T \perp U$, and if $W \sqsubseteq V$ and $W \perp U$, then $W \sqsubseteq T$.

First formalised in [ABD21], this property will often be assumed in later sections, and is defined again later, where we start to use it. This property implies in particular that for each $U \in \mathfrak{F}$, there is a unique $U^\perp \in \mathfrak{F}$ such that $V \perp U$ if and only if $V \sqsubseteq U^\perp$. In this situation, Proposition 5.11 from [BHS19] holds as written, and in fact $\mathfrak{F}_A = \mathfrak{F}_{U^\perp}$ and $E_U = F_{U^\perp}$.

The reader should compare the notion of clean containers in an HHS structure to the identical notion for real cubings from Definition 4.40. □

Remark 15.5 (Visitor from the future: wedges)**.** As for real cubings, one can ask that an HHS $(\mathcal{X},\mathfrak{F})$ has *wedges*, i.e. if $U,V\in\mathfrak{F}$ have the property that there is some W with $W\sqsubseteq U, W\sqsubseteq V$, then there is a unique W that is $\sqsubseteq$–maximal with this property. This is completely analogous to the notion for real cubings (Definition 4.39). We will not need the notion of wedges in the next few sections — though some of the questions in Section 21 mention wedges — but we will use this notion (and remind the reader about it) in Section 3. □

Remark 15.6. Elsewhere in the literature, one works mainly with $(F_U,\mathfrak{F}_U)$, so the incorrect statement of Proposition 5.11 in [BHS19] does not have serious effects. We take the opportunity to point out the places in the literature where one should really use Proposition 15.3 above instead:

- In [BHS17b], following Remark 3.5, one should use $(E_U,\mathfrak{F}_A)$ rather than $(E_U,\mathfrak{F}_U^{\perp})$.
- In [BHS17a], on page 14, one should again use $(E_U,\mathfrak{F}_A)$ rather than $(E_U,\mathfrak{F}_U^{\perp})$. The point is that the sets $\mathfrak{F}_A$ and $\mathfrak{F}_U^{\perp}$ differ only on elements U in which E_U has bounded image in $\mathcal{C}U$, so passing from the latter to the former has no geometric effect. However, $\mathfrak{F}_A$ satisfies the orthogonality axiom, while $\mathfrak{F}_U^{\perp}$, as defined in Section 5 of [BHS19], might not.
- In the proof of Theorem 3.4 of [DHS17], $(E_U,\mathfrak{F}_A)$ should be used, to obtain a lower-complexity HHS structure.
- In several places in [DHS17], one has a group G acting on an HHS $(\mathcal{X},\mathfrak{F})$ by HHS automorphisms, and one wishes to consider an action of $\mathrm{Stab}_G(U)$ on E_U by HHS automorphisms. The question is which index set to use for the HHS structure on E_U. In all but two places, it suffices to use $(E_U,\mathfrak{F})$ as in Proposition 15.3 instead of the HHS structure mentioned in Section 1.3 of [DHS17]. In the proofs of Theorem 9.13 and Theorem 9.20, we wish to have $\mathrm{Stab}_G(U)$ act on an HHS with underlying space E_U and strictly lower complexity. There are ways to correct this using an application of [DHS17, Proposition 9.2], but we note that those two theorems have recently been given a simpler proof by Petyt and Spriano [PS20] and so do not pursue the matter further here.

All the above proofs work as written under the extra assumption of clean containers. □

By the previous proposition and Proposition 8.27 of [BHS19], $F_U\times E_U$ is a hierarchically hyperbolic space, where the index set has the property that every element whose associated hyperbolic space is not a single point belongs to $\mathfrak{F}_U\sqcup\mathfrak{F}_U^{\perp}$, and each $F_U\times\{e\},\{f\}\times E_U$ is uniformly hierarchically quasiconvex.

The metric on $F_U\times E_U$ is the ℓ_1–metric (where the factors still have the subspace metric from $\mathcal{X}$), and the coarse median $\mu_\times$ on $F_U\times E_U$ is given by (μ_F,μ_E), where μ_F,μ_E are the coarse media on F_U,E_U coming from their hierarchically hyperbolic structures.

Define a map $\phi: F_U\times E_U\to\mathcal{X}$ as follows. Given $x\in F_U, y\in E_U$, let $\vec{x}\in F_U^{\kappa}$ be $(\pi_V(x))_V$ and let $\vec{y}\in E_U^{\kappa}$ be $(\pi_V(y))_V$. Let $\phi(x,y)=f(\vec{x},\vec{y})$.

Proposition 15.7. *There exist constants C_1,C_2,C_3, depending on E but independent of U and x_0, such that ϕ is C_1–quasimedian (C_2,C_2)–quasi-isometric embedding whose image is at Hausdorff distance C_3 from P_U.*

Remark 15.8 (Quasimedian)**.** By C_1–*quasimedian*, we mean that for any triple in $F_U\times E_U$, the coarse median is taken C_1–close to the coarse median of the images of the points in the triple. In [Bow13], such maps are called *quasimorphisms*, but this word has another common meaning, so we changed terminology. □

Proof of Proposition 15.7. Let $(x,y),(s,t),(w,z)\in F_U\times E_U$. Let $m\in F_U\times E_U$ be the coarse median of these three points. Then for each $V\sqsubseteq U$, the V–coordinate of m is the coarse median of $\pi_V(x),\pi_V(s),\pi_V(w)$, and for $V\perp U$, the V–coordinate of m is the coarse median of $\pi_V(y),\pi_V(t),\pi_V(z)$, where coarse medians are taken in the hyperbolic spaces $\mathcal{C}V$. (Here, we are working in the hierarchically hyperbolic structure on $F_U\times E_U$, where the projection of (x,y) to $\mathcal{C}V, V\sqsubseteq U$ is $\pi_V(x)$ and the projection to $\mathcal{C}W, W\perp U$ is $\pi_W(y)$.)

Now, for each $V\sqsubseteq U$, we have that $\pi_V(\phi(x,y))$ is $r_0(\kappa+E)$–close to $\pi_V(x)$, and for $W\perp U$, $\pi_W(\phi(x,y))$ is $r_0(\kappa+E)$–close to $\pi_W(y)$. For all other W, $\pi_W(\phi(x,y))$ is $r_0(\kappa+E)$–close to ρ^U_W. Analogous facts hold for (s,t) and (w,z).

So, the coarse median m' (in $\mathcal{X}$) of $\phi(x,y),\phi(s,t),\phi(w,z)$ has the following V–coordinates, for $V\in\mathfrak{F}$:

- If $V\sqsubseteq U$, then $\pi_V(m')$ is C_1'–close to the geodesic from $\pi_V(x)$ to $\pi_V(s)$, and similarly when either of x or s is replaced by w; here $C_1'=C_1'(r_0(\kappa+E),E)$.
- If $V\perp U$, then the same holds, but with x,s,w replaced by y,t,z.
- Otherwise, $\pi_V(m')$ is $r_0(\kappa+E)$–close to ρ^U_W.

Hence, by the distance formula (Theorem 10.7), there exists C_1 depending only on C_1' such that $\mathsf{d}_{\mathcal{X}}(m',\phi(m))\leqslant C_1$, i.e. ϕ is C_1–quasimedian.

Next, we show that ϕ is a quasi-isometric embedding. Let s_0 be as in Theorem 10.7 (i.e. the smallest valid distance formula threshold) and let $\zeta\geqslant s_0$ be a constant to be determined.

Let $(x,y),(s,t)\in F_U\times E_U$. Since $x,s\in F_U$, we have $\mathsf{d}_V(x,x_0)\leqslant r_0(\kappa+E)$ if $V\perp U$ and $\mathsf{d}_V(x,\rho^U_V)\leqslant r_0(\kappa+E)$ if $V\pitchfork U$ or $U\subsetneq V$. So, letting $\zeta\geqslant 10r_0(\kappa+E)$, we have $A,B\geqslant 1$ (depending only on ζ) so that, by the distance formula,

$$\mathsf{d}_{F_U}(x,s)\asymp_{A,B}\sum_{V\sqsubseteq U,\mathsf{d}_V(x,s)\geqslant\zeta}\mathsf{d}_V(x,s).$$

Similarly,

$$\mathsf{d}_{E_U}(y,t)\asymp_{A,B}\sum_{V\perp U,\mathsf{d}_V(y,t)\geqslant\zeta}\mathsf{d}_V(y,t).$$

Hence

$$\mathsf{d}_{F_U\times E_U}((x,y),(s,t))\asymp_{A,2B}\sum_{V\sqsubseteq U,\mathsf{d}_V(x,s)\geqslant\zeta}\mathsf{d}_V(x,s)+\sum_{V\perp U,\mathsf{d}_V(y,t)\geqslant\zeta}\mathsf{d}_V(y,t).$$

Now, if $V\sqsubseteq U$, then $|\mathsf{d}_V(x,s)-\mathsf{d}_V(\phi(x,y),\phi(s,t))|\leqslant 2r_0(\kappa+E)$. So, if $\mathsf{d}_V(x,s)\geqslant\zeta>20r_0(\kappa+E)$, we have

$$\frac{9}{10}\mathsf{d}_V(x,s)\leqslant\mathsf{d}_V(\phi(x,y),\phi(s,t))\leqslant\frac{11}{10}\mathsf{d}_V(x,s).$$

When $V\perp U$, the same inequalities hold with x replaced by y and s replaced by t on the far left and far right.

So,

$$\sum_{V\sqsubseteq U,\mathsf{d}_V(x,s)\geqslant\zeta}\mathsf{d}_V(x,s)\leqslant\frac{10}{9}\sum_{V\sqsubseteq U,\mathsf{d}_V(\phi(x,y),\phi(s,t))>9\zeta/10}\mathsf{d}_V(\phi(x,y),\phi(s,t))$$

and similarly

$$\sum_{V\perp U,\mathsf{d}_V(y,t)\geqslant\zeta}\mathsf{d}_V(y,t)\leqslant\frac{10}{9}\sum_{V\perp U,\mathsf{d}_V(\phi(x,y),\phi(s,t))>9\zeta/10}\mathsf{d}_V(\phi(x,y),\phi(s,t)).$$

Now, applying the distance formula with threshold $9\zeta/10$, having choosing ζ sufficiently large in terms of $r_0(\kappa+E)$ and s_0, we obtain uniform constants A',B' such that

$$\mathsf{d}_{F_U\times E_U}((x,y),(s,t))\leqslant A'\mathsf{d}_{\mathcal{X}}(\phi(x,y),\phi(s,t))+B'.$$

A similar argument gives the lower bound, so ϕ is a uniform quasi-isometric embedding.

Finally, by construction, $\mathrm{im}(\phi) \subseteq P_U$. On the other hand, let $f(\vec{x}, \vec{y}) \in P_U$. Then $(\vec{x}, \vec{q_0}) \in F_U^\kappa \times \{\vec{q_0}\}$, and $(\vec{p_0}, \vec{y}) \in \{\vec{p_0}\} \times E_U^\kappa$, so $x' = f(\vec{x}, \vec{q_0}) \in F_U$ and $y' = f(\vec{p_0}, \vec{y}) \in E_U$. A simple application of the distance formula now shows that $\mathsf{d}_{\mathcal{X}}(f(\vec{x}, \vec{y}), \phi(x', y'))$ is uniformly bounded, so P_U is Hausdorff-close to $\mathrm{im}\,\phi$. □

So, in summary, for each $U \in \mathfrak{F}$, we have an associated subspace $P_U \subset \mathcal{X}$ such that:

- P_U is hierarchically quasiconvex, with hierarchical quasiconvexity function $\kappa^\times$ independent of U (it depends only on the constants from Definition 10.1).
- For each $W \in \mathfrak{F}$ such that $U \subsetneq W$ or $U \pitchfork W$, we have $\mathsf{diam}(\pi_W(P_U) \cup \rho_W^U) \leqslant \kappa^\times(0)$.
- P_U is the image of a uniform quasi-isometric embedding $\phi : F_U \times E_U \to \mathcal{X}$ such that each $F_U \times \{e\}$ and $\{f\} \times E_U$ has $\kappa^\times$–hierarchically quasiconvex image. We abuse notation and let $F_U \times \{e\}$ denote this image (and we just say F_U when the particular parallel copy has been fixed).
- Each parallel copy F_U has the property that π_V is $\kappa^\times(0)$–coarsely surjective when $V \sqsubseteq U$, and, when $V \perp U$, the set $\pi_V(F_U)$ has uniformly bounded diameter, by $\kappa^\times(0)$. The reverse holds when F_U is replaced by E_U.
- If $V \sqsubseteq U$, then for each parallel copy F_U, we can choose a parallel copy F_V such that F_V is contained in a $\kappa^\times(0)$–neighbourhood of F_U (this is Proposition 5.16 from [BHS19]).

Remark 15.9. If $U \perp V$, then P_U and P_V have coarse intersection containing a quasi-isometrically embedded copy of $F_U \times F_V$. Choosing some x in this subspace, we see that if $W \in \mathfrak{F}$ has the property that each of U, V is transverse to W or properly nested in W, then ρ_W^V, ρ_W^U both uniformly coarsely coincide with $\pi_W(x)$ and therefore are coarsely equal. This reflects the following lemma. □

Lemma 15.10. *Let $W, U, V \in \mathfrak{F}$. Suppose that $U \perp V$, and each of U, V is either properly nested in W, or transverse to W. Then*

$$\mathsf{d}_W(\rho_W^U, \rho_W^V) \leqslant 3E.$$

Proof. This is Lemma 1.5 in [DHS17]. Fix $p \in \pi_U(\mathcal{X}), q \in \pi_V(\mathcal{X})$. By Definition 10.1.(8), there exists $x \in \mathcal{X}$ such that

- $\mathsf{d}_U(x, p) \leqslant E$,
- $\mathsf{d}_V(x, q) \leqslant E$, and, most importantly,
- $\mathsf{d}_W(x, \rho_W^U) \leqslant E$ and
- $\mathsf{d}_W(x, \rho_U^V) \leqslant E$.

The triangle inequality, plus $\mathsf{diam}(\pi_W(x)) \leqslant E$, now yields the lemma. □

Finally, we mention how to make product regions function nicely with actions by HHS automorphisms:

Remark 15.11 (Equivariant product regions in HHG, first version)**.** Later, the following flexibility will be useful. If P_U is as above, and L is some constant, and P_U' is any subspace that is L–Hausdorff close to P_U, then P_U' has the same properties as P_U listed above, except the uniform constants and hierarchical quasiconvexity functions now also depend on L.

In particular, if $(G, \mathfrak{F})$ is a hierarchically hyperbolic group, then for each $U \in \mathfrak{F}$ and $g \in G$, we have the following. Recall the map $f : F_U^\kappa \times E_U^\kappa \to G$ whose image is P_U. Suppose that $x \in P_U$. Define $gf : F_U^\kappa \times E_U^\kappa \to G$ by $(gf)(p, q) = g \cdot f(p, q)$. We also have $f^g : F_{gU}^\kappa \times E_{gU}^\kappa \to G$ as defined above. By Definition 10.11, we have an isometry $g : F_U^\kappa \times E_U^\kappa \to F_{gU}^\kappa \times E_{gU}^\kappa$ arising from the isometries $g : \mathcal{C}V \to \mathcal{C}gV$ for $V \sqsubseteq U$ or $V \perp U$. Since $g\rho_V^U = \rho_{gV}^{gU}$ whenever $U \pitchfork V$ or $U \subsetneq V$, we have a uniform constant L, depending only on the HHS constants, such that

$\mathsf{d}_G(gf(p,q), f^g(g(p,q)) \leqslant L$ for all p, q. In particular, gP_U and P_{gU} are at bounded Hausdorff distance.

Now let $\mathfrak{F}_1 \subset \mathfrak{F}$ contain exactly one $U \in \mathfrak{F}$ in each G–orbit. For $U \in \mathfrak{F}_1$, let $P_U^0 = P_U$, and redefine P_U to be $\bigcup_{g \in \mathrm{Stab}_G(U)} gP_U^0$. Then, for arbitrary $U \in \mathfrak{F}$, we can write $U = g\bar{U}$ for $\bar{U} \in \mathfrak{F}$, with g in a fixed left coset $g\,\mathrm{Stab}_G(U)$. Let $P_U = gP_{\bar{U}}$. Since $P_{\bar{U}}$ is $\mathrm{Stab}_G(U)$–invariant, this definition is independent of the choice of g in the given coset. Finally, by the preceding discussion, P_U is at bounded Hausdorff distance from the original U–standard product region.

Hence, when dealing with HHGs, we **can and shall assume** that the product regions have the additional property that $P_{gU} = gP_U$ for all $U \in \mathfrak{F}$ and $g \in G$. □

Remark 15.12 (Equivariant product regions in HHG II)**.** There is another way to arrange for the set of product regions in an HHG $(G, \mathfrak{F})$ to be G–invariant. For each $U \in \mathfrak{F}$, for suitably large (in terms of the HHS constants) κ, let P_U be the set of $x \in G$ such that

$$\mathsf{d}_V(x, \rho_V^U) \leqslant \kappa$$

whenever $U \subsetneq V$ or $U \pitchfork V$. It is now immediate from the definition of an HHG that $gP_U = P_{gU}$, and in particular $\mathrm{Stab}_G(U)$ acts on P_U, for $g \in G$ and $U \in \mathfrak{F}$. On the other hand, the product region P_U as defined above is at uniformly bounded Hausdorff distance from the image of ϕ, and hence has all of the properties required of a product region (at the expense of a uniform enlargement of the hierarchical quasiconvexity function). □

16. The cubical approximation theorem

The coarse median map $\mu : \mathcal{X}^3 \to \mathcal{X}$ actually enjoys a stronger property than merely making $\mathcal{X}$ a coarse median space, expressed by the following proposition, which states that hulls of finite sets in $\mathcal{X}$ can be approximated by $CAT(0)$ cube complexes.

The idea is to generalise the fact that in a Gromov-hyperbolic space, any finite set of points can be approximated by a tree in the following sense: if δ is the hyperbolicity constant, and $x_1, \ldots, x_n$ are points, then there is a constant $C = C(\delta, n)$ and a $(1, C)$–quasi-isometric embedding of a tree T, whose leaves get sent to some of the x_i, and whose image coarsely coincides with the quasiconvex hull of $\{x_1, \ldots, x_n\}$.

The following is Theorem 2.1 in [BHS17c], and has been reproved (and generalised) by Bowditch in [Bow18a, Theorem 1.3]. A more refined version, under extra hypotheses on the HHS structure, was recently established by Durham-Minsky-Sisto [DMS20].

Proposition 16.1 (Cubulation of hulls)**.** *Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space. Then there exists M_0 (depending on the HHS constants) such that the following holds. Let $\theta \geqslant \theta_0$ and $n \in \mathbb{N}$.*

There exists constants $M_1 \geqslant 100nM_0$ such that for all $M \geqslant M_1$, we have a constant $C \geqslant 1$ satisfying the following. Let $A = \{x_1, \ldots, x_n\} \subset \mathcal{X}$ and let $\mathcal{U}$ be the set of $U \in \mathfrak{F}$ such that $\mathsf{d}_U(x_i, x_j) \geqslant M$ for some i, j. Then there exists a finite CAT(0) cube complex $\mathbf{Y}$ and a map $f : \mathbf{Y} \to \mathcal{X}$ such that:

- *f is a (C, C)–quasi-isometric embedding whose image is at finite Hausdorff distance (depending only on the HHS structure and M, n, θ) from $H_\theta(A)$.*
- *There exist $\hat{x}_{i_1}, \ldots, \hat{x}_{i_s} \in \mathbf{Y}$ such that $\mathsf{d}_{\mathcal{X}}(f(\hat{x}_{i_j}), x_j) \leqslant C$ for all j and $\mathbf{Y}$ is equal to the convex hull in $\mathbf{Y}$ of $\{\hat{x}_{i_1}, \ldots, \hat{x}_{i_s}\}$.*
- *Let $\hat{a}, \hat{b}, \hat{c} \in \mathbf{Y}$ be vertices and let $\hat{m} \in \mathbf{Y}^{(0)}$ be their median. Then $\mathsf{d}_{\mathcal{X}}(f(\hat{m}), m) \leqslant C$, where $m = \mu(f(\hat{a}), f(\hat{b}), f(\hat{c}))$ denotes the coarse median. (In other words, f is C–quasimedian.)*
- *Each hyperplane h of $\mathbf{Y}$ is labelled by an element of $\mathcal{U} \subset \mathfrak{F}$ such that hyperplanes h, h' cross if and only if their labels U, U' satisfy $U \perp U'$. In particular,* $\dim \mathbf{Y} \leqslant \chi$.

Finally, if γ is a combinatorial geodesic in $\mathbf{Y}$, then $f \circ \gamma$ is a (D, D)–hierarchy path in $\mathcal{X}$, where D depends only on C.

We say more about the labels below.

Remark 16.2. The entire proposition is [BHS17c, Theorem 2.1], except for the final statement about hierarchy paths, which follows from the quasimedian quasi-isometric embedding statement, together with Lemma 1.37 of [BHS17c]. □

The proposition directly strengthens the property of being a coarse median space; the definition of a coarse median space asks that any finite set of points is contained in the image of a quasimedian map from the 0–skeleton of a finite CAT(0) cube complex. For any finite set in $\mathcal{X}$, the corresponding cube complex $\mathbf{Y}$ satisfies the required properties from [Bow13, Section 8], plus more: its image is coarsely median convex, by the above proposition combined with Proposition 13.3.

Later we will mainly be interested in the case of Proposition 16.1 where $n = 2$, i.e. in using CAT(0) cube complexes that are intervals for their intrinsic median to approximate hulls of pairs of points in $\mathcal{X}$. It will be useful to use some parts of the proof of the proposition — i.e. some extra properties of the cubical approximation that follow from the construction in [BHS17c, Section 2].

Accordingly, we now discuss the construction of $\mathbf{Y}$ in the simpler setting where $n = 2$.

Remark 16.3 (Hyperbolicity unnecessary when $n = 2$)**.** Because we are working with $n = 2$, we will not actually use that $\mathcal{C}U, U \in \mathfrak{F}$ is hyperbolic during the construction. We will comment on this at the end, and on a more general statement of the proposition that is therefore true when $n = 2$. □

Fix $x, y \in \mathcal{X}$. Fix $\theta \geqslant \max\{\theta_0, E\}$, so that for some function κ depending on θ, Lemma 6.3 of [BHS19] implies that $H_\theta(\{x, y\})$ is κ–hierarchically quasiconvex.

Let $M_0 = M_0(E, \theta)$ be a constant to be determined, let $M_1 \geqslant 200M_0$, and let $M \geqslant M_1$.

Let $\mathrm{Rel}_M(x, y)$ be the set of $U \in \mathfrak{F}$ such that $\mathsf{d}_U(x, y) \geqslant M$. Recall from Lemma 11.4 that $\mathrm{Rel}_M(x, y)$ is finite as long as we chose $M_0 > E$.

For each $U \in \mathrm{Rel}_M(x, y)$, let γ_U be a geodesic joining $\pi_U(x)$ to $\pi_U(y)$. Let $\mathrm{Rel}^U_M(x, y)$ be the set of $V \in \mathrm{Rel}_M(x, y)$ with $V \subsetneq U$. By the consistency and bounded geodesic image axioms, $V \in \mathrm{Rel}^U_M(x, y)$ implies that there is a point $r^V_U \in \gamma_U$ with $\mathsf{d}_U(r^V_U, \rho^V_U) \leqslant E$.

For each $U \in \mathrm{Rel}_M(x, y)$, choose a (finite) set of points $\{p^U_i\}_{i \in I_U}$ in γ_U with the following properties:

- $\mathsf{d}_U(\{x, y\}, p^U_i) \geqslant M_0$ for all $i \in I_U$;
- $\mathsf{d}_U(p^U_i, p^U_j) \geqslant M_0$ for $i \neq j$ in I_U;
- $\mathsf{d}_U(p^U_i, r^V_U) \geqslant M_0$ for $i \in I_U$ and $V \in \mathrm{Rel}^U_M(x, y)$;
- each component of $\gamma_U - \{p^U_i, r^V_U : i \in I_U, V \in \mathrm{Rel}^U_M(x, y)\}$ has diameter at most $20M_0$.

Figure 19 shows a heuristic picture of the above ingredients, using the same x, y and elements of $\mathfrak{F}$ as in Figure 11.

We are now ready to define *walls* in $H_\theta(\{x, y\})$. For each $U \in \mathrm{Rel}_M(x, y)$ and each $i \in I_U$, let $\overleftarrow{w}(U, i)$ be the set of $z \in H_\theta(\{x, y\})$ such that every point on γ_U that is θ–close to $\pi_U(z)$ lies in the same component of $\gamma_U - \{p^U_i\}$ as the endpoint of γ_U in $\pi_U(x)$. Let $\overrightarrow{w}(i, U) = H_\theta(\{x, y\}) - \overleftarrow{w}(i, U)$. The sets $\overleftarrow{w}(i, U), \overrightarrow{w}(i, U)$ are the *p^U_i–halfspaces.*

Then the pair $w(i, U) = (\overleftarrow{w}(i, U), \overrightarrow{w}(i, U))$ is a *wall* in $H_\theta(\{x, y\})$, i.e. a bipartition. We say that $w(i, U)$ *separates* z, z' if z, z' lie in distinct p^U_i–halfspaces. Since there are finitely many p^U_i for each U, and finitely many $U \in \mathrm{Rel}_M(x, y)$, we have defined finitely many walls. Accordingly, $H_\theta(\{x, y\})$, with the given set of walls, is a *wallspace* in the sense of [HP98, Nic04].

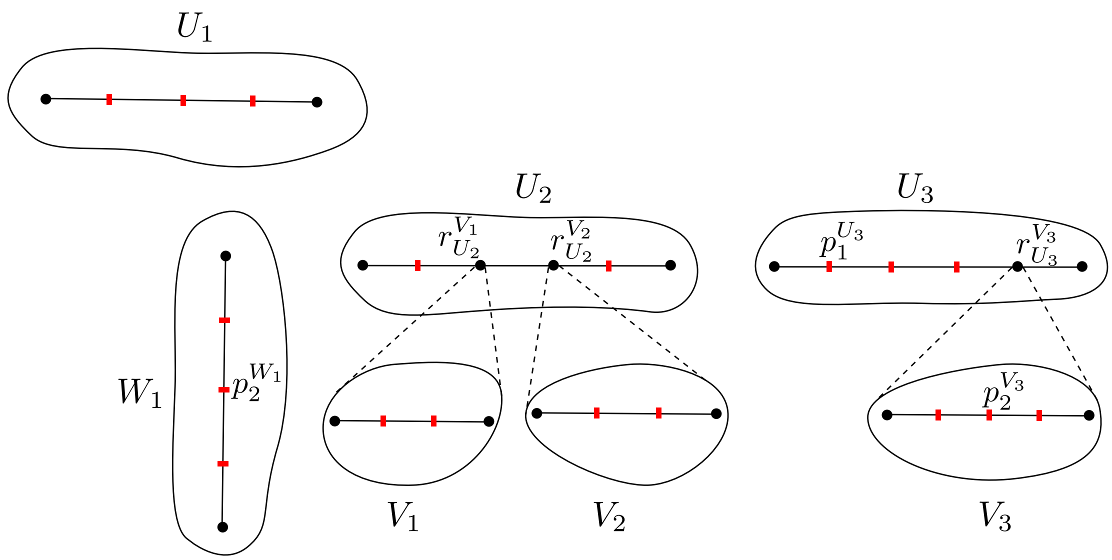

FIGURE 19. The geodesics, points $r^{V_i}_{U_j}$, and extra "wall-points" $p^{W_i}_j$, shown in the example from Figure 11.

Following [Nic04, CN05], define the CAT(0) cube complex $\mathbf{Y}$ dual to this wallspace as follows. A 0–cube $\hat{y} \in \mathbf{Y}$ is *coherent orientation* of the walls, i.e. a map from the set of walls to the set of halfspaces such that

- for all $w(i, U)$, the halfspace $\hat{y}(w(i, U))$ is one of the two p^U_i–halfspaces;
- for all U, V and all $i \in I_U, j \in I_V$, we have

$$\hat{y}(w(i, U)) \cap \hat{y}(w(j, V)) \neq \varnothing.$$

We join $\hat{y}, \hat{y}'$ by a 1–cube if they differ on exactly one wall, and then fill in all cubes whose 1–skeleta appear.

The hyperplane $h(i, U)$ of $\mathbf{Y}$ corresponding to the wall $w(i, U)$ is given a *label*, $Lab(h(i, U)) = U$.

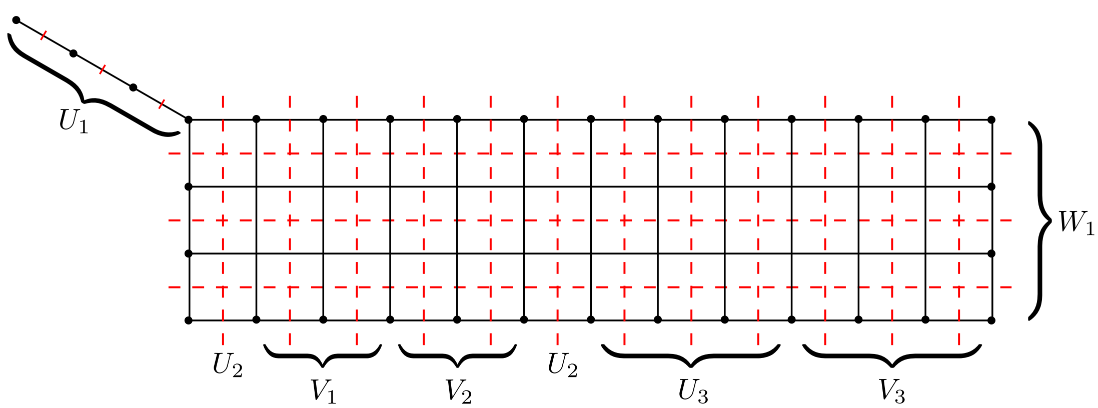

FIGURE 20. The approximating cube complex in the example from Figure 19, with the hyperplanes labelled.

Having defined the finite CAT(0) cube complex $\mathbf{Y}$, we now define the map $f : \mathbf{Y} \to \mathcal{X}$ and establish the properties mentioned in the proposition, along with some other properties we will use.

Fix a 0–cube $\hat{y} \in \mathbf{Y}$. We define $f(\hat{y}) \in \mathcal{X}$ as follows.

For each $U \in \mathrm{Rel}_M(x,y)$ and each $i \in I_U$, we have a p_i^U–halfspace $\hat{y}_{i,U}$ chosen by the coherent orientation $\hat{y}$.

For each $V \in \mathrm{Rel}_M(x,y)$, let $S(i,U,V)$ be the intersection of all closed subintervals of γ_V that contain all points in γ_V that are θ–close to $\pi_V(\hat{y}_{i,U})$.

Fix V. For all U, U' and i, i', coherence provides $a \in \hat{y}(i,U) \cap \hat{y}(i',U')$. Now, choose $\bar{a} \in \gamma_V$ that is θ–close to $\pi_V(a)$, which is possible since $a \in H_\theta(\{x,a\})$ by definition. So $\bar{a} \in S(i,U,V) \cap S(i',U',V)$.

We have shown that for any V, the subintervals $S(i,U,V)$ pairwise intersect as i,U vary. So, there exists $b_V \in \bigcap_{i,U} S(i,U,V)$.

For $V \notin \mathrm{Rel}_M(x,y)$, let $b_V \in \pi_V(x)$.

We next observe that, for each $V \in \mathfrak{F}$, the set of possible b_V has diameter bounded in terms of M. When $V \notin \mathrm{Rel}_M(x,y)$, this is clear.

Suppose that $V \in \mathrm{Rel}_M(x,y)$. Suppose that $\bar{a}, \bar{b} \in \gamma_V$ satisfy $\mathsf{d}_V(\bar{a},\bar{b}) > 50M_0$, where $\bar{a}, \bar{b}$ are respectively θ–close to $\pi_V(a), \pi_V(b)$ for some $a, b \in \mathcal{X}$. Then one of the following holds:

- Some p_j^V separates $\bar{a}, \bar{b}$ in γ_V. Then, provided M_0 is sufficiently large in terms of θ, we have that $w(j,V)$ separates a, b, and $\bar{a}, \bar{b}$ cannot both belong to b_V.
- For some $U \in \mathrm{Rel}_M^V(x,y)$, the point r_V^U separates $\bar{a}, \bar{b}$, and no p_j^V has this property.

  Our choice of $50M_0$ then implies that there are at least two such U, and one such U has both of $\bar{a}, \bar{b}$ at least M_0–far from r_V^U.

  Hence, provided M_0 is large enough in terms of E and θ, consistency and bounded geodesic image imply that $\mathsf{d}_U(a,b) > 50M_0$.

  Moreover, by Definition 10.1.(4), we can assume that U is $\sqsubseteq$–minimal in $\mathrm{Rel}_M(x,y)$.

  Choose $\bar{a}', \bar{b}' \in \gamma_U$ respectively θ–close to $\pi_U(a), \pi_U(b)$. Then $\bar{a}', \bar{b}'$ are separated by some p_i^U, and so $\bar{a}, \bar{b}$ cannot both lie in b_V.

We have shown that the set of possible b_V has diameter bounded by some B_0 depending only on M_0, provided M_0 is sufficiently large in terms of E and θ. (This was a version of Lemma 2.6 in [BHS17c], simplified by the assumption $n = 2$.)

Exactly as in Lemma 2.7 of [BHS17c], there exists $B_1 = B_1(M_0)$ such that $(b_V)_{V \in \mathfrak{F}}$ is a B_1–consistent tuple, so realisation (Theorem 10.5) provides a point $f(\hat{y}) \in \mathcal{X}$ such that $\mathsf{d}_V(f(\hat{Y}), b_V) \leqslant r_0 B_1$ for all $V \in \mathfrak{F}$.

This defines the map $f : \mathbf{Y} \to \mathcal{X}$.

The proofs that f is a quasimedian quasi-isometric embedding whose image coarsely coincides with $H_\theta(\{x,y\})$ do not seem to simplify significantly when $n = 2$, so we refer the interested reader to [BHS17c].

We now discuss the remaining statements from Proposition 16.1 and some additional facts about the hyperplanes in $\mathbf{Y}$ that follow fairly easily from the construction in the case $n = 2$ but which are not stated exportably in [BHS17c].

First, let $\hat{x}$ be the 0–cube of $\mathbf{Y}$ obtained by setting $\hat{x}(w(i,U)) = \overleftarrow{w}(i,U)$ for all i, U, and similarly define $\hat{y}$ by $\hat{y}(w(i,U)) = \overrightarrow{w}(i,U)$. Note that $x \in \hat{x}(w(i,U))$ and $y \in \hat{y}(w(i,U))$ for all i, U, so these are really coherent orientations.

Note that every hyperplane of $\mathbf{Y}$ separates $\hat{x}$ from $\hat{y}$, so $\mathbf{Y}$ is the convex hull in $\mathbf{Y}$ of $\{\hat{x}, \hat{y}\}$. By uniformly enlarging the constant C from Proposition 16.1 and perturbing f, we can assume $f(\hat{x}) = x, f(\hat{y}) = y$.

Accordingly, letting $\mathrm{Hyp}(\mathbf{Y})$ denote the set of hyperplanes in $\mathbf{Y}$, we can put a partial order on $\mathrm{Hyp}(\mathbf{Y})$, denoted $<$, such that $h < v$ if h separates v from $\hat{x}$. Given $h, v \in \mathrm{Hyp}(\mathbf{Y})$, exactly one of the following holds:

- $h = v$;

- h, v cross;
- $h < v$ or $v < h$.

The next statement relates $<$ to the partial order $\prec$ on $\mathrm{Rel}_M(x,y)$ from Lemma 11.6. Recall that distinct U, V satisfy $U \prec V$ if $U \pitchfork V$ and $\mathsf{d}_V(x, \rho^U_V) \leqslant E$.

Proposition 16.4 (Extra properties of the cubical approximation when $n = 2$)**.** *The map $f : \mathbf{Y} \to \mathcal{X}$, and the map $\mathrm{Hyp}(\mathbf{Y}) \to \mathrm{Rel}_M(x,y)$ given by $h \mapsto Lab(h)$, have the following properties:*

(I) $Lab(h) \bot Lab(v)$ if and only if h and v cross (i.e. h, v are distinct and $<$–incomparable).
(II) Suppose that $h, v \in \mathrm{Hyp}(\mathbf{Y})$ satisfy $v < h$ and let $Lab(v) = V, Lab(h) = U$. Suppose that $U \pitchfork V$. Then $V \prec U$.
(III) Suppose that $h, v \in \mathrm{Hyp}(\mathbf{Y})$ and $Lab(v) = V, Lab(h) = H$. Suppose that $V \subsetneq H$. Let $p^H_i \in \gamma_H$ be the point determining the wall corresponding to h. Then
- *if $v < h$, then r^V_H lies in the same component of $\gamma_H - \{p^H_i\}$ as the endpoint in $\pi_H(x)$;*
- *if $h < v$, then r^V_H lies in the same compoonent of $\gamma_H - \{p^H_i\}$ as the endpoint in $\pi_H(y)$.*

(IV) There exists $B_2 = B_2(M)$ such that the following holds. Let $h_1, \ldots, h_k \in \mathrm{Hyp}(\mathbf{Y})$. Let $\hat{\alpha}$ be a combinatorial geodesic in $\mathbf{Y}$ that joins some 0–cubes $\hat{a}, \hat{b}$ and crosses exactly the hyperplanes h_i. Then for all $V \notin \{Lab(h_1), \ldots, Lab(h_k)\}$, we have

$$\mathsf{d}_V(f(\hat{a}), f(\hat{b})) \leqslant B_2.$$

Proof. Assertion (I) is just Lemma 2.12 from [BHS17c].

We now prove assertion (II). Suppose that $v < h$ and let $Lab(v) = V, Lab(h) = U$. Suppose that $U \pitchfork V$. Then either $V \prec U$ or $U \prec V$. Suppose that $U \prec V$, i.e. $\mathsf{d}_V(x, \rho^U_V) \leqslant E$. Let $i \in I_U$ and $j \in I_V$ be such that the hyperplanes v, h respectively correspond to the walls $w(j, V), w(i, U)$. Since $v < h$, we have that x is separated from h by v, i.e. $\overleftarrow{w}(j, V) \subset \overleftarrow{w}(i, U)$.

By Lemma 16.5 below, there thus exists $z \in \overleftarrow{w}(i, U) - \overleftarrow{w}(j, V)$. So, letting $\bar{z} \in \gamma_U$ be θ–close to $\pi_U(z)$, we have $\mathsf{d}_U(\bar{z}, \rho^V_U) \geqslant \mathsf{d}_U(p^U_i, \rho^V_U) \geqslant M_0 - E$, since $\mathsf{d}_U(y, \rho^V_U) \leqslant E$ by consistency and $U \prec V$. Hence $\mathsf{d}_U(z, \rho^V_U) \geqslant M_0 - \theta - E > E$, whence $\mathsf{d}_V(z, \rho^U_V) \leqslant E$. Thus $\mathsf{d}_V(z, x) \leqslant 3E$. So, $z \in \overleftarrow{w}(j, V)$, a contradiction. Hence $V \prec U$.

Next, we prove (III). Suppose that $v < h$ and $V \subsetneq H$. Choose j, V and i, H so that v, h correspond to $w(j, V), w(i, H)$. Since $v < h$, we have $\overleftarrow{w}(j, V) \subset \overleftarrow{w}(i, H)$. Let $z \in \overleftarrow{w}(i, H) - \overleftarrow{w}(j, V)$ (using Lemma 16.5). Since $z \in \overrightarrow{w}(j, V)$, we have that $\mathsf{d}_V(x, z) > E$, so by consistency and bounded geodesic image, ρ^V_H lies E–close to the subpath of γ_H joining $\pi_H(x)$ to p^H_i, as required. The proof of the statement about $h < v$ is identical.

We now prove assertion (IV). Fix $V \in \mathfrak{F}$. If $V \notin \mathrm{Rel}_M(x,y)$, then, since the image of f is uniformly close to $H_\theta(\{x, y\})$, the points $\pi_V(f(\hat{a}))$ and $\pi_V(f(\hat{b}))$ lie uniformly close to a geodesic in $\mathcal{C}V$ of length at most M, so $\mathsf{d}_V(f(\hat{a}), f(\hat{b}))$ is bounded in terms of the constants from Proposition 16.1, as required.

So, suppose $V \in \mathrm{Rel}_M(x,y)$. For $1 \leqslant i \leqslant k$, let $U_i = Lab(h_i)$ (note that these labels need not all be distinct).

For each $W \in \mathrm{Rel}_M(x,y)$, let $b_W(\hat{a}), b_W(\hat{b})$ be the W–coordinates of the tuples used above to define $f(\hat{a}), f(\hat{b})$ respectively.

By hypothesis, $b_W(\hat{a}) = b_W(\hat{b})$ unless $W = U_i$ for some i, so in particular, $b_V(\hat{a}) = b_V(\hat{b})$. Since $b_V(\hat{a})$ and $b_V(\hat{b})$ are respectively r_0B_1–close to $\pi_V(f(\hat{a}))$ and $\pi_V(f(\hat{b}))$, the claim follows, with $B_2 = 2r_0B_1$. □

The auxiliary lemma needed above was:

Lemma 16.5. *Suppose that $(i,U),(j,V)$ are distinct. Then $\overleftarrow{w}(i,U) \neq \overleftarrow{w}(j,V)$.*

Proof. If $U \perp V$, then by [BHS17c, Lemma 2.12], the corresponding walls cross and are in particular distinct.

If $U \pitchfork V$, then suppose that $\overleftarrow{w}(i,U) = \overleftarrow{w}(j,V)$. Without loss of generality, $U \prec V$. Choose $z \in \mathcal{X}$ such that $\pi_U(z)$ is θ–close to a point $\bar{z} \in \gamma_U$ such that $\bar{z}$ is in the same component of $\gamma_U - \{p_i^U\}$ as the endpoint in $\pi_U(y)$, but $\mathsf{d}_U(p_i^U, z) \leqslant 10(E+\theta)$. Note that $z \in \overrightarrow{w}(i,U)$.

Provided M_0 is sufficiently large in terms of E, θ, we have $\mathsf{d}_U(z, \rho_U^V) > E$, so $\mathsf{d}_V(z,x) \leqslant 3E$, by consistency, so $z \in \overleftarrow{w}(j,V)$, as required.

Suppose that $U \subsetneq V$ and that $\overleftarrow{w}(i,U) = \overleftarrow{w}(j,V)$. Without loss of generality, $p_{j,V}$ lies between the endpoint of γ_U in $\pi_U(x)$ and the point r_V^U. So, by consistency and bounded geodesic image, $\pi_U(\overleftarrow{w}(j,V))$ is contained in the E–neighbourhood of $\pi_U(x)$. On the other hand, we can choose $z \in \overleftarrow{w}(i,U) = \overleftarrow{w}(j,V)$ so that $\pi_U(z)$ is $10(E+\theta)$–close to p_i^U, contradicting that $\mathsf{d}_U(x, p_i^U) > M_0$ provided M_0 is sufficiently large.

A similar argument works when $U = V$. $\square$

Remark 16.6 (The relative HHS case)**.** Two generalisations of HHSes were introduced in [BHS19] and warrant mention in connection with the cubical approximation theorem. First is the notion of a *hierarchical space* $(\mathcal{X}, \mathfrak{F})$. This is defined exactly as in Definition (10.1), with the following changes:

- We still require each $\mathcal{C}U, U \in \mathfrak{F}$ to be a geodesic space, but we do not require it to be hyperbolic;
- for convenience, we require the projections π_U to be E–coarsely surjective (although this is not strictly necessary, there is little loss of generality in practice and it simplifies explanations).

Hierarchical spaces are too general to be able to prove the distance formula along the lines of [BHS19]. However, the key point is that the realisation theorem, Theorem 10.5, makes no use of hyperbolicity and therefore holds in the context of hierarchical spaces (see Theorem 3.1 in [BHS19] and the paragraph preceding it).

A *relative* HHS is a hybrid of the two notions: $(\mathcal{X}, \mathfrak{F})$ is a relative HHS if Definition 10.1 holds, except that we allow the geodesic space $\mathcal{C}U, U \in \mathfrak{F}$ to be non-E–hyperbolic if U is $\sqsubseteq$–minimal (see [BHS19, Definition 6.8].

The statement of Proposition 16.1 and that of Proposition 16.4 continue to hold in the relative HHS context, when $n = 2$, in the following sense.

Fix $x, y \in \mathcal{X}$. Given $\theta \geqslant 0$, define $H_\theta(\{x,y\})$ as in the HHS case, with the following modification: recall that $H_\theta(\{x,y\})$ is the set of points z that project θ–close to every $\mathcal{C}U$–geodesic from $\pi_U(x)$ to $\pi_U(y)$ for all $U \in \mathfrak{F}$. We modify the definition as follows: if $U \in \mathfrak{F}$ is $\sqsubseteq$–minimal and $\mathcal{C}U$ is not E–hyperbolic, we fix a geodesic γ_U in $\mathcal{C}U$ from $\pi_U(x)$ to $\pi_U(y)$, and require $\pi_U(z)$ to be θ–close to the geodesic γ_U only.

Lemma 6.12 of [BHS19] says that $H_\theta(\{x,y\})$ is a coarsely lipschitz coarse retract of $\mathcal{X}$, via a map that, at the level of $\mathcal{C}U$, is the coarse closest-point projection when $\mathcal{C}U$ is E–hyperbolic, and is otherwise a coarsely lipschitz map to γ_U. Proposition 6.15 of [BHS19] then shows that $H_\theta(\{x,y\})$ inherits a hierarchical space structure from $(\mathcal{X}, \mathfrak{F})$, and that this is actually an HHS structure, because each non-hyperbolic $\mathcal{C}U$ has been replaced by a single geodesic γ_U.

We can now apply Proposition 16.1 to build a uniformly quasimedian uniform quasi-isometry $f : \mathbf{Y} \to H_\theta(\{x,y\})$, where $\mathbf{Y}$ is a CAT(0) cube complex and $H_\theta(\{x,y\})$ is given the above HHS structure. Composing with the inclusion into $\mathcal{X}$ gives a quasi-isometric embedding $f : \mathbf{Y} \to \mathcal{X}$. The quasimedian part no longer makes sense, since $\mathcal{X}$ need not be coarse median in the relative case, but the conclusion of Proposition 16.4 continues to hold (the $\prec$–ordering on relevant elements works for any hierarchical space, and the inclusion induces uniform quasi-isometric embeddings at the level of the $\mathcal{C}U$). $\square$

17. ☺ Sample application of cubical approximation: hierarchically hyperbolic cone-off

A key feature of the theory of HHSes is the *factored space* construction from [BHS17a]. Since we appeal to it once, later in this section, we mention it here. Moreover, as an illustration of the utility of the cubical approximation, we take the opportunity to give a more conceptual proof than is given in [BHS17a].

The statement requires some preparation. Let $(\mathcal{X}, \mathfrak{F})$ be a (relative) HHS. Let $\mathfrak{U} \subset \mathfrak{F}$ be a set of elements such that:

- $U \in \mathfrak{U}$ and $V \sqsubseteq U$ imply $V \in \mathfrak{U}$, and
- in the relative case, $\mathfrak{U}$ contains every (necessarily $\sqsubseteq$–minimal) V such that $\mathcal{C}V$ is not E–hyperbolic.

We denote the metric on $\mathcal{X}$ by $\mathsf{d}_{\mathcal{X}}$, and define a new metric $\hat{\mathsf{d}}$ as follows. Given $x, y \in \mathcal{X}$, let $D(x,y) = \mathsf{d}_{\mathcal{X}}(x,y)$, unless there exists $U \in \mathfrak{U}$ and $e \in E_U$ such that $x, y \in F_U \times \{e\}$, in which case we set $D(x,y) = \min\{1, \mathsf{d}_{\mathcal{X}}(x,y)\}$. Then let $\hat{\mathsf{d}}$ be the length metric induced by D. Let $\widehat{\mathcal{X}} = (\mathcal{X}, \hat{\mathsf{d}})$.

Our goal is to build an HHS structure $(\widehat{\mathcal{X}}, \mathfrak{F} - \mathfrak{U})$. For each $V \in \mathfrak{F} - \mathfrak{U}$, we keep the same (necessarily E–hyperbolic) space $\mathcal{C}V$, and we keep the same relations, $\rho^{\bullet}_{\bullet}$ points. The projections $\widehat{\mathcal{X}} \to \mathcal{C}V$ are just compositions of the set-theoretic identity $\widehat{\mathcal{X}} \to \mathcal{X}$ with the projections $\pi_V : \mathcal{X} \to \mathcal{C}V$.

The main statement, Proposition 2.4 of [BHS17a], says:

Proposition 17.1 (Cone-off)**.** *$(\widehat{\mathcal{X}}, \mathfrak{F} - \mathfrak{U})$ is a hierarchically hyperbolic space.*

Moreover, if a group G acts by HHS automorphisms on $(\mathcal{X}, \mathfrak{F})$, and $\mathfrak{U}$ is G–invariant, then the induced action on $(\widehat{\mathcal{X}}, \mathfrak{F} - \mathfrak{U})$ is also by HHS automorphisms.

Proof. The statement about group actions is immediate from the construction, once we note that projections have not changed, and the G–action on $\mathcal{X}$ (as a set) has not changed when we changed the metric to produce $\widehat{\mathcal{X}}$. In the new metric, the action of G on $\mathcal{X}$ will be an action by uniform quasi-isometries, in view of the distance formula, once we prove that $\widehat{\mathcal{X}}$ is an HHS, which we do presently.

One must check that $(\widehat{\mathcal{X}}, \mathfrak{F} - \mathfrak{U})$ satisfies the axioms from Definition 10.1. Most of this is straightforward and we refer the reader to [BHS17a, Section 2]. The exception is the uniqueness axiom; we verify this here using the cubical approximation, instead of the argument in [BHS17a], which relies on gate maps to product regions.

Recall that we have to prove the following: there exists a function $\theta_u : \mathbb{R}^+ \to \mathbb{R}^+$, just depending on the HHS constants for $(\mathcal{X}, \mathfrak{F})$, such that for any $\kappa > 0$, if $x, y \in \mathcal{X}$ satisfy $\mathsf{d}_V(x,y) \leqslant \kappa$ for all $V \in \mathfrak{F} - \mathfrak{U}$, then $\hat{\mathsf{d}}(x,y) \leqslant \theta_u(\kappa)$.

Fix κ, let M_0 be the constant from Proposition 16.1. (M_0 depends on a choice of θ for defining hulls. We choose θ large enough in terms of E that θ–hulls inherit HHS structures from $\mathcal{X}$, using Proposition 6.15 of [BHS19], and base our choice of M_0 on this.)

Let $M_1 = 200M_0$, and let $M = 10(\max\{M_1, \kappa\} + E)$. Fix $x, y \in \mathcal{X}$ such that $\mathsf{d}_V(x,y) \leqslant \kappa$ for all $V \in \mathfrak{F} - \mathfrak{U}$. Let $\mathrm{Rel}_M(x,y)$ be the set of all $V \in \mathfrak{F}$ such that $\mathsf{d}_V(x,y) \geqslant M$. Since $M > \kappa$, we have $\mathrm{Rel}_M(x,y) \subseteq \mathfrak{U}$.

Let $\max \mathrm{Rel}_M(x,y)$ be the set of $V \in \mathrm{Rel}_M(x,y)$ that are $\sqsubseteq$–maximal among elements of $\mathrm{Rel}_M(x,y)$.

By Lemma 11.1, there exists $N = N(M)$ such that $|\max \mathrm{Rel}_M(x,y)| \leqslant N$.

Apply Proposition 16.1 to produce a constant $C = C(M)$ and a (C,C)–quasi-isometric embedding $f : \mathbf{Y} \to \mathcal{X}$ of a finite CAT(0) cube complex $\mathbf{Y}$, satisfying the conclusion of that proposition and Proposition 16.4 (see Remark 16.6 for discussion of $\mathbf{Y}$ in the relative case).

Given $U \in \mathrm{Rel}_M(x,y)$, a combinatorial geodesic α in $\mathbf{Y}$ is called a *U–path* if, for all hyperplanes h crossing α, the label satisfies $Lab(h) \sqsubseteq U$.

Given a constant L, the U–path α is called *L–long* if $\mathsf{d}_U(f(\hat{a}), f(\hat{b})) \geqslant L$, where the 0–cubes $\hat{a}, \hat{b}$ are the endpoints of α.

Claim 12. There exists ν such that the following holds. Let $U \in \max \mathrm{Rel}_M(x,y)$ and suppose $\hat{a}, \hat{b} \in \mathbf{Y}^{(0)}$ are joined by a $10E$–long U–path γ. Then $\mathsf{diam}_{\hat{\mathcal{X}}}(f \circ \gamma) \leqslant \nu$.

Proof of Claim 12. Let $A \in \mathfrak{F}$ and suppose that $U \subsetneq A$ or $U \pitchfork A$. We will bound $\mathsf{d}_A(f(\hat{a}), \rho^U_A)$ uniformly, and same for $f(\hat{b})$.

Since $f(\mathbf{Y})$ is at uniformly bounded distance C from $H_\theta(x,y)$, we have that $\pi_A(f(\mathbf{Y}))$ is contained in the $(EC + \theta + E)$–neighbourhood of any $\mathcal{C}A$–geodesic from $\pi_A(x)$ to $\pi_A(y)$ (or, in the relative case when A is $\sqsubseteq$–minimal, our designated geodesic γ_A). In particular, $\pi_A(f(\hat{a}))$ and $\pi_A(f(\hat{b}))$ are $(EC + \theta + E)$–close to such a geodesic.

Bound when $U \subsetneq A$: If $U \subsetneq A$, we have $\mathsf{d}_A(x,y) \leqslant M$, because $U \in \max \mathrm{Rel}_M(x,y)$. So $\mathsf{d}_A(f(\hat{a}), x) \leqslant EC + E + \theta + M$, and the same holds for $f(\hat{b})$. By bounded geodesic image, $\mathsf{d}_A(x, \rho^U_A) \leqslant E + M$, so $\mathsf{d}_A(f(\hat{a}), \rho^U_A) \leqslant EC + E + \theta + 2M$, and the same holds for $f(\hat{b})$. Let $T_0 = EC + E + \theta + 2M$ and note that T_0 depends only on κ and the HHS constants.

Transverse bound for "small" A: Similarly, if A is such that $U \pitchfork A$, and $\mathsf{d}_A(x,y) \leqslant 50M + 2(EC + \theta + C)$, then $\mathsf{d}_A(f(\hat{a}), \rho^U_A) \leqslant T_0$, after enlarging T_0 by a uniform amount depending only on M, C and θ, and hence only on κ and the HHS constants.

Transverse bound for "large" A: Next suppose that $U \pitchfork A$ and $\mathsf{d}_A(x,y) > 50M + 2(EC + \theta + E)$. By consistency, we have (say) $\mathsf{d}_A(y, \rho^U_A) \leqslant E$ and $\mathsf{d}_U(x, \rho^A_U) \leqslant E$. Since $f(\hat{a}), f(\hat{b})$ map C–close to $H_\theta(x,y)$, we have that $\pi_A(f(\hat{a}), \pi_A(f(\hat{b}))$ are $(EC + E + \theta)$–close to our $\mathcal{C}A$–geodesic from $\pi_A(x)$ to $\pi_A(y)$.

We claim that $\mathsf{d}_A(f(\hat{a}), f(\hat{b})) \leqslant 50M + 2(EC + E + \theta)$. If not, then the points $p, q \in [\pi_A(x), \pi_A(y)]_{\mathcal{C}A}$ respectively $(EC + E + \theta)$–close to $\pi_A(f(\hat{a}))$ and $\pi_A(f(\hat{b}))$ are at distance at least $50M$.

By the construction of walls in $H_\theta(x,y)$ used to produce $\mathbf{Y}$, and the fact that $\hat{a}, \hat{b}$ are not separated by A–labelled hyperplanes, $f(\hat{a})$ and $f(\hat{b})$ are separated by a wall corresponding to a hyperplane h such that $Lab(h) \subsetneq A$ and $\mathsf{d}_A(y, \rho^{Lab(h)}_A) > 10M$. But since $\mathsf{d}_A(y, \rho^W_A) \leqslant 10E$ whenever $W \sqsubseteq U$ and $W \subsetneq A$, we have that $Lab(h) \not\sqsubseteq U$. Hence $\hat{a}, \hat{b}$ are separated by a hyperplane not labelled by an element nested in U, a contradiction. So, $\mathsf{d}_A(f(\hat{a}), f(\hat{b})) \leqslant 50M + 2(EC + E + \theta)$.

Now, if $\pi_A(f(\hat{a})), \pi_A(f(\hat{b}))$ are both $10(50M + (EC + E + \theta))$–far from ρ^U_A, then the geodesic in $\mathcal{C}A$ joining them stays E–far from ρ^U_A, and the same is true of the geodesic from $\pi_A(x)$ to either $\pi_A(f(\hat{a}))$ or $\pi_A(f(\hat{b}))$. Hence, applying consistency to the transverse pair U, A, we have that $\pi_U(x), \pi_U(f(\hat{a})), \pi_U(f(\hat{b}))$ are all E–close to ρ^A_U, contradicting that γ is $10E$–long as a U–path.

Close to P_U: Hence we can assume that $\mathsf{d}_A(f(\hat{a}), \rho^U_A) \leqslant T_1$, and the same holds for $f(\hat{b})$, whenever $U \subsetneq A$ or $U \pitchfork A$; here T_1 is a constant depending only on κ and the HHS constants. By uniform hierarchical quasiconvexity of P_U, we thus get a constant η_1, depending only on κ and the HHS constants, such that $\mathsf{d}(f(\hat{a}), P_U), \mathsf{d}(f(\hat{b}), P_U) \leqslant \eta_1$.

Orthogonality bound: We now produce T_2 such that $\mathsf{d}_A(f(\hat{a}), f(\hat{b})) \leqslant T_2$ whenever $A \perp U$. If $A' \sqsubseteq A$, then A' is also orthogonal to U. For such A', no hyperplane v with $Lab(v) = A'$ can separate $\hat{a}$ from $\hat{b}$. Proposition 16.4 thus yields T_2.

Conclusion: Hence there exists η_2, depending on T_2 and the HHS constants, and a point $e \in E_U$, such that $\mathsf{d}(f(\hat{a}), F_U \times \{e\}) \leqslant \eta_2$ and $\mathsf{d}(f(\hat{b}), F_U \times \{e\}) \leqslant \eta_2$. Since f is uniformly quasimedian and uniformly a quasi-isometric embedding, and $F_U \times \{e\}$ is uniformly quasiconvex, the image of $f \circ \gamma$ is thus contained in a uniform d–neighbourhood of $F_U \times \{e\}$ and thus has $\hat{\mathsf{d}}$–diameter bounded uniformly in terms of κ and the HHS constants, as required. $\square$

Claim 13. There exists $n \leqslant N$ and elements $U_1, \ldots, U_n \in \max \mathrm{Rel}(x, y)$ such that $\mathbf{Y}$ contains a combinatorial geodesic γ joining $\hat{x}, \hat{y}$, where:

- $\hat{x}, \hat{y}$ are 0–cubes respectively sent by f to x, y;
- $\gamma = \gamma_1 \cdots \gamma_n$, where each γ_i is a U_i–path; and
- each γ_i traverses an edge dual to a hyperplane labelled U_i.

Proof of Claim 13. Let $\hat{x}, \hat{y}$ be the 0–cubes in $\mathbf{Y}$ mapping to x, y respectively. Recall from Proposition 16.1 that $\mathbf{Y}$ is the cubical convex hull in $\mathbf{Y}$ of $\{\hat{x}, \hat{y}\}$, and recall that there is a partial order $<$ on the hyperplanes of $\mathbf{Y}$ so that $h < v$ if and only if h separates $\bar{x}$ from v, and $<$–incomparability is equivalent to crossing.

Fix $U \in \max \mathrm{Rel}_M(x, y)$, and let $\mathcal{H}_1$ be the set of hyperplanes h of $\mathbf{Y}$ with $Lab(h) \sqsubseteq U$. Note that at least one element of $\mathcal{H}_1$ is labelled U.

Define a 0–cube p as follows. For each hyperplane h, we choose an associated halfspace (component of $\mathbf{Y} - h$) as follows:

- if $h \in \mathcal{H}_1$, we choose the halfspace containing $\hat{x}$;
- if $h \notin \mathcal{H}_1$ has exactly one associated halfspace containing a hyperplane $v \in \mathcal{H}_1$, choose that halfspace;
- if $h \notin \mathcal{H}_1$ and h crosses every element of $\mathcal{H}_1$, choose the halfspace containing $\hat{x}$.

We need to know that we have oriented all the hyperplanes, and that this orientation is coherent. Equivalently, we claim that, if there exist $u, v \in \mathcal{H}_1$ separated by h, then $h \in \mathcal{H}_1$.

We now verify this. Let h, u, v be as above, with $u < h < v$. Since u and h do not cross, $Lab(u)$ and $Lab(h)$ are not orthogonal, because of Proposition 16.4. If $Lab(h) \sqsubseteq U$, we are done, so we can assume $Lab(h) \not\sqsubseteq Lab(u)$, since $Lab(u) \sqsubseteq U$.

If $Lab(u) \pitchfork Lab(u)$, then by Proposition 16.4, we have $Lab(u) \prec Lab(h)$, i.e. $\mathsf{d}_{Lab(h)}(x, \rho^{Lab(u)}_{Lab(h)}) \leqslant E$. Similarly, if $Lab(v) \pitchfork Lab(h)$, then $\mathsf{d}_{Lab(h)}(y, \rho^{Lab(v)}_{Lab(h)}) \leqslant E$.

In particular, letting $p_i^{Lab(h)} \in \gamma_{Lab(h)}$ be the point used to define h, we have that $\rho^{Lab(v)}_{Lab(h)}$ and $\rho^{Lab(u)}_{Lab(h)}$ are respectively E–close to points $r^u_h = r^{Lab(u)}_{Lab(h)}, r^v_h = r^{Lab(v)}_{Lab(h)}$ in $\gamma_{Lab(h)}$ that are separated by, and each M_0–far, from p_i^U.

Similarly, if $Lab(u) \sqsubsetneq Lab(h)$, then by Proposition 16.4, the points r^u_h, r^v_h are separated by, and M_0–far, from p_i^U.

So, $\mathsf{d}_{Lab(h)}(\rho^{Lab(u)}_{Lab(h)}, \rho^{Lab(v)}_{Lab(h)}) \geqslant 2M_0 - 2E$.

Now, $Lab(h) \not\perp U$ since that would imply $Lab(h) \perp Lab(u)$. If $Lab(h) \sqsubseteq U$, then $h \in \mathcal{H}_1$, as needed, so assume not. If $U \sqsubsetneq Lab(h)$, then we would contradict $U \in \max \mathrm{Rel}_M(x, y)$. So, $U \pitchfork Lab(h)$, and $\rho^U_{Lab(h)}$ is a bounded set that is E–close to $\rho^{Lab(v)}_{Lab(h)}$ and $\rho^{Lab(u)}_{Lab(h)}$, which contradicts the inequality established above provided $M_0 > 5E/2$.

Thus, if h separates two elements of $\mathcal{H}_1$, then $h \in \mathcal{H}_1$.

Thus p is a well-defined 0–cube of $\mathbf{Y}$. We can likewise define a 0–cube q (by changing the orientations of exactly the hyperplanes in $\mathcal{H}_1$) so that the hyperplanes separating p, q are precisely those in $\mathcal{H}_1$.

Let β be any combinatorial geodesic from p to q. Then β is a U–path that crosses a hyperplane labelled U.

Now, we could have chosen U so that $p = x$. Indeed, just choose a hyperplane v dual to an edge incident to x and let $U \in \max \mathrm{Rel}_M(x, y)$ be such that $Lab(v) \sqsubseteq U$.

Let h be a hyperplane separating $\hat{y}$ from q. Then $Lab(h) \not\sqsubseteq U$, so $Lab(h) \sqsubseteq W$ for some $W \in \max \mathrm{Rel}_M(x, y) - \{U\}$. In other words, letting the *complexity* of a pair p', q' of 0–cubes be the number of elements $W \in \max \mathrm{Rel}_M(x,y)$ such that p', q' are separated by some h with $Lab(h) \sqsubseteq W$, we have that $(q,\hat{y})$ has strictly lower complexity than $(\hat{x},\hat{y})$. So, we can join q to $\hat{y}$ by a path α, each of which is the concatenation of W–paths with the required properties, for a total of at most $|\max \mathrm{Rel}_M(x,y)| - 1 \leqslant N-1$ values of W. Thus $\beta\alpha$ is a geodesic with the desired property, since β is a U–path crossing a hyperplane labelled U. □

Let $\gamma = \gamma_1 \cdots \gamma_n$ be as in Claim 13. Let $\hat{a}_i, \hat{b}_i$ be the initial and terminal 0–cubes of γ_i, for each $i \leqslant n$. (So, $\hat{b}_i = \hat{a}_{i+1}$.)

Let I be the set of $i \leqslant n$ such that the U_i–path γ_i is $10E$–long.

For $i \in I$, Claim 12 yields $\hat{\mathsf{d}}(f(\hat{a}_i), f(\hat{b}_i)) \leqslant \nu$. Thus

$$\hat{\mathsf{d}}(x,y) \leqslant N\nu + \sum_{i \notin I} \hat{\mathsf{d}}(f(\hat{a}_i), f(\hat{b}_i)).$$

Let $i \notin I$. Then $\mathsf{d}_{U_i}(f(\hat{a}_i), f(\hat{b}_i)) \leqslant 10E$, by the definition of I. Since γ_i is a U–path, every hyperplane crossing γ_i is labelled by an element of $\mathrm{Rel}_M(x,y)$ that is nested in U_i. Hence, by Proposition 16.4, we have for all $V \not\sqsubseteq U_i$ or $V \notin \mathrm{Rel}_M(x,y)$ that $\mathsf{d}_V(f(\hat{a}_i), f(\hat{b}_i)) \leqslant B_2 = B_2(M,E,\theta)$, where B_2 is the constant from the proposition. Let $M^1 = \max\{B_2, M\}$.

So, $\mathrm{Rel}_{M^1}(f(\hat{a}_i), f(\hat{b}_i))$ consists of elements of $\mathrm{Rel}_M(x,y)$ properly nested in U_i. Hence we can apply the above argument with the following changes:

- x and y are respectively replaced by $f(\hat{a}_i), f(\hat{b}_i)$;
- M is replaced by M^1;
- $N(M)$ is replaced by $N(M^1)$.

Letting $\mathbf{Y}^1$ be the approximating cube complex, we see that the highest level of a label of a hyperplane of $\mathbf{Y}^1$ is strictly lower than the level of U_i.

So, by induction on the level, there exists ν^1, depending on M^1 and the HHS constants, and hence only on κ and the HHS constants, such that $\hat{\mathsf{d}}(f(\hat{a}_i), f(\hat{b}_i)) \leqslant \nu^1$.

The base case, where U_i is $\sqsubseteq$–minimal in $\mathrm{Rel}_M(x,y)$, follows from the distance formula and the fact that $\mathsf{d}_{\mathcal{X}} \geqslant \hat{\mathsf{d}}$.

(In the induction, the constants involve increase in a uniform way at each step, but there are at most χ steps.)

Hence

$$\hat{\mathsf{d}}(x,y) \leqslant N\nu + N\nu^1,$$

and we are done. □

Remark 17.2 (Relative HHS version)**.** In [BHS17a], the lemma is proved for relative HHSes, and the preceding proof works in that case in view of Remark 16.6. In the present paper, we will only apply the HHS version. □

Remark 17.3. Proposition 17.1 implies a useful fact supporting the intuition that an HHS is obtained from products of simpler HHSes — standard product regions — "arranged hyperbolically". Indeed, it yields Corollary 2.9 of [BHS17a], which says that, letting $S \in \mathfrak{F}$ be the $\sqsubseteq$–maximal element, $\mathcal{C}S$ is quasi-isometric to the space obtained from $\mathcal{X}$ by coning off the product regions. □

18. Hierarchy paths and product regions

We now briefly discuss how hierarchy paths pass through product regions. Fix an HHS $(\mathcal{X},\mathfrak{F})$ and let $x,y \in \mathcal{X}$. Given a constant κ, we say that $U \in \mathfrak{F}$ is *κ–relevant* for x,y if $\mathsf{d}_U(x,y) \geqslant \kappa$.

The next proposition says that if U is κ–relevant, for sufficiently large κ, then any hierarchy path from x to y passes close to P_U.

It also corrects Proposition 5.17 in [BHS19], which is misstated — see Remark 18.3, which explains that the appearances of [BHS19, Proposition 5.17] elsewhere in the literature can be replaced by Proposition 18.1 without further fuss.

Proposition 18.1. *For all sufficiently large D (in terms of the HHS constants), there exists ν such that the following holds. Let $x, y \in \mathcal{X}$, let γ be a (D, D)–hierarchy path from x to y, and let $U \in \mathfrak{F}$ be $200DE$–relevant for the points x, y. Then γ has a subpath β such that*

- *$\beta \subset \mathcal{N}_\nu(P_U)$, and*
- *π_U is ν–coarsely constant on any subpath of γ disjoint from β.*

Remark 18.2. One could give a more satisfying proof of Proposition 18.1 using the cubical approximation, or by using gates, but we give a proof imitating [BHS19] as far as possible. □

Proof of Proposition 18.1. We can assume that $\gamma : \{0, \ldots, n\} \to \mathcal{X}$ is a $2D$–discrete path, and write $x_i = \gamma(i)$ for $0 \leqslant i \leqslant n$. Hence $\mathsf{d}_U(x_i, x_{i+1}) \leqslant 2DE + E$ for all i, and each x_i lies D–close to a fixed geodesic γ_U from $\pi_U(x_0)$ to $\pi_U(x_n)$.

Choose i, i' such that $0 \leqslant i < i' \leqslant n$ and

- i is minimal with the property that $\mathsf{d}_U(x_0, x_i) > 10(DE + E)$;
- i' is maximal with the property that $\mathsf{d}_U(x_{i'}, x_n) > 10(DE + E)$.

Suppose that $U \subsetneq V$. Since $\mathsf{d}_U(x_0, x_i) > 10(DE + E) > E$, consistency and bounded geodesic image demand that ρ^U_V lies E–close to the geodesic in $\mathcal{C}V$ between $\pi_V(x_0)$ and $\pi_V(x_i)$. Similarly, ρ^U_V lies E–close to the geodesic in $\mathcal{C}V$ between $\pi_V(x_{i'})$ and $\pi_V(x_n)$. Hence there exists $K(D, E)$ such that $\mathsf{d}_V(\rho^U_V, x_i) \leqslant K(D, E)$ and $\mathsf{d}_V(\rho^U_V, x_{i'}) \leqslant K(D, E)$.

Suppose that $U \pitchfork V$. We wish to bound $\mathsf{d}_V(x_i, \rho^U_V)$ and $\mathsf{d}_V(x_{i'}, \rho^U_V)$. There are two cases.

- Suppose $\mathsf{d}_V(x_0, x_n) > E$. Then by consistency we have, say, $\mathsf{d}_V(x_0, \rho^U_V) \leqslant E$ and $\mathsf{d}_U(x_n, \rho^V_U) \leqslant E$. Because of our discrete path assumption, we have $\mathsf{d}_U(x_0, x_i) \leqslant \mathsf{d}_U(x_0, x_{i-1}) + 2DE + E \leqslant 12(DE + E)$. Similarly, $\mathsf{d}_U(x_{i'}, x_n) \leqslant 12(DE + E)$.
  Since $\mathsf{d}_U(x_i, x_0) \leqslant 12(DE + E)$ and $\mathsf{d}_U(x_0, \rho^V_U) > 200DE - 10E$, we have $\mathsf{d}_U(x_i, \rho^V_U) > E$, whence by consistency $\mathsf{d}_V(x_i, \rho^U_V) \leqslant E$. We also have $\mathsf{d}_U(x_{i'}, x_n) > 10(DE + E)$, so $\mathsf{d}_U(x_{i'}, \rho^V_U) > E$, whence $\mathsf{d}_V(x_{i'}, \rho^U_V) \leqslant E$. A symmetric argument works if $\mathsf{d}_V(x_n, \rho^U_V) \leqslant E$, which is the other option provided by the initial application of consistency.
- Now suppose that $\mathsf{d}_V(x_0, x_n) \leqslant E$. Now, since $\mathsf{d}_U(x_i, x_{i'}) \geqslant 200DE - 2(12(DE + E))$, at least one of $\pi_U(x_i), \pi_U(x_{i+1})$ is E–far from ρ^V_U, so by consistency, we have that, say, $\mathsf{d}_V(x_i, \rho^U_V) \leqslant E$. But $\mathsf{d}_V(x_i, x_{i'}) \leqslant 2DE + E$, so $\mathsf{d}_V(x_{i'}, \rho^U_V) \leqslant 2DE + 3E$.

So, we have $K' = K'(D, E)$ so that $\mathsf{d}_V(x_i, \rho^U_V), \mathsf{d}_V(x_{i'}, \rho^U_V) \leqslant K'(D, E)$ for $V \pitchfork U$ or $U \subsetneq V$. So, letting $\nu_1 = \kappa^\times(K'(D, E))$, we have that $x_i, x_{i'} \in \mathcal{N}_{\nu_1}(P_U)$. By hierarchical quasiconvexity and the fact that γ is a (D, D)–hierarchy path, we can increase ν_1 by a uniformly bounded amount (depending on D and the HHS constants) to obtain ν such that $x_j \in \mathcal{N}_\nu(P_U)$ for $i \leqslant j \leqslant i'$.

Letting $\beta = \gamma|_{i,\ldots,i'}$, we have found ν and a subpath β in $\mathcal{N}_\nu(P_U)$.

By construction, for $j < i$, the points $\pi_U(x_j, x_0)$ are $10DE$–close, and similarly for x_j, x_n when $j > i'$. Hence π_U is coarsely constant on any subpath of γ disjoint from β. □

Remark 18.3. The places in the literature where the misstated Proposition 5.17 of [BHS19] is used are:

- In the proof of Lemma 2.8 of [BHS17a]. This was proven independently above, in Proposition 17.1, using the cubical approximation theorem. Alternatively, the use of [BHS19, Proposition 5.17] in [BHS17a, Lemma 2.8] can be replaced by Proposition 18.1.

- In [ABD21], the misstated proposition is used, but in each case Proposition 18.1 can be used verbatim since it is only being used to make π_U coarsely constant (the misstatement in [BHS19] was the unproved assertion that certain other $\pi_V, V \neq U$, are coarsely constant).

It is also used in an early version of [RST18], whose authors have corrected the statement. $\square$

Usually, the following statement about hierarchy paths and product regions is all one really needs. We will not use it later, so we refer the reader to [RST18, Proposition 4.24] for the proof.

Proposition 18.4. *Let $(\mathcal{X}, \mathfrak{F})$ be an HHS. Then for all sufficiently large M, there exist constants ν, D such that the following holds. Let $x, y \in \mathcal{X}$. Let $U \in \mathrm{Rel}_M(x, y)$. Then there exists a (D, D)–hierarchy path γ joining x to y such that γ has a subpath β with the following properties:*

- *$\beta \subset \mathcal{N}_\nu(P_U)$;*
- *the endpoints of β are respectively ν–close in $\mathcal{X}$ to $\mathfrak{g}_U(x), \mathfrak{g}_U(y)$;*
- *for any subpath α of γ disjoint from β, and any $V \sqsubseteq U$ or $V \perp U$, the map π_V is ν–coarsely constant on α.*

We use hierarchy paths occasionally later, although we expect most results could be proved using the cubical approximation of hulls instead. (This parallels the fact that, in a CAT(0) cube complex, it is often more natural to consider the entire median interval between a pair of 0–cubes, rather than privileging one of the many combinatorial geodesics.)

19. Bigsets and all that

In this subsection, consider an HHS $(\mathcal{X}, \mathfrak{F})$ and a group G acting on $(\mathcal{X}, \mathfrak{F})$ by HHS automorphisms (Definition 10.9). Specifically, we recall some notions from [DHS17, Section 6] that will arise later. These pertain to the action of a cyclic subgroup $\langle g \rangle \leqslant G$ on the HHS structure. We slightly strengthen a special case of Lemma 6.6 of [DHS17]. This statement will be used in the proof of Lemma 41.44.

Lemma 19.1 (Simple projection bounds for stabilisers)**.** *Fix $x \in \mathcal{X}$. Let $U \in \mathfrak{F}$ and let $g \in \mathrm{Stab}_G(U)$. Then for all $V \in \mathfrak{F}$ such that $V \pitchfork U$ or $U \sqsubsetneq V$, we have*

$$\mathsf{diam}(\pi_V(\langle g \rangle \cdot x)) \leqslant B_{aut},$$

where $B_{aut} = B_{aut}(x, U)$ is a constant depending on the HHS structure, the element U, and the point x, but not on g.

Proof. Let P_U be the standard product region associated to U and let $\mathfrak{g}_U : \mathcal{X} \to P_U$ be the gate map. Let $D_0 = \mathsf{d}_{\mathcal{X}}(x, \mathfrak{g}_U(x))$. Recall that π_V is (E, E)–coarsely lipschitz for all V. Suppose that $V \pitchfork U$ or $U \sqsubsetneq V$, so that ρ^U_V is defined and bounded. Then by the definition of the gate map to P_U, we have $\mathsf{d}_V(x, \rho^U_V) \leqslant ED_0 + E + \kappa^\times(0)$.

Now, for all $n \in \mathbb{Z}$, we have $g^{-n}V \pitchfork U$ or $U \sqsubsetneq g^{-n}V$, since G acts on $\mathfrak{F}$ preserving the relations $\sqsubseteq, \perp, \pitchfork$ and $gU = U$. So, as above, we have

$$\mathsf{d}_{g^{-n}V}(x, \rho^U_{g^{-n}V}) \leqslant ED_0 + E + \kappa^\times(0).$$

Applying the isometry $g^n : \mathcal{C}g^{-n}V \to \mathcal{C}V$, we get $\mathsf{d}_V(g^n x, \rho^U_V) \leqslant ED_0 + E + \kappa^\times(0)$. So $\mathsf{diam}_{\mathcal{C}V}(\pi_V(\langle g \rangle \cdot x))$ is bounded in terms of D_0 and the HHS constants ($\kappa^\times(0)$ depends only on the HHS constants). Since D_0 depends only on x and U, we are done. $\square$

Recall from [DHS17, Section 6] that the *bigset* *Big*(g) of $g \in G$ is the set of $U \in \mathfrak{F}$ such that $\pi_U(\langle g \rangle \cdot x)$ is unbounded in $\mathcal{C}U$ (for some, and hence any, $x \in \mathcal{X}$). Lemmas 6.3 and 6.7 of [DHS17] show that *Big*(g) consists of pairwise orthogonal elements and hence has cardinality at most χ, and each element of $Big(g)$ is thus stabilised by a uniformly bounded positive power of g.

The following lemma combines (and slightly strengthens) Proposition 6.4 and Lemma 6.6 of [DHS17]. We repeat the proof for self-containment. This statement will be used in Section 35. The proof is essentially the same as in [DHS17]; we explain the extra observation needed for the slight strengthening. We also explain how Proposition 17.1 is used. Some details that are exactly the same as in [DHS17] are omitted.

Lemma 19.2. *Let $x \in X$ and $g \in G$. Let $U \in Big(g)$. Then there exist constants $B_0^{aut} = B_0^{aut}(x, U, g)$ and $B_1^{aut} = B_1^{aut}(x, U, g)$ such that the following hold:*

- $\mathsf{diam}(\pi_V(\langle g\rangle \cdot x)) \leqslant B_0^{aut}$ *whenever $U \subsetneq V$ or $U \pitchfork V$.*
- $\mathsf{diam}(\pi_V(\langle g\rangle \cdot x)) \leqslant B_1^{aut}$ *whenever $V \subsetneq U$ or $V \perp U$ and $V \notin Big(g)$.*

Moreover, if g fixes U (for example if elements of $\mathfrak{F}$ are not orthogonal to their g–translates), then B_0^{aut} can be taken to be independent of g.

Finally, if $Big(g) = \varnothing$, then $\langle g\rangle$ has bounded orbits in $\mathcal{X}$, and the second item above holds for all $V \in \mathfrak{F}$.

Proof. Since U is stabilised by a uniformly bounded power of g, Lemma 19.1 implies both the first item and the "moreover" clause.

Let $\{U_1, \ldots, U_k\} = Big(g)$ (say, $U = U_1$), which is a pairwise-orthogonal set of elements of $\mathfrak{F}$ invariant under $\langle g\rangle$. Let $\mathfrak{T}$ be the set of $V \in \mathfrak{F}$ such that either $V \subsetneq U_i$ for some i, or $V \perp U_i$ for all i.

Note that by the first item, it suffices to bound the diameter of $\pi_V(\langle g\rangle \cdot x)$ for $V \in \mathfrak{T}$. By passing to a uniform power, we can assume that $\langle g\rangle$ fixes each U_i. Now, if the unique $\sqsubseteq$–maximal $S \in \mathfrak{F}$ is not in $Big(g)$, then $\mathfrak{F}_{U_i}$ has strictly lower complexity than $\mathfrak{F}$. Considering the action of $\langle g\rangle$ on $(F_{U_i}, \mathfrak{F}_{U_i})$ by HHS automorphisms (see [DHS20, Section 2] for a precise description of how to make $\langle g\rangle$ act on F_{U_i}), we have by induction on complexity a bound on the diameter of $\pi_V(\langle g\rangle \cdot x)$ for $V \subsetneq U_i$. In the base case, $\mathfrak{F}_{U_i} = \{U_i\}$ and the bound holds vacuously.

Hence the desired conclusion holds for all $V \sqsubseteq U_i$ and all i, unless $Big(g) = \{S\}$.

Let $\mathfrak{T}'$ be the set of V such that $V \sqsubseteq U_i$ for some i. Since $\mathfrak{T}'$ is $\langle g\rangle$–invariant and downward-closed, Proposition 17.1 provides an HHS $(\widehat{\mathcal{X}}, \mathfrak{F} - \mathfrak{T}')$ where g is an HHS automorphism. (The action on $\widehat{\mathcal{X}}$ is the same as the original action on $\mathcal{X}$, but it might now be an action by uniform quasi-isometries instead of isometries. This does not affect the remainder of the argument, since the only genuine isometries needed are at the level of the maps $g : \mathcal{C}U \to \mathcal{C}gU$, and these have not changed.)

In other words, we are considering an action of g on an HHS structure where the bigset of g is empty. Proposition 6.4 of [DHS17] implies that $\langle g\rangle$ has bounded orbits in $\mathcal{X}$ in this case, and hence $\langle g\rangle \cdot x$ has uniformly bounded projections to $\mathcal{C}V$ (i.e. the bound depends on g, x but not V) in this case.

To conclude, we need to prove the second item in the statement in the case where $Big(g) = \{S\}$. This is essentially a bounded geodesic image argument, and we refer the reader to the last two paragraphs of the proof of Lemma 6.6 in [DHS17]. □

Remark 19.3. In our applications, we have an HHG $(G, \mathfrak{F})$, an element $U \in \mathfrak{F}$ such that P_U is at bounded distance in G to 1 (bounded independently of U), and $g \in \mathrm{Stab}_G(U)$. Lemma 19.1 shows that $\mathsf{diam}(\pi_V(\langle g\rangle)) \leqslant B_0^{aut}$, where B_0^{aut} is independent of g and depends on U to the extent that it depends on $\mathsf{d}_G(1, P_U)$, as long as $U \subsetneq V$ or $V \pitchfork U$.

If $U \in Big(g)$, then the diameter of the projection of $\langle g\rangle$ to $\mathcal{C}V$ is bounded in terms of g when $V \subsetneq U$ by Lemma 19.2.

Finally, in practice, the diameter of the projection of $\langle g\rangle$ to elements orthogonal to U will often be bounded through additional assumptions on g arising from extra algebraic information about G. □

20. ☺ Discrete real cubings and hierarchical hyperbolicity

This section is unnecessary for either the subsequent applications to asymptotic cones, or for an understanding of real cubings or hierarchically hyperbolic spaces.

The goal is to relate conditions in [BHS17b, HS20] guaranteeing that a CAT(0) cube complex is a hierarchically hyperbolic space to finite depth of the orthogonal poset-colouring (and hence to "interesting" real cubing structures on the cube complex).

Recall that a real cubing $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is discrete if each $\mathcal{T}^{\bullet}\mathbf{W}, \mathbf{W} \in \mathfrak{F}^{\bullet}$ is a simplicial tree, and $\rho^{\mathbf{V}}_{\mathbf{W}}$ is a vertex of $\mathcal{T}^{\bullet}\mathbf{W}$ whenever it is defined and a single point.

In Theorem 4.16, we have proven that every discrete real cubing is median-preservingly, ℓ_1–isometric to a finite-dimensional CAT(0) cube complex and conversely, every finite-dimensional CAT(0) cube complex is median-preservingly ℓ_1–isometric to a discrete real cubing.

This theorem is not much more informative than the fact that any CAT(0) cube complex is an isometrically embedded median subspace of an (infinite) cube. In the rest of the section, we show that one gets a more useful real cubing structure on $\mathbf{X}$ when the orthogonal poset-colouring of the hyperplanes in $\mathbf{X}$ has finite depth, and we relate this to conditions in [BHS17b] ensuring that $\mathbf{X}$ is a hierarchically hyperbolic space.

20.1. ☺ Factor systems.

Fix a CAT(0) cube complex $\mathbf{X}$. We recall the following definition from [HS20], see also [BHS17b, Section 8]:

Definition 20.1 (Hyperclosure, factor system, weak factor system)**.** Let $\mathfrak{H}_1$ be the smallest collection of convex subcomplexes of $\mathbf{X}$ satisfying the following conditions:

- $\mathbf{X} \in \mathfrak{H}_1$;
- each combinatorial hyperplane is in $\mathfrak{H}_1$ (recall that if $\hat{h}$ is a hyperplane, the union of the closed cubes intersecting $\hat{h}$ is a cube complex isomorphic to $\hat{h} \times [-\frac{1}{2}, \frac{1}{2}]$, and a *combinatorial hyperplane* is a subcomplex of the form $\hat{h} \times \{\pm\frac{1}{2}\}$);
- if $F, F' \in \mathfrak{H}_1$, then $\mathfrak{g}_F(F') \in \mathfrak{H}_1$, where $\mathfrak{g}_F : \mathbf{X} \to F$ is the gate map;
- if $F \in \mathfrak{H}_1$ and F' is a subcomplex parallel to F, then $F' \in \mathfrak{H}_1$.

We let $\mathfrak{H}$ be the set of parallelism classes of subcomplexes in $\mathfrak{H}_1$. The set $\mathfrak{H}_1$ is called the *hyperclosure* . Any set of subcomplexes that is closed under parallelism and gates in the preceding sense, and contains $\mathfrak{H}_1$, is called a *candidate factor system*.

If $\mathfrak{H}_1$ is a uniformly locally finite collection of subcomplexes (every point is contained in boundedly many elements of $\mathfrak{H}$), then $\mathfrak{H}$ is a *factor system*.

If $\mathfrak{H}$ has the property that there exists $N < \infty$ such that N bounds the lengths of chains in the partial ordering of $\mathfrak{H}_1$ by inclusion, then $\mathfrak{H}$ is a *weak factor system*. □

The original intent of the definition of a factor system in [BHS17b] was to provide an HHS structure on $\mathbf{X}$. See [BHS17b, Remark 13.2].

Specifically, given parallelism classes $[F], [F'] \in \mathfrak{H}$, where $\mathfrak{H}$ is a candidate factor system, we write:

- $[F] \sqsubseteq [F']$ (nested) if representatives F, F' of the parallelism classes can be chosen so that $F \subseteq F'$; equivalently, for any representatives F, F' of the given parallelism classes, F is parallel to a subcomplex of F';
- $[F] \perp [F']$ (orthogonal) if the representatives F, F' can be chosen so that the inclusions $F, F' \to \mathbf{X}$ extend to a convex embedding $F \times F' \to \mathbf{X}$;

- $[F] \pitchfork [F']$ (transverse) otherwise.

From the definitions, we immediately get:

Lemma 20.2. *Every factor system is a weak factor system. If $\mathfrak{H}$ is a weak factor system, then the partially ordered set $(\mathfrak{H}, \sqsubseteq)$ has a uniform bound on the length of chains.*

Moreover:

Lemma 20.3. *Let $\mathbf{X}$ be a CAT(0) cube complex and let $\mathfrak{H}$ be a weak factor system, and let N be the bound on the length of $\sqsubseteq$–chains. Then $\mathbf{X}$ is finite-dimensional and there exists $M < \infty$ such that if $[F_1], \ldots, [F_m] \in \mathfrak{H}$ are pairwise orthogonal, then $m \leqslant M$.*

Proof. We first verify finite dimension. If c is a d–cube in $\mathbf{X}$, then each $(\dim c - k)$–dimensional face of c is contained in a codimension–k combinatorial hyperplane, which necessarily belongs to $\mathfrak{H}_1$, and we obtain a $\sqsubseteq$–chain of length d. So the dimension is bounded in terms of N.

Let $[F_1], \ldots, [F_m]$ be pairwise-orthogonal. Then by induction on m, one can choose $F_1, \ldots, F_m$ representing the given parallelism classes, such that $F_i \hookrightarrow \mathbf{X}$ extends to a convex embedding $F_1 \times \cdots \times F_m \to \mathbf{X}$. (Indeed, by induction, we have $P = F_1 \times \cdots \times F_{m-1}$, and every hyperplane crossing P crosses every hyperplane crossing F_m, by the definition of orthogonality. Now apply Proposition 2.22.)

At most one of the F_i is a single point (there is only one parallelism class of vertices), so without loss of generality, none is. Thus P is contained in a combinatorial hyperplane H, and the set $\mathfrak{H}_H$ of elements of $\mathfrak{H}$ nested in H is a weak factor system on H with $\sqsubseteq$–chains bounded by $N-1$, where N is the bound for $\mathfrak{H}$, since $H \subsetneq \mathbf{X}$. By induction on N, there exists M' such that pairwise-orthogonal subsets of $\mathfrak{H}_H$ have cardinality at most M', so since $F_1, \ldots, F_{m-1} \sqsubseteq H$, we have $m - 1 \leqslant M'$ and hence $m \leqslant M' + 1$. So, in fact, $M \leqslant N$ suffices. □

Observe that if $[F], [F'] \in \mathfrak{H}$, where $\mathfrak{H}$ is *any* weak factor system, then $[\mathfrak{g}_F(F')] \in \mathfrak{H}$ has the property that $[F''] \sqsubseteq [F], [F']$ implies that $[F''] \sqsubseteq [\mathfrak{g}_F(F')]$. So $\mathfrak{H}$ automatically has the *wedge property.*

Now, if $F \in \mathfrak{H}_1$, then there is a maximal convex subcomplex $F^\perp$ of $\mathbf{X}$, unique up to parallelism, such that $F \to \mathbf{X}$ extends to $F \times F^\perp \hookrightarrow \mathbf{X}$. If F, F_1 are parallel, then so are $F^\perp, F_1^\perp$. If $\mathfrak{H}$ is a weak factor system with the property that $F^\perp \in \mathfrak{H}_1$ whenever $F \in \mathfrak{H}_1$, then $\mathfrak{H}$ is a *weak factor system with clean containers.*

In [BHS17b], the authors work with factor systems, but one could achieve the same with weak factor systems:

Proposition 20.4. *Let $\mathbf{X}$ be a CAT(0) cube complex with a weak factor system $\mathfrak{H}$. Then $(\mathbf{X}, \mathfrak{H})$ is a hierarchically hyperbolic space, where the hyperbolic spaces $\mathcal{C}F, F \in \mathfrak{H}$ and $\pi_F : \mathbf{X} \to \mathcal{C}F$ are as in* [BHS17b, Remark 13.2] *(and in particular are uniformly quasi-isometric to trees). Moreover, this HHS structure has* wedges *in the sense of Definition 35.1 and, if $\mathfrak{H}$ has clean containers, then the HHS has clean containers in the sense of Definition 35.3.*

Proof. Remark 13.2 of [BHS17b] provides the desired hierarchically hyperbolic structure when $\mathfrak{H}$ is a factor system. We first remark that when $\mathbf{X}$ is a uniformly locally finite CAT(0) cube complex — equivalently, every vertex is contained in at most d hyperplane carriers for some $d < \infty$ — then any weak factor system is actually a factor system, and we are done. This can be seen from the fact that $\mathfrak{H}_1$ is closed under taking intersections of subcomplexes (because $\mathfrak{g}_F(F') = F \cap F'$ when the latter is nonempty) and a pigeonhole argument. One then bounds the number of elements of $\mathfrak{H}_1$ that contain a given vertex in terms of d and the length N of a longest $\sqsubseteq$–chain in $\mathfrak{H}_1$.

When $\mathbf{X}$ is not uniformly locally finite, we instead check directly the HHS axioms from Definition 10.1 below (see also Definition 1.1 in [BHS19], which is an equivalent formulation of the definition of an HHS from [BHS17b]).

Axioms (1)-(4) follow exactly as in [BHS17b, Remark 13.2] — the arguments do not use local finiteness of the factor system. Axiom (5) simply asserts that the length of $\sqsubseteq$–chains is bounded, which is part of the definition of a weak factor system. Axiom (6) (large links) is [BHS17b, Proposition 9.4] and Axiom (7) (bounded geodesic image) follows from [BHS17b, Proposition 8.20], neither of which use local finiteness.

Axiom (9) (uniqueness) follows from the stronger *distance formula*, which is Theorem 9.1 in [BHS17b]. This is proved by induction on the multiplicity of $\mathfrak{H}_1$, so it uses local finiteness, but the induction goes through with no significant change if one inducts instead on the maximal length of a $\sqsubseteq$–chain, observing that if $F \in \mathfrak{H}_1$, then the set of $F' \in \mathfrak{H}_1$ that are parallel to subcomplexes of F forms a weak factor system in F in which chains have length bounded by $N-1$, where N is the bound for $\mathfrak{H}_1$.

Finally, Axiom (8) (partial realisation) follows from [BHS17b, Theorem 12.4]. Again, in the proof of the latter theorem, one must replace an induction on multiplicity by an induction on the length of $\sqsubseteq$–chains exactly as for Axiom (9). One can also verify partial realisation directly from the construction. Let $F_1, \ldots, F_n \in \mathfrak{H}_1$ be pairwise orthogonal. By the definition of orthogonality and the Helly property for convex subcomplexes, we can choose the F_i in their parallelism classes so that the convex hull of their union is a convex subcomplex of the form $F_1 \times \cdots \times F_n$. Choose a point x_i in the factored contact graph $\mathcal{C}F_i$ for each i. This amounts to choosing, for each i, a hyperplane h_i crossing F_i. These hyperplanes pairwise cross, and hence mutually intersect in a point $y \in F_1 \times \cdots \times F_n$. By construction, y projects uniformly close to x_i in $\mathcal{C}F_i$, for each i, and the definition of the points $\rho_U^{F_i}$ (for $U \in \mathfrak{H}_1$ for which $\rho_U^{F_i}$ is defined) from [BHS17b] now shows that y is the desired partial realisation point. □

20.2. ☺ **Orthogonal poset-colouring and factor systems.** Parallel convex subcomplexes cross the same walls, and by the next lemma, this amounts to saying that they cross the same hyperplanes. Since we are working with convex subcomplexes, which correspond in the natural way to median-convex subsets of the vertex set of $\mathbf{X}$, we will use the terms "hyperplane" and "wall" interchangeably in this section.

Lemma 20.5 (Walls in a cube complex)**.** *Let $\hat{w} = \{w, w^*\}$ be a wall in the CAT(0) cube complex $\mathbf{X}$. Then there is a unique hyperplane $\hat{h}$ such that the partition of $\mathbf{X}^{(0)}$ induced by $\hat{h}$ is $\{w^{(0)}, \mathbf{X}^{(0)} - w^{(0)}\}$.*

Proof. Let $x \in w, y \in w^*$ be vertices. Let γ be a path in $\mathbf{X}^{(1)}$ joining x to y. Then γ must have an edge with one endpoint in w and one endpoint in w^*, so we can assume that x, y are the endpoints of some edge e.

Let $\hat{h}$ be the hyperplane dual to e. Let h, h^* be the halfspaces associated to $\hat{h}$ and containing x, y respectively. So, $\hat{h}$ separates x, y. On the other hand, $\hat{w}$ separates x, y. Now note that in $e^{(0)}$, there is a unique wall separating x, y, so by Lemma 2.12, there is a unique wall in $\mathbf{X}$ separating x, y, so $h = w$ and $h^* = w^*$. □

Finally, let $\mathfrak{F}^\bullet$ be the set of colours used in the canonical orthogonal poset-colouring of the median space $\mathbf{X}^{(0)}$. Recall the construction of $\mathfrak{F}^\bullet$ from Section 3.3:

- start with the set $\mathfrak{F}_0^\bullet$ consisting of all non-empty sets of hyperplanes of the form $\bigcap_{i \in I} \mathcal{W}(\hat{h}_i)$, where $\hat{h}_i$ is a hyperplane and $\mathcal{W}(\hat{h}_i)$ is the set of hyperplanes crossing it; we include $\mathcal{W}$;

- let $\mathfrak{F}^\bullet_1 \subset \mathfrak{F}^\bullet_0$ consist of $\mathcal{W}$, together with any $\mathcal{U} \in \mathfrak{F}^\bullet_0$ that is orthogonal to some subset $\mathcal{V} \in \mathfrak{F}^\bullet_0$ (i.e. every hyperplane in $\mathcal{U}$ crosses every hyperplane in $\mathcal{V}$);
- finally, let $\mathfrak{F}^\bullet$ be the set containing the inclusion–maximal element of each equivalence class in $\mathfrak{F}^\bullet_1$, where $\mathcal{U}$ and $\mathcal{V}$ are equivalent if they are orthogonal to the same elements of $\mathfrak{F}^\bullet_1$.

The map $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ takes each hyperplane $\hat{w}$ to the set $Col(\hat{w})$ of hyperplanes that cross all hyperplanes crossing $\hat{w}$ (so e.g. $\hat{w} \in Col(\hat{w})$).

We now relate finite depth of the orthogonal poset-colouring of $\mathbf{X}$ to the existence of a weak factor system (with wedges and clean containers).

Theorem 20.6. *Let $\mathbf{X}$ be a CAT(0) cube complex. Suppose that the orthogonal poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ has finite depth. Then $\mathbf{X}$ has a weak factor system $\mathfrak{H}$ with wedges and clean containers. Hence $(\mathbf{X}, \mathfrak{H})$ is a hierarchically hyperbolic space with wedges and clean containers, and $(\mathbf{X}, \mathfrak{F}^\bullet)$ is a real cubing with wedges and clean containers.*

Conversely, suppose that $\mathbf{X}$ admits a weak factor system $\mathfrak{H}$. Then $\mathfrak{F}^\bullet$ has finite depth, and hence $\mathbf{X}$ admits a weak factor system with wedges and clean containers.

Thus, in particular, $\mathbf{X}$ has a weak factor system if and only if it has a weak factor system with wedges and clean containers.

Two remarks before proving the theorem:

Remark 20.7. The converse direction of Theorem 20.6 strengthens a result in [HS20, Proposition 5.1], which shows that a CAT(0) cube complex with a factor system *and a proper co-compact group action* admits a factor system with clean containers (and, as remarked above, the wedge property is automatic because we can take gates). A careful look at the argument in [HS20] reveals, however, that the group action isn't used for clean containers if one already knows the hyperclosure is a factor system. □

Remark 20.8. The hierarchically hyperbolic structure $(\mathbf{X}, \mathfrak{H})$ and the ℝ–cubing structure $(\mathbf{X}, \mathfrak{F}^\bullet)$ are closely related, but with important differences. For example, if $\mathbf{X}$ is the universal cover of the wedge of two tori with the usual cubical structure, then the $\sqsubseteq$–maximal element of $\mathfrak{H}$ is $\mathbf{X}$, and the associated hyperbolic space is $\mathcal{C}\mathbf{X}$, which is quasi-isometric to the Bass-Serre tree of the free product splitting of $\mathbb{Z}^2 * \mathbb{Z}^2$. On the other hand, the $\sqsubseteq$–maximal element of $\mathfrak{F}^\bullet$ is $\mathcal{W}$, and the associated real tree is a single point, by the construction in the proof of Theorem 5.1, because every wall crosses one of the constituent flats of $\mathbf{X}$, and hence has properly nested colour. Now, for any $x, y \in \mathbf{X}$, the quantities

$$\sum_{[F] \in \mathfrak{H}} \mathsf{d}_{\mathcal{C}[F]}(x, y)$$

and

$$\sum_{\mathbf{U} \in \mathfrak{F}^\bullet} \mathsf{d}_{\mathcal{T}^\bullet \mathbf{U}}(x, y)$$

differ by a bounded multiplicative and additive error, in view of Theorem 10.7 and Definition 4.2.(5), which relate these quantities to $\mathsf{d}_{\mathbf{X}}(x, y)$. The reader can resolve cognitive dissonance by noting that, to travel in $\mathbf{X}$ in a way that makes progress in the Bass-Serre tree, one must cross walls, and hence accumulate distance in the various real trees $\mathcal{T}^\bullet\mathbf{U}$. If we replaced the wedge of two tori by a pair of tori joined by an edge, the universal cover would be a quasi-isometric CAT(0) cube complex $\mathbf{X}'$. The HHS structure will not have changed, but the $\sqsubseteq$–maximal ℝ–tree in the real cubing structure will now be a copy of the Bass-Serre tree. □

Proof of Theorem 20.6. Suppose that $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ has finite depth. Let $\mathbf{U} \in \mathfrak{F}^\bullet$.

Claim 14. There is a finite set $\hat{h}_1, \ldots, \hat{h}_n$ such that the $\sim$–class $\mathbf{U}$ is represented by $\bigcap_{i=1}^n \mathcal{W}(\hat{h}_i)$, and this set is maximal in the $\sim$–class. Moreover, n is bounded by the depth of the orthogonal poset-colouring.

This set is empty if and only if $\mathbf{U} \in \mathfrak{F}^\bullet$ is the unique $\sqsubseteq$–maximal element.

Proof of Claim 14. Let $\mathcal{U} = \bigcap_{i \in I} \mathcal{W}(\hat{h}_i)$ be a set in $\mathfrak{F}_0^\bullet$ representing the $\sim$–class $\mathbf{U}$. Then $\mathbf{U}$ is also represented by the maximal element $(\mathcal{U}^\perp)^\perp$.

In fact, $\mathcal{U} = (\mathcal{U}^\perp)^\perp$. Indeed, we have $\mathcal{U}^\perp = \bigcap_{\hat{w} \in \mathcal{U}} \mathcal{W}(\hat{w})$. Every wall $\hat{w} \in \mathcal{U}$ crosses every wall in $\mathcal{U}^\perp$, by definition, so $\mathcal{U} \subset (\mathcal{U}^\perp)^\perp$. On the other hand, if $\hat{w} \in (\mathcal{U}^\perp)^\perp$, then $\hat{w}$ crosses every wall in $\mathcal{U}^\perp$. In particular, $\hat{w}$ crosses every $\hat{h}_i$, so $\hat{w} \in \mathcal{U}$.

So, in view of Lemma 3.23, we can take $\mathcal{U}$ to be the $\sqsubseteq$–maximal representative of the $\sim$–class $\mathbf{U}$.

Let $(I_n)_{n \geqslant 0}$ be a sequence of finite subsets of I with $I_n \subset I_{n+1}$ for all n. Let $\mathcal{U}_n \in \mathfrak{F}_0^\bullet$ be given by

$$\mathcal{U}_n = \bigcap_{i \in I_n} \mathcal{W}(\hat{h}_i).$$

So, we have $\mathcal{U}_n \supseteq \mathcal{U}_{n+1} \supseteq \mathcal{U}$ for all $n \geqslant 0$. As above, $\mathcal{U}_n$ is maximal in its $\sim$–class. So, if the sequence $(\mathcal{U}_n)_{n \geqslant 0}$ fails to stabilise, finite depth of $(\mathfrak{F}^\bullet, \sqsubseteq)$ is contradicted.

Hence there exists N such that $n \geqslant N$ implies $\mathcal{U}_n = \mathcal{U}_N$. This completes the proof. □

Claim 15. For each $\hat{h}_i$ as in Claim 14, let H_i be one of the combinatorial hyperplanes bounding the carrier of $\hat{h}_i$. Let

$$F_{\mathbf{U}} = \mathfrak{g}_{H_n}(\mathfrak{g}_{H_{n-1}}(\cdots \mathfrak{g}_{H_1}(H_1)) \cdots).$$

Then $F_{\mathbf{U}}$ is well-defined up to parallelism (independent of the choice of the $\hat{h}_i$ satisfying the conclusion of Claim 14 and independent of their order).

Proof of Claim 15. From the definition of $F_{\mathbf{U}}$, we get

$$\mathcal{W}(F_{\mathbf{U}}) = \bigcap_{i=1}^n \mathcal{W}(\hat{h}_i),$$

which is the unique maximal element of the $\sim$–class $\mathbf{U}$. Since this element is independent of the $\hat{h}_i$, the same is true of $\mathcal{W}(F_{\mathbf{U}})$, so $F_{\mathbf{U}}$ is uniquely determined up to parallelism. □

We now verify that the set $\mathcal{H}_1$ of convex subcomplexes parallel to some element of $\{F_{\mathbf{U}} : \mathbf{U} \in \mathfrak{F}^\bullet\}$ is a candidate factor system and that the quotient $\mathcal{H}$ of $\mathcal{H}_1$ by the parallelism relation is a weak factor system with clean containers (wedges are automatic, as mentioned above).

Indeed, $\mathbf{X} = F_{\mathbf{S}}$, where $\mathbf{S}$ is the $\sim$–class of the entire set $\mathcal{W}$ of walls (the $\sqsubseteq$–maximal element of $\mathfrak{F}^\bullet$). If $F \in \mathfrak{H}_1$, then the same is true for any subcomplex parallel to F, by definition of $\mathcal{H}_1$. By construction, each combinatorial hyperplane H has the form $F_{\mathbf{U}}$, where $\mathbf{U} = \mathcal{W}(\hat{h})$, where $\hat{h}$ is the hyperplane whose carrier contains H as a bounding copy of $\hat{h}$. If $F, F' \in \mathcal{H}_1$, then, up to parallelism, we can write $F = F_{\mathbf{U}}, F' = F_{\mathbf{U}'}$. Then $\mathfrak{g}_{F'}(F) = F_{\mathbf{U} \wedge \mathbf{U}'}$. This shows that $\mathcal{H}_1$ is a candidate factor system, and the fact that $\mathcal{H}_1$ is a weak factor system now follows from finite depth. Indeed, if $F_{\mathbf{U}_1} \subsetneq \cdots \subsetneq F_{\mathbf{U}_n}$, then $\mathbf{U}_1 \subsetneq \cdots \subsetneq \mathbf{U}_n$.

From Proposition 20.4, we thus get the claimed HHS structure $(\mathbf{X}, \mathfrak{H})$.

Conversely, suppose that $\mathfrak{H}_1$ is a candidate factor system such that $\mathfrak{H}$ is a weak factor system. Then we can assume that $\mathfrak{H}_1$ is the hyperclosure, since a sub-weak factor system of a weak factor system is a weak factor system, by the definitions. We now check that the orthogonal poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^\bullet$ has finite depth. The first part of the theorem then shows that $\mathbf{X}$ has a weak factor system with wedges and clean containers.

Suppose that the orthogonal poset-colouring has infinite depth. Let $\mathfrak{F}^{\bullet}_{fin} \subset \mathfrak{F}^{\bullet}$ be the set of $\mathbf{U}$ represented by sets of walls of the form $\mathcal{U} = \bigcap_{i\in J} \mathcal{W}(\hat{h}_i)$ with J finite. If there is a bound on the length of $\subsetneq$–chains in $\mathfrak{F}^{\bullet}_{fin}$, then we obtain a corresponding bound in $\mathfrak{F}^{\bullet}$. So, we can assume that for any $n \geqslant 0$, $\mathfrak{F}^{\bullet}_{fin}$ contains a chain $\mathbf{U}_1 \subsetneq \mathbf{U}_2 \subsetneq \cdots \subsetneq \mathbf{U}_n$.

But for each $\mathbf{U}_i$, we can construct $F_{\mathbf{U}_i} \in \mathfrak{H}_1$ as above, and thus obtain a length–n chain of proper inclusions (up to parallelism) in $\mathfrak{H}_1$. For sufficiently large n, this contradicts that the latter is a weak factor system. Hence the orthogonal poset-colouring has finite depth.

In the situation where the depth of the orthogonal poset-colouring is finite, the weak factor system $\mathfrak{H}_1$ has wedges (the wedge of $F_{\mathbf{U}}, F_{\mathbf{V}}$ is $\mathfrak{g}_{F_{\mathbf{U}}}(F_{\mathbf{V}})$). The clean containers property follows since $\mathbf{X}$ contains a convex product region $F_{\mathbf{U}} \times F_{\mathbf{U}^{\perp}}$ for each $\mathbf{U}$, and $\mathbf{U} \perp \mathbf{V}$ is equivalent to $\mathbf{V} \sqsubseteq \mathbf{U}^{\perp}$ is equivalent to the existence of a convex product region $F_{\mathbf{U}} \times F_{\mathbf{V}}$.

(Somewhat more precisely, let $\mathbf{U} \in \mathfrak{F}^{\bullet}$ and let $F_{\mathbf{U}}$ be as above, so that the walls crossing $F_{\mathbf{U}}$ are exactly those in the set $\mathcal{U} = \bigcap_{i=1}^{k} \mathcal{W}(\hat{h}_i)$. Let $F^{\perp}$ be the maximal convex subcomplex such that $F_{\mathbf{U}}$ lies, up to parallelism, in a convex product region $F_{\mathbf{U}} \times F^{\perp}$. Then the set of walls crossing $F^{\perp}$ is exactly the set of walls $\hat{w}$ such that $\hat{w}$ crosses all walls crossing $F_{\mathbf{U}}$, i.e. $\hat{w}$ crosses all elements of $\mathcal{U}$. In other words, $\mathcal{W}(F^{\perp}) = \bigcap_{\hat{u}\in\mathcal{U}} \mathcal{W}(\hat{u})$. Apply Claim 14 to the latter set, so that we can construct $F_{\mathcal{W}(F^{\perp})}$ as an element of $\mathfrak{H}_1$. Then observe that $F_{\mathcal{W}(F^{\perp})} = F^{\perp}$.)

This completes the proof. □

Remark 20.9. The preceding theorem does not quite characterise those CAT(0) cube complexes that are hierarchically hyperbolic spaces in terms of finite depth of the orthogonal poset-colouring, because $\mathbf{X}$ can admit hierarchically hyperbolic structures different from the on from Proposition 20.4. This is true even if we restrict to hierarchically hyperbolic structures giving rise to a coarse median operator (see Section 12) compatible with the median coming from the cubical structure.

For example, any hyperbolic CAT(0) cube complex is an HHS in a trivial way, by virtue of being hyperbolic — but any hyperbolic CAT(0) cube complex also admits a factor system. □

21. ☺ Questions and remarks

21.1. ☺ Characterisation of CAT(0) cube complexes that admit an HHS structure. In Theorem 20.6, we provided sufficient conditions for a CAT(0) cube complex to admit an HHS structure, namely existence of a weak factor system or equivalently, an orthogonal poset-colouring of finite depth. However, this is not a necessary condition in general.

The following is an example of a CAT(0) cube complex which admits an HHS structure but does not admit a weak factor system (i.e. the orthogonal poset-colouring has infinite depth).

Example 21.1 (Infinite-depth cube complex with an HHS structure)**.** Let X be the CAT(0) cube complex defined as follows. For $i \in \mathbb{N}$, let s_i and t_j be the segments $[0, i]$ and $[0, j]$.

For all $i \in \mathbb{N}$ consider the product $P_i = s_i \times St_i$ where St_i is the star formed from the segments $t_1, \ldots, t_i$, i.e. the quotient of the union of the segments $t_1, \ldots, t_i$ by identifying the point 0 in all the segments $t_1, \ldots, t_j$.

Then X is obtained as the quotient of the union of P_i, $i \in \mathbb{N}$ after identifying the copies of the segments $\{0\} \times t_j$ in the pieces P_i, $j \leqslant i \in \mathbb{N}$.

Let $\mathfrak{F}$ be the set $\mathfrak{F} = \{U_i, V_i, U, V, S \mid i \in \mathbb{N}\}$ with the following relations:

- $U_i \perp V_j$ for all $i \in \mathbb{N}$ and $j \leqslant i$;
- $U_i \sqsubseteq U$ for all $i \in \mathbb{N}$;
- $V_i \sqsubseteq V$ for all $i \in \mathbb{N}$;
- $U_i, V_i, U, V \sqsubseteq S$ for all $i \in \mathbb{N}$;

- otherwise the elements are transverse.

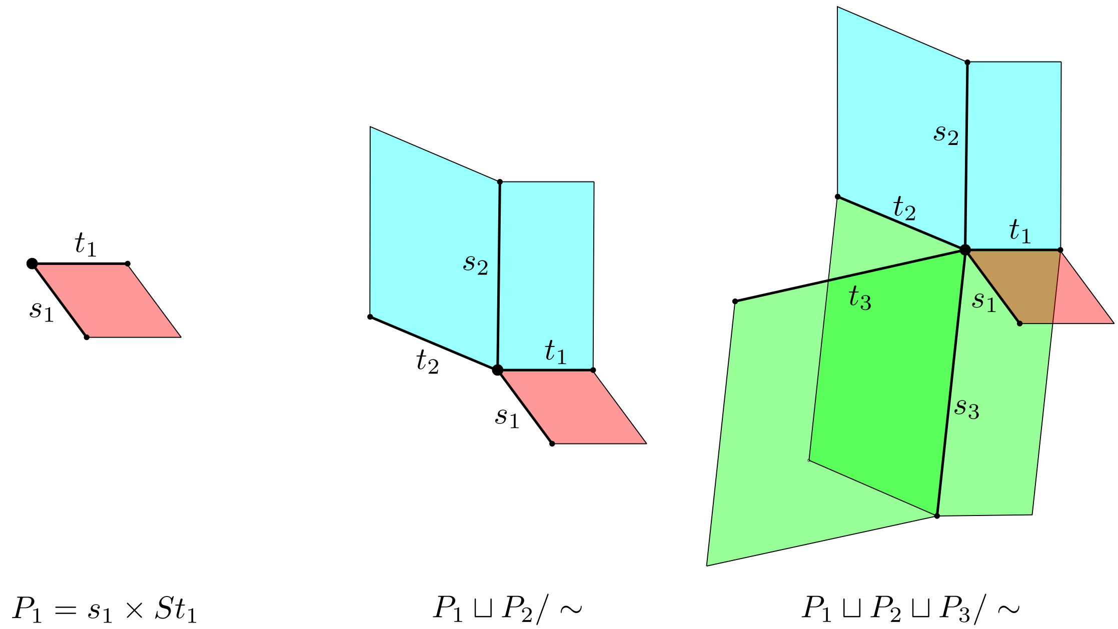

FIGURE 21. First three steps of the construction of X

We associate to U_i and V_i a segment $[0, i]$ and to U, V, S a point. We identify the segments with s_i and t_i in X respectively. Then the map π is defined as the closest point projection.

For $U, V \in \mathfrak{F}$, if ρ^U_F is defined, we defined it as $\rho^U_F = 0$.

One can check that $(X, \mathfrak{F})$ is an HHS structure.

However, the orthogonal poset-colouring has infinite depth. Indeed let V_i' be the set of walls that cross s_i and U_i' the set of walls that cross each wall in V_i'. Then since each wall that crosses s_i also crosses s_j for all $j > i$, from the construction of the orthogonal poset-colouring, we have that $U_j' \sqsubseteq U_i'$ for all $j > i$ and $i \in \mathbb{N}$. Therefore, the depth is infinite. □

The key reason that allows X to admit an HHS structure but not a factor system, is the fact that each point in X and the basepoint 0 have at most two different projections in the segments associated to the index set, i.e. their projections differ in at most one segment associated to U_i $i \in \mathbb{N}$ and one associated to V_i: for all $x \in X$, there exists a unique s_i and t_j such that $\pi_{s_i}(x) \neq \pi_{s_i}(0)$ and $\pi_{t_j}(x) \neq \pi_{t_j}(0)$. In other words, there are infinitely many transverse elements in the index set but the "partially ordered chains" determined by pairs of points have bounded length.

This motivates:

Question 21.2. Is the following true?

The CAT(0) cube complex X admits an HHS structure (inducing a coarse median at bounded distance from the median coming from the cubical structure) if and only if, for any infinite $\sqsubseteq$-chain in the canonical orthogonal system, there exists W in the chain and $N = N(W)$ such that for all pairs of points in F_W, the number of $\sqsubseteq$–maximal elements on which the projections of the two points differ (by more than a bounded amount) is bounded by N.

We believe that, for locally finite cube complexes admitting proper cocompact group actions, one may be able to make an argument based on the passing-up lemma (Lemma 11.1) to show that essentially any HHS structure must come from a factor system, i.e. from the orthogonal poset-colouring:

Question 21.3. Let X be a locally finite CAT(0) cube complex that admits a proper cocompact action of a (necessarily finitely generated) group. Is it true that X admits an HHS structure if and only if X has a finite depth orthogonal poset-colouring if and only if X has a weak factor system?

The related question — whether every locally finite CAT(0) cube complex admitting a proper cocompact group action has a factor system — was first posed in [BHS17b] and was studied in [HS20]. At present, all proper cocompact CAT(0) cube complexes of which we are aware have factor systems, and the question seems quite difficult. Hence it might be useful to consider Question 21.3, which is easier in principle because it allows one to assume there is some HHS structure.

The following example (of a CAT(0) cube complex with a proper cocompact group action and no weak factor system — and, it seems, no HHG structure) illustrates why one should assume local finiteness in the question from [BHS17b] and hence in the easier version stated above:

Example 21.4 (Infinitely generated cubical group which is not an HHG)**.** The universal cover of the Salvetti complex of an infinitely generated RAAG $\mathbb{G}$ constructed below is a locally infinite CAT(0) cube complex (the 1–skeleton is a median graph), but this RAAG is not an HHG. Note that the Salvetti complex of $\mathbb{G}$ is two-dimensional.

Let $\mathbb{G}$ be defined by the following presentation. Let $A = \{a_i\}_{i\in\mathbb{N}}$ and $B = \{b_i\}_{i\in\mathbb{N}}$. Let

$$\mathbb{G} = \langle A, B \mid [a_i, b_j] = 1 \text{ for } i \geqslant k, j < k, \text{ for all } k\rangle.$$

Since $\mathbb{G}$ is the vertex set of the universal cover of the Salvetti complex, it is a (discrete) median metric space. But the *hyperclosure* construction does not yield an HHG structure, since the finite complexity axiom would be violated. By considering *standard product regions*, it seems that in fact there is no HHG structure on $\mathbb{G}$ that coarsely induces the median associated to the cubical structure of the Salvetti complex. □

Relatedly, it would be interesting to answer the following question:

Question 21.5. Is there a locally finite CAT(0) cube complex which admits an HHS structure but not a weak factor system? The HHS structure should induce a coarse median that is bounded distance from the median coming from the cubical structure. (Otherwise, simple "staircase" constructions provide examples of cube complexes with no factor system that are nonetheless HHS because they are quasi-isometric to $[0, \infty) \times [0, \infty)$.)

21.2. ☺ **Characterisation of real cubings that admit an HHS structure.** In Theorem 20.6, we provided a sufficient condition for a CAT(0) cube complex to admit an HHS structure, namely to have a weak factor system or equivalently, an orthogonal poset-colouring of finite depth.

We believe that the proof for CAT(0) cube complexes can be generalised to real cubings. Namely, if a real cubing has a poset-colouring of finite depth, then it admits a structure of HHS. Recall that having a poset-colouring of finite depth is equivalent to have an index set with clean containers. More precisely, we formulate the following question:

Question 21.6. Let $(\mathbf{X}, \mathfrak{F}^\bullet)$ be a real cubing with finite depth orthogonal poset-colouring. Does there exists an index set $\mathfrak{F}'$ such that $(\mathbf{X}, \mathfrak{F})$ is an HHS, with the associated coarse median at bounded distance from the median coming from the real cubing structure?

Without loss of generality, one can assume that $\mathfrak{F}^{\bullet}$ is the index set defined by the finite depth orthogonal poset-colouring, see Section 3.3. In particular, the index set has wedges, joins, clean containers and a maximal $\sqsubseteq$–element.

Notice that with this index set, $(\mathbf{X}, \mathfrak{F}^{\bullet})$ is not an HHS since it may fail the uniqueness axiom. Indeed, one can consider Example 4.27 with the orthogonal index set. The tree associated to the maximal element is a point and so there are points at distance as large as we want in the space whose projection are small in all the trees. Take for instance 1 and the element $(abc)^n$ - the points are at distance $3n$ but all the projection into trees of the index set are at distance at most 1.

In order to construct the trees given the orthogonal poset-colouring of walls, we consider the subspace $\mathbf{F_U}$, define a pseudo-metric on it and quotiented the space to obtain a metric space which we prove to be a real tree. Instead of that approach, one needs to follow the HHS structure on a CAT(0) cube complex and consider the analog of the contact graph associated to $\mathbf{F_U}$, i.e. one needs to cone–off some product regions and obtain a hyperbolic space.

More precisely, given a colour $\mathbf{U}$ (a subset of walls), let $\mathcal{C}_0\mathbf{U}$ have a vertex for each wall whose colour is nested in $\mathbf{U}$, with adjacency of two walls if the fio-measure of the halfspaces separating them is at most some global constant. Let w_1, w_n be two walls. Let $w_1, w_2, \ldots, w_n$ be a path in $\mathcal{C}_0\mathbf{U}$ joining them. Let v be any wall separating w_1, w_n (necessarily the colour of v is nested in $\mathbf{U}$, so it is a vertex of $\mathcal{C}_0\mathbf{U}$). Then some w_i has to come close in $\mathbf{X}$ to v, and hence close in $\mathcal{C}_0\mathbf{U}$.

The idea is that some stronger version of Manning's Bottleneck Criterion [Man05] should imply that $\mathcal{C}_0\mathbf{U}$ is a quasi-tree, as for cube complexes in [Hag14].

Now, $\mathcal{C}\mathbf{U}$ is obtained from $\mathcal{C}_0\mathbf{U}$ by coning off each $\mathcal{C}_0\mathbf{V}$ with $\mathbf{V}$ nested in $\mathbf{U}$. An argument like the above used for CAT(0) cube complexes should show that these subgraphs are quasi-convex in $\mathcal{C}_0\mathbf{U}$. Then general facts about quasi-trees should show that the cone-off is still a quasi-tree.

In order to prove uniqueness, one can try to use the finite complexity and the arguments to prove the distance formula (which implies uniqueness) from [BHS17b]. The real cubing structure should help to verify the partial realisation axiom.

Part 3. Real cubing structures on asymptotic cones

In this part, we first briefly recall the definition of an asymptotic cone. The main work, starting in Section 26, is to prove that, given a hierarchically hyperbolic space, every asymptotic cone is bilipschitz equivalent to an ℝ–cubing. This ℝ–cubing depends, a priori, on the choices inherent in the asymptotic cone (ultrafilter, rescaling, and in the absence of a proper cobounded group action, observation point). This generalises and strengthens some of the results about the mapping class group in [BDS11b, BDS11a].

The strategy is to first appeal to a result of Bowditch to find a complete, connected, finite rank median space bilipschitz equivalent to the asymptotic cone, and then use the hierarchically hyperbolic structure to find a finite-depth poset-colouring of the walls in this median space, satisfying the tangible filter condition. Then we will apply Theorem 5.1. Using information specific to the cone, we will relate the real cubing data (the index set, nesting, orthogonality) to the hierarchically hyperbolic space data, without having to mention the colouring in our final statement.

In the asymptotic cone, there are "product regions" that are ultralimits of standard product regions in the hierarchically hyperbolic space, and the real trees in the real cubing structure are obtained by collapsing these product regions.

The chain of events is different from that in [BDS11b]: they produce their ℝ–trees without reference to a median on the asymptotic cone, embed the asymptotic cone in the product of the ℝ–trees, and restrict the median. We *start* with a median on the asymptotic cone, provided by a theorem of Bowditch [Bow18b], and work with this to build the ℝ–trees and verify the ℝ–cubing axioms from Definition 4.2 (via Theorem 5.1).

We also consider the special case where the underlying hyperbolic space is the Cayley graph of a hierarchically hyperbolic group G. Here, we discuss how the ultrapower of G acts by automorphisms on the ℝ–cubing structure of the asymptotic cone. In order to extract an action by genuine automorphisms (rather than by K–automorphisms for some K possibly greater than 1), we need to appeal to a variant of Bowditch's theorem, due to Zeidler [Zei16].

Having done this, we discuss the local structure of the asymptotic cone of G. Specifically, we consider the grove at each point (as in Section 4.9). Although the ℝ–trees in the ℝ–cubing structure are not in general complete, it turns out that the local ℝ–cubing at each point of the asymptotic cone of G has image in each ℝ–tree a subtree isometric to either a point, a line, or the universal $2^{\aleph_0}$–tree defined in [DP01]. This will play an important role in Part 5.

22. Asymptotic cones

We briefly review asymptotic cones, referring the reader to [DK18] for a comprehensive discussion.

Let $(\mathcal{X}, \mathsf{d})$ be a metric space and let ω be a non-principal ultrafilter on ℕ. Let $(p_n)_n$ be a sequence of points in $\mathcal{X}$ and let $(j_n)_n$ be a *scaling sequence* of positive integers with $j_n \xrightarrow{n} \infty$.

The sequence $(x_n)_n$ in $\mathcal{X}$ is *admissible* if $\mathsf{d}(x_n, p_n)/j_n$ is bounded ω–almost everywhere in ℕ. Observe that if $\mathbf{x} = (x_n), \mathbf{y} = (y_n)$ are admissible, then the sequence $\mathsf{d}(x_n, y_n)/j_n$ is bounded and hence the ω–limit $\lim_\omega \mathsf{d}(x_n, y_n)/j_n = \hat{\mathsf{d}}_\infty(\mathbf{x}, \mathbf{y})$ exists.

Declare $\mathbf{x}, \mathbf{y}$ to be equivalent – written $\mathbf{x} \sim \mathbf{y}$ – if $\hat{\mathsf{d}}_\infty(\mathbf{x}, \mathbf{y}) = 0$. A sequence $\mathbf{x}$ is *negligible* if $\mathbf{x} \sim \mathbf{p}$, where $\mathbf{p} = (p_n)$. A sequence $\mathbf{x}$ is *admissible* if $\hat{\mathsf{d}}_\infty(\mathbf{x}, \mathbf{p}) < \infty$. We let $\mathcal{X}^*_{ad}$ denote the set of admissible sequences.

The *asymptotic cone* of $\mathcal{X}$ with respect to $\omega, (p_n)$, and (j_n) is the metric space $(\mathrm{Cone}^\omega(\mathcal{X}), \hat{\mathsf{d}}_\infty)$ where $\mathrm{Cone}^\omega(\mathcal{X}) = \mathcal{X}^*_{ad}/\sim$, with metric $\hat{\mathsf{d}}_\infty$; it is easy to verify this is well-defined and is a metric on $\mathrm{Cone}^\omega(\mathcal{X})$.

In the case where $\mathcal{X}$ is a (locally finite Cayley graph of a) finitely-generated group G with a word-metric, it can be shown that the asymptotic cone is independent of the choice of basepoint $\mathbf{p}$.

It is shown in [vdDW84] that asymptotic cones of metric spaces are always complete. Similarly, if $(\mathcal{Y}_n)_n$ is a sequence of subspaces of $\mathcal{X}$, then the ultralimit of the $\mathcal{Y}_n$ is a closed subspace of $\mathrm{Cone}^\omega(\mathcal{X})$ (provided it is at finite distance from the observation point). We will use these facts freely below.

In our applications, $\mathcal{X}$ will always be a (D, D)–quasigeodesic space, for some fixed D. Hence, any two points in any asymptotic cone of $\mathcal{X}$ can be joined by a D–bilipschitz path. Moreover, in many applications, $\mathcal{X}$ will be a Cayley graph, so asymptotic cones will be geodesic spaces [DK18, Corollary 7.67].

We will often be interested in the situation where $\mathcal{X} = G$, a finitely generated group equipped with the word metric d. Letting G^* denote the set of ω–equivalence classes of sequences in G, we have that G^* is a group with multiplication defined termwise on representative sequences.

Given an admissible sequence $(x_n)_n$ in G, and an element $(g_n) \in G^*$, the sequence $(g_n x_n)_n$ is admissible with respect to the translated basepoint $(g_n o_n)_n$, where $(o_n)_n$ is the original basepoint.

So, we can think of (g_n) as an isometry $\mathrm{Cone}^\omega(G)_{(o_n)} \to \mathrm{Cone}^\omega(G)_{(go_n)}$. We usually just care about the subgroup $G^*_{ad} \leqslant G^*$ of admissible sequences, which then acts by isometries on $\mathrm{Cone}^\omega(G)$. There is a further subgroup $G^*_{neg} \leqslant G_{ad}$ consisting of *negligible* sequences, defined as above for the given observation point.

When considering asymptotic cones of groups, we always put the basepoint at the point $\mathbf{1}$ represented by the constant sequence whose terms are all $1 \in G$. There is no loss of generality in doing this since, as mentioned above, every asymptotic cone of G is naturally isometric to an asymptotic cone based at $\mathbf{1}$, see e.g. [DK18, Exercise 7.65].

23. Median metric on the asymptotic cone of an HHS

Fix an HHS $(\mathcal{X}, \mathfrak{F})$. Our proof that the asymptotic cones of $\mathcal{X}$ are bilipschitz homeomorphic to ℝ–cubings will take as an *input* the fact that the asymptotic cone is bilipschitz homeomorphic to a median metric space, where the median comes from the coarse median on $\mathcal{X}$. This fact is due to Bowditch, and we elaborate on it in this section. In order to accommodate isometries of the asymptotic cone, we will also use a variant of the result, due to Zeidler. The results of Bowditch and Zeidler work for finite-rank coarse median spaces — a more general class than HHSes — and we will see that their results apply in our setting due to facts from Section 12 and Section 16.

Throughout the section, let ω be a non-principal ultrafilter on $\mathbb{N}$, and let $(j_n)_n$ be an increasing sequence of natural numbers. Let $(o_n)_n$ be a sequence of observation points in $\mathcal{X}$. Let $\mathrm{Cone}^\omega(\mathcal{X})$ be the asymptotic cone of $\mathcal{X}$ determined by the observation point $(o_n)_n$, the scaling sequence (j_n), and the ultrafilter ω.

23.1. The median algebra structure on $\mathrm{Cone}^\omega(\mathcal{X})$.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be points in $\mathrm{Cone}^\omega(\mathcal{X})$ represented by admissible sequences $(x_n)_n, (y_n)_n, (z_n)_n$ respectively. For each n, let $m_n = \mu(x_n, y_n, z_n)$ be their coarse median. Since $\mathsf{d}_{\mathcal{X}}(m_n, o_n)$ can be bounded in terms of the distances from o_n to x_n, y_n, z_n, admissibility of $(x_n)_n, (y_n)_n, (z_n)_n$ imply admissibility of $(m_n)_n$. Thus $(m_n)_n$ represents some point $\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathrm{Cone}^\omega(\mathcal{X})$.

The function $\boldsymbol{\mu} : \mathrm{Cone}^\omega(\mathcal{X})^3 \to \mathrm{Cone}^\omega(\mathcal{X})$ makes $\mathrm{Cone}^\omega(\mathcal{X})$ into a topological median algebra, by [Bow13, Proposition 9.1].

This does **not** mean that $\mathrm{Cone}^\omega(\mathcal{X})$, equipped with the metric $\hat{\mathsf{d}}_\infty$ and the median μ, is a median metric space. In the next subsection, we see how Bowditch adapts $\hat{\mathsf{d}}_\infty$ within its bilipschitz class to make it a median metric, without changing $\boldsymbol{\mu}$.

23.2. Bilipschitz median metrics.

We now modify $\hat{\mathsf{d}}_\infty$ to a median metric. The first step is to bound the coarse median rank of $\mathcal{X}, \mathsf{d}_{\mathcal{X}}, \mu)$.

Lemma 23.1. *Letting μ be the coarse median from Section 12, the coarse median space $(\mathcal{X}, \mathsf{d}_{\mathcal{X}}, \mu)$ has rank at most χ, the HHS complexity.*

Proof. Let $A \subset \mathcal{X}$ be a set with $|A| < \infty$. Proposition 16.1 provides a constant C, depending on the HHS constants and on $|A|$, but not on the specific points in A, such that there is a finite CAT(0) cube complex $\mathbf{Y}$ and a C–quasimedian, (C, C)–quasi-isometric embedding $f : \mathbf{Y} \to \mathcal{X}$ whose image is C–Hausdorff close to $H_\theta(A)$. Hence the finite median algebra, the map f, and the quasi-inverse $\bar{f} : H_\theta(A) \to \mathbf{Y}$ satisfy Condition (C2) from the definition of a coarse median space [Bow13, p. 4]. Moreover, $\dim \mathbf{Y} \leqslant \chi$ by Proposition 16.1. So, by the definition of the rank of a coarse median space [Bow13, p. 4], the coarse median rank of $\mathcal{X}$ is at most χ. □

From Lemma 23.1 and [Bow13, Proposition 9.3], the metric space $(\mathrm{Cone}^\omega\mathcal{X}, \hat{\mathsf{d}}_\infty)$, equipped with the median $\boldsymbol{\mu}$, is a topological median algebra of rank at most χ.

Lemma 23.2. *There exists k_0 such that for all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in \mathrm{Cone}^\omega(\mathcal{X})$,*

$$\mathsf{d}_{\mathrm{Cone}^\omega(\mathcal{X})}(\boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{w})) \leqslant k_0 \mathsf{d}_{\mathrm{Cone}^\omega\mathcal{X}}(\mathbf{z}, \mathbf{w}).$$

Proof. This follows from the proof of [Bow13, Proposition 9.1]; indeed the exact inequality appears at the end of that proof. ☐

The modification of $\hat{\mathsf{d}}_\infty$ uses a proposition of Bowditch [Bow18b, Proposition 2.4]:

Proposition 23.3. *Let (M, ν) be a median algebra with rank at most r. Let d be a metric on M such that, for some $k \geqslant 1$, we have*

$$\mathsf{d}(\nu(a,b,c), \nu(a,b,d) \leqslant k\mathsf{d}(c,d))$$

for all $a,b,c,d \in M$. Then M has a metric d' such that

- *(M, d', ν) is a median metric space of rank at most r, and*
- *(M, d) and (M, d') are L–bilipschitz equivalent, where $L = L(r,k)$.*

We now sketch Bowditch's proof of the preceding proposition.

Sketch of Proposition 23.3. Given any $a, b, c \in M$, we define

$$(a,b)^{\mathsf{d}}_c = \frac{1}{2}(\mathsf{d}(a,c) + \mathsf{d}(b,c) - \mathsf{d}(a,b)).$$

We then let $I(a,b)^{\mathsf{d}}$ be the set of c for which $(a,b)^{\mathsf{d}}_c = 0$. This "generalised median interval" measures the failure of d to be a median metric in the sense that $I(a,b) = I(a,b)^{\mathsf{d}}$ for all a, b if and only if d is a median metric for the median μ (here $I(a,b)$ denotes the μ–median interval between a and b).

For any finite median subalgebra A of M, we can assign a metric d_A as follows. If $\hat{w}$ is a wall in A, then we assign it a *width* $\#(\hat{w})$ as follows: take the minimum of the values $\mathsf{d}(a,b)$ as a, b vary over pairs in A that are separated by exactly the wall $\hat{w}$ (in other words, we are assigning weights to hyperplanes in a finite cube complex according to the distance in M between the endpoints of the various edges dual to the given hyperplane).

For $a, b \in A$, we define $\mathsf{d}_A(a,b) = \sum_{\hat{w} \in \mathcal{W}(a,b)} \#(\hat{w})$. This is a median metric on A, and it is bilipschitz equivalent to the restriction of d to A, with constant depending on k_0 and the rank (details of this are in [Bow14, Section 5]).

The set of finite subalgebras of M, partially ordered by inclusion, is a cofinal subset of the set of all finite subsets of M. An application of Tychonov's theorem then gives a metric d', bilipschitz to d, such that $\mathsf{d}_A(a,b)$ converges to $\mathsf{d}'(a,b)$ (as A grows), for any $a, b \in M$. By construction, for an $a, b, c \in M$, we have that a, b, c, and $m = \mu(a,b,c)$ all lie in A for each finite subalgebra A in a cofinal subset. For any $c \in I(a,b)$, we have $(a,b)^{\mathsf{d}_A}_c = 0$ for such A, so $I(a,b) \subset I(a,b)^{\mathsf{d}'}$. On the other hand, if $c \in I(a,b)^{\mathsf{d}'}$, then $\mathsf{d}_A(c,m) \to 0$. So $I(a,b) = I(a,b)^{\mathsf{d}'}$, whence d' is a median metric for μ. ☐

The preceding proposition has been strengthened by Zeidler to account for isometries [Zei16]. In the following lemma, we summarise everything we will need later:

Lemma 23.4. *Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space. Let $\mathrm{Cone}^\omega(\mathcal{X})$ be an asymptotic cone of $\mathcal{X}$, and let $\boldsymbol{\mu}$ be the median on $\mathrm{Cone}^\omega(\mathcal{X})$ arising from the coarse median on $\mathcal{X}$.*

Then there is a metric $\mathbf{D} : \mathrm{Cone}^\omega(\mathcal{X})^2 \to \mathbb{R}_+$ such that:

- *the identity map $(\mathrm{Cone}^\omega(\mathcal{X}), \mathsf{d}_{\mathrm{Cone}^\omega(\mathcal{X})}) \to (\mathrm{Cone}^\omega(\mathcal{X}), \mathbf{D})$ is K_0–bilipschitz, for some $K_0 \geqslant 1$ depending only on $(\mathcal{X}, \mathfrak{F})$;*
- *$(\mathrm{Cone}^\omega(\mathcal{X}), \mathbf{D}, \boldsymbol{\mu})$ is a finite-rank complete connected geodesic median metric space. In particular, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{Cone}^\omega(\mathcal{X})$, we have:*
  1. *$\mathbf{D}(\mathbf{z}, \boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z})) \leqslant \mathbf{D}(\mathbf{z}, \mathbf{x})$.*
  2. *$\mathbf{D}(\mathbf{x}, \mathbf{y}) = \mathbf{D}(\mathbf{x}, \mathbf{z}) + \mathbf{D}(\mathbf{y}, \mathbf{z}) - 2\mathbf{D}(\mathbf{z}, \boldsymbol{\mu}(\mathbf{x}, \mathbf{y}, \mathbf{z}))$.*
  3. *Median intervals in $\mathcal{X}$ are compact.*

If in addition $\mathcal{X} = G$ and $(G, \mathfrak{F})$ is an HHG, then we can choose the metric $\mathbf{D}$ *on* $\mathrm{Cone}^\omega(G)$ *with the following additional property: for each $a \in G^*_{ad}$, the isometry $a : \mathrm{Cone}^\omega(G) \to \mathrm{Cone}^\omega(G)$ remains an isometry when we equip* $\mathrm{Cone}^\omega(G)$ *with the metric* $\mathbf{D}$.

Proof. The first assertion, and the fact that $\mathbf{D}$ is a median metric for $\boldsymbol{\mu}$, follow from Proposition 23.3, as does finiteness of the rank.

The original space $(\mathrm{Cone}^\omega(\mathcal{X}), \mathsf{d}_{\mathrm{Cone}^\omega(\mathcal{X})})$ is complete and connected, so by the first assertion, $(\mathrm{Cone}^\omega(\mathcal{X}), \mathbf{D})$ is also complete and connected. By finiteness of rank and completeness, intervals are compact [Bow20, Corollary 5.2]. Complete connected median metric spaces are geodesic [Bow16b, Lemma 4.6]. The enumerated claims follow immediately from the definition of a median metric.

It remains to prove the statement about isometries. Since $\boldsymbol{\mu}$ is continuous with respect to the $\hat{\mathsf{d}}_\infty$ topology, $(\mathrm{Cone}^\omega(G), \hat{\mathsf{d}}_\infty, \boldsymbol{\mu})$ is a finite rank *metric median algebra*, in the language of Zeidler [Zei16, Definition 3.1]. Since $\mu : G^3 \to G$ is G–equivariant (see Section 12), $\boldsymbol{\mu} : \mathrm{Cone}^\omega(G)^3 \to \mathrm{Cone}^\omega(G)$ is G^*_{ad}–equivariant, so the G^*_{ad}–action is by *isometric automorphisms* in Zeidler's language. By [Zei16, Proposition 5.1], $(\mathrm{Cone}^\omega(G), \hat{\mathsf{d}}_\infty, \boldsymbol{\mu})$ is *uniformly rectifiable*, in the sense of [Zei16, Section 3]. So, using [Zei16, Proposition 3.3] in place of Proposition 23.3 yields a metric $\mathbf{D}$ with all of the itemised properties that additionally satisfies the assertion about the action of G^*_{ad}. □

Remark 23.5 (BBF colourings and finite products of ℝ–trees)**.** In many examples, $\mathbf{D}$ can be chosen to arise from a bilipschitz embedding of $\mathrm{Cone}^\omega(\mathcal{X})$ into a finite product of ℝ–trees. In the case where $\mathcal{X}$ is a mapping class group, this was done by Behrstock-Drutu-Sapir [BDS11b, BDS11a], Bowditch [Bow13, Section 12], and Bestvina-Bromberg-Fujiwara [BBF15, Section 5]. This type of construction involves "colouring" the walls in the median space with finitely many colours, such that like-coloured walls do not cross. In the mapping class group setting, this can be done using the result of Bestvina-Bromberg-Fujiwara that the subsurfaces of a surface can be coloured (by orbits under the action of a finite-index subgroup of the mapping class group) in such a way that like-coloured subsurfaces are not disjoint (i.e. not orthogonal as index set elements in the HHS structure). We will not use this now, but will return to this idea in Section 35. □

24. Ultralimits of product regions

In Section 26, where we show that $(\mathrm{Cone}^\omega(\mathcal{X}), \mathbf{D})$ is isometric to an ℝ–cubing with $\boldsymbol{\mu}$ the associated median, subspaces of $\mathrm{Cone}^\omega(\mathcal{X})$ arising as ultralimits of sequences of product regions will be important and the following proposition supports their use.

In the proposition, given a product region P_U, we let F_U denote a subspace of the form given in Section 15; recall that this involves an implicit choice of basepoint in E_U, which we suppress. The different possible subspaces F_U, as that basepoint varies, will be called *parallel copies* of F_U, reflecting the fact that P_U is quasimedian quasi-isometric to $F_U \times E_U$ (Proposition 15.7).

Proposition 24.1. *Let $\mathrm{Cone}^\omega(\mathcal{X})$ be an asymptotic cone of $\mathcal{X}$, with observation point $(o_n)_n$, ultrafilter ω, and scaling sequence $(j_n)_n$.*

Let $(U_n)_n, (V_n)_n$ be sequences of elements of $\mathfrak{F}$ such that

- $\mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{P_{U_n}}(o_n))/j_n$ *and*
- $\mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{P_{V_n}}(o_n))/j_n$ *are ω–a.e. bounded, and*
- $V_n \sqsubseteq U_n$ *for ω–a.e. n.*

Let $F_{U_n} \subseteq P_{U_n}$ be a sequence of spaces such that $\mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{F_{U_n}}(o_n))/j_n$ is ω–a.e. bounded. Then for ω–a.e. n, there is a parallel copy F_{V_n} such that F_{V_n} is uniformly coarsely contained in F_{U_n} and $\mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{F_{V_n}}(o_n))/j_n$ is ω–a.e. bounded.

Hence $\lim_\omega F_{V_n} \subseteq \lim_\omega F_{U_n}$.

Proof. We first deduce the final "hence" statement from the previous part of the lemma. Suppose $(F_{U_n})_n$ is as in the statement, and $(F_{V_n})_n$ is such that, for some $\nu < \infty$, we have $F_{V_n} \subseteq \mathcal{N}_\nu(F_{U_n})$ for ω–a.e. n, and $\mathsf{d}_\mathcal{X}(F_{V_n}, o_n)/j_n$ is ω–a.e. bounded. Then, taking (rescaled) ultralimits, we have $\lim_\omega F_{V_n} \subset \lim_\omega F_{U_n}$, as required. So, given $(F_{U_n})_n$ as in the statement, it suffices to find the $(F_{V_n})_n$ with the claimed property.

By hypothesis, $\mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{P_{V_n}}(o_n))/j_n$ is bounded ω–a.e.

Moreover, $\mathfrak{g}_{F_{U_n}}(\mathfrak{g}_{P_{V_n}}(o_n))$ lies uniformly close to a parallel copy of F_{V_n} that is coarsely contained in F_{U_n}, since $V_n \sqsubseteq U_n$.

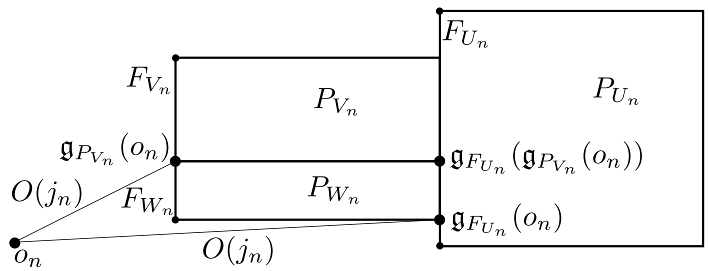

FIGURE 22. Proof of Proposition 24.1

We wish to show that

$$\lim_\omega(\mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{F_{U_n}}(\mathfrak{g}_{P_{V_n}}(o_n)))/j_n < \infty,$$

since this will imply that F_{V_n} contains points at linear distance from o_n, and hence (by e.g. Lemma 1.27 in [BHS17c]), the point $\mathfrak{g}_{F_{V_n}}(o_n)$ is at linear distance from o_n, as required.

Now, by the definition of the gate, we have that $\mathfrak{g}_{F_{U_n}}(\mathfrak{g}_{P_{V_n}}(o_n))$ lies on a uniform-quality hierarchy path from $\mathfrak{g}_{P_{V_n}}(o_n)$ to any point in F_{U_n}, and in particular $\mathfrak{g}_{F_{U_n}}(o_n)$. Since hierarchy paths are quasigeodesics, an application of the triangle inequality shows that this hierarchy path has length bounded above by a linear function of

$$\mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{P_{V_n}}(o_n)) + \mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{F_{U_n}}(o_n)),$$

which is in turn ω–a.e. bounded by a linear function of j_n in view of our hypotheses.

Hence $\mathfrak{g}_{F_{U_n}}(\mathfrak{g}_{P_{V_n}}(o_n))$ is joined to $\mathfrak{g}_{F_{U_n}}(o_n)$, and hence to o_n, by a path of length $O(j_n)$, as required. So $(F_{V_n})_n$ is visible in $\mathrm{Cone}^\omega(\mathcal{X})$, as required. □

We also need the following preparatory material about ultralimits of product regions.

Definition 24.2 (Legal sequence, $\mathfrak{F}^\infty$)**.** We say that the sequence $(U_n)_n$ of elements of $\mathfrak{F}$ is *legal* if

$$\lim_\omega \mathsf{d}_\mathcal{X}(o_n, \mathfrak{g}_{P_{U_n}}(o_n))/j_n < \infty.$$

The legal sequences $(U_n)_n, (V_n)_n$ are *equivalent* if $V_n = U_n$ for ω–a.e. n. Let $\mathfrak{F}^\infty$ denote the set of equivalence classes of legal sequences in $\mathfrak{F}$. □

Definition 24.3 (Standard product regions in asymptotic cones)**.** Let $\mathbf{U} \in \mathfrak{F}^\infty$, and let $(U_n)_n$ be a legal sequence representing $\mathbf{U}$. Let

$$\mathbf{P_U} = \lim_\omega P_{U_n} \subset \mathrm{Cone}^\omega(\mathcal{X}).$$

We refer to $\mathbf{P_U}$ as a *standard product region* in $\mathrm{Cone}^\omega(\mathcal{X})$. □

Lemma 24.4 (Product regions in $\mathrm{Cone}^\omega(\mathcal{X})$ are naturally gated)**.** *Let* $\mathbf{U} \in \mathfrak{F}^\infty$ *be represented by a legal sequence* $(U_n)_n$*. Then:*

(1) $\mathbf{P_U}$ *is a closed, median convex subspace of the median metric space* $(\mathrm{Cone}^\omega(\mathcal{X}),\mathbf{D},\boldsymbol{\mu})$.
(2) $\mathbf{P_U}$ *is geodesically convex in* $(\mathrm{Cone}^\omega(\mathcal{X}),\mathbf{D})$.
(3) For each n, *let* $\mathfrak{g}_{P_{U_n}}:\mathcal{X}\to P_{U_n}$ *be the gate map (see Section 15). Define* $\mathfrak{g}_\mathbf{U}:\mathrm{Cone}^\omega(\mathcal{X})\to\mathbf{P_U}$ *by* $\mathfrak{g}_\mathbf{U}((x_n)_n)=\lim_\omega\mathfrak{g}_{P_{U_n}}(x_n)$. *Then* $\mathfrak{g}_\mathbf{U}$ *is well-defined, and is the gate map to* $\mathbf{P_U}$, *in the median sense.*

In particular, $\mathfrak{g}_\mathbf{U}$ *is a* 1*–lipschitz median homomorphism and* $\mathfrak{g}_\mathbf{U}(\mathbf{x})$ *is the unique* $\mathbf{D}$*–closest point of* $\mathbf{P_U}$ *to* $\mathbf{x}$, *for all* $\mathbf{x}\in\mathrm{Cone}^\omega(\mathcal{X})$.

Proof. Let $\mathbf{p},\mathbf{p}'\in\mathbf{P_U}$ and let $\mathbf{x}\in\mathrm{Cone}^\omega(\mathcal{X})$. Choose admissible sequences $(p_n)_n,(p'_n)_n,(x_n)_n$ in $\mathcal{X}$ respectively representing these three points. By Proposition 13.3 and Lemma 15.1, there exists Q, depending only on the HHS constants, such that for ω–a.e. n, we have $\mu(p_n,p'_n,q_n)\in\mathcal{N}_Q(P_{U_n})$. Hence $\boldsymbol{\mu}(\mathbf{p},\mathbf{p}',\mathbf{x})\in\mathbf{P_U}$, so $\mathbf{P_U}$ is median-convex. As a rescaled ultralimit of subspaces of $\mathcal{X}$, $\mathbf{P_U}$ is automatically closed. This proves (1).

Since $(\mathrm{Cone}^\omega(\mathcal{X}),\mathbf{D},\boldsymbol{\mu})$ is a geodesic median metric space, median-convexity of $\mathbf{P_U}$ implies geodesic convexity. Indeed, if γ is a geodesic of $\mathrm{Cone}^\omega(\mathcal{X})$ joining points $\mathbf{x},\mathbf{y}\in\mathbf{P_U}$, then for any $\mathbf{z}\in\gamma$, we have $\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z})=\mathbf{z}$, so $\mathbf{z}\in\mathbf{P_U}$ by median convexity. This proves (2).

Let $\mathbf{x}=(x_n)_n\in\mathrm{Cone}^\omega(\mathcal{X})$. We need to show that $(\mathfrak{g}_{P_{U_n}}(x_n))_n$ is an admissible sequence. But since $(x_n)_n$ is admissible, $\lim_\omega \mathsf{d}_\mathcal{X}(o_n,x_n)/j_n<\infty$. Since $\mathfrak{g}_{P_{U_n}}:\mathcal{X}\to P_{U_n}$ is (K,K)–coarsely lipschitz for all n, where K depends on $\kappa^\times(0)$ and the HHS constants, we have

$$\mathsf{d}_\mathcal{X}(o_n,\mathfrak{g}_{P_{U_n}}(x_n))\leqslant\mathsf{d}_\mathcal{X}(o_n,\mathfrak{g}_{P_{U_n}}(o_n))+K\mathsf{d}_\mathcal{X}(o_n,x_n)+K,$$

which is ω–a.e. bounded after rescaling — the second term was dealt with immediately above and the first term is dealt with by legality of $(U_n)_n$. So, $(\mathfrak{g}_{P_{U_n}}(x_n))_n$ is admissible, and limits to a point $\mathfrak{g}_\mathbf{U}(\mathbf{x})\in\mathbf{P_U}$ that is independent of the choice of representative $(x_n)_n$, by a similar computation.

So, we have a well-defined retraction map $\mathfrak{g}_\mathbf{U}:\mathrm{Cone}^\omega(\mathcal{X})\to\mathbf{P_U}$, which we need to check is the gate map in the median sense (and hence the closest-point projection for $\mathbf{D}$).

Let $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathrm{Cone}^\omega(\mathcal{X})$ be represented by sequences $(x_n),(y_n),(z_n)$. Then for all n, the definition of μ and $\mathfrak{g}_{P_{U_n}}$ implies that $\mu(\mathfrak{g}_{P_{U_n}}(x_n),\mathfrak{g}_{P_{U_n}}(y_n),\mathfrak{g}_{P_{U_n}}(z_n))$ is uniformly close to $\mathfrak{g}_{P_{U_n}}(\mu(x_n,y_n,z_n))$. Hence

$$\mathfrak{g}_\mathbf{U}(\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z}))=\boldsymbol{\mu}(\mathfrak{g}_\mathbf{U}(\mathbf{x}),\mathfrak{g}_\mathbf{U}(\mathbf{y}),\mathfrak{g}_\mathbf{U}(\mathbf{z})).$$

In particular, if $\mathbf{x}\in\mathrm{Cone}^\omega(\mathcal{X})$ and $\mathbf{z}\in\mathbf{P_U}$, then, letting $\mathbf{y}=\mathfrak{g}_\mathbf{U}(\mathbf{x})$, we have the following. First, $\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbf{P_U}$ by median-convexity, so $\mathfrak{g}_\mathbf{U}(\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z}))=\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z})$. On the other hand, $\mathfrak{g}_\mathbf{U}(\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z}))=\boldsymbol{\mu}(\mathbf{y},\mathbf{y},\mathbf{z})=\mathbf{y}$. So, $\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{z})=\mathbf{y}$. In other words, $\mathfrak{g}_\mathbf{U}(\mathbf{x})$ is in the median interval between $\mathbf{x}$ and any point in $\mathbf{P_U}$. Since this property characterises gate maps in the median sense, we have proved (3).

The closest-point projection statement follows from the above characterisation of gates in median metric spaces (see e.g. [Bow20, Section 4]) and the lipschitz statement follows from e.g. [CDH10, Lemma 2.13]. □

Lemma 24.5. *Let* $\mathbf{U}\in\mathfrak{F}^\infty$ *and let* $(U_n)_n$ *be a legal sequence representing* $\mathbf{U}$. *For each* n, *let* $\phi_n:F_{U_n}\times E_{U_n}\to P_{U_n}$ *be the map from Section 15. Let* $\mathbf{F_U}$ *and* $\mathbf{E_U}$ *be the rescaled ultralimits of* $(F_{U_n})_n$ *and* $(E_{U_n})_n$, *and let* $\phi:\mathbf{F_U}\times\mathbf{E_U}\to\mathbf{P_U}$ *be the limit of the maps* ϕ_n. *Then:*

- ϕ *is an isometry, where* $\mathbf{F_U}\times\mathbf{E_U}$ *is given the* ℓ_1*–metric, where the metric on each factor is the restriction of* $\mathbf{D}$*;*
- ϕ *is a median isomorphism, where* $\mathbf{F_U}\times\mathbf{E_U}$ *is given the product median.*

Abusing notation, let $\mathbf{F_U}$ *denote the image in* $\mathbf{P_U}$ *of any parallel copy* $\mathbf{F_U}\times\{\mathbf{e}\},\mathbf{e}\in\mathbf{E_U}$. *Then* $\mathbf{F_U}$ *is closed and median-convex in* $\mathrm{Cone}^\omega(\mathcal{X})$, *and the gate map* $\mathfrak{h}:\mathrm{Cone}^\omega(\mathcal{X})\to\mathbf{F_U}$

factors as $\mathfrak{h} = p \circ \mathfrak{g}_{\mathbf{U}}$*, where* $p : \mathbf{P_U} \to \mathbf{F_U}$ *is the natural projection to* $\mathbf{F_U} \times \{\mathbf{e}\}$*. Again,* $\mathfrak{h}$ *is a* 1*–lipschitz retraction and is the closest-point projection to the given parallel copy of* $\mathbf{F_U}$*.*

In particular, if $\mathbf{x}, \mathbf{y} \in \mathrm{Cone}^\omega(\mathcal{X})$*, then* $\mathbf{D}(\mathfrak{h}(\mathbf{x}), \mathfrak{h}(\mathbf{y}))$ *is independent of the choice of* $\mathbf{e}$*.*

Proof. By Proposition 15.7, ϕ_n is a quasimedian quasi-isometry, with constants depending on the HHS structure (but independent of U_n). So, $\phi : \mathbf{F_U} \times \mathbf{E_U} \to \mathbf{P_U}$ is a median isomorphism. Moreover, since $\mathbf{P_U}$ is median convex in $\mathrm{Cone}^\omega(\mathcal{X})$, median convexity of $\mathbf{F_U} \times \{\mathbf{e}\}$ follows from median convexity of $\mathbf{F_U}$ in $\mathbf{F_U} \times \mathbf{E_U}$. Equipping each parallel copy $\mathbf{F_U} \times \{\mathbf{e}\}$ with the subspace metric inherited from $(\mathbf{P_U}, \mathbf{D})$ makes ϕ an isometry. The statements about $\mathfrak{h}$ then follow as before from [Bow20, Section 4] and the fact that the gate map in $\mathbf{P_U}$ to any parallel copy of $\mathbf{F_U}$ comes from natural projection. □

In summary, we have the following. Let $\phi(\mathbf{f}, \mathbf{e}) = \mathbf{x} \in \mathbf{P_U}$, so that $\mathbf{f}$ is contained in a unique parallel copy $\mathbf{F_U} = \mathbf{F_U} \times \{\mathbf{e}\}$. For each n, choose $f_n \in F_{U_n}, e_n \in E_{U_n}$ so that $\lim_\omega \phi_n(f_n, e_n) = x_n$, where (x_n) represents $\mathbf{x}$. Then $\phi_n(F_{U_n} \times \{e_n\})$ limits to $\mathbf{F_U}$. In other words, each parallel copy of $\mathbf{F_U}$ in $\mathbf{P_U}$ is an ultralimit of parallel copies of F_{U_n}.

As we discuss in Section 34.2, $\mathbf{F_U}$ can be parallel to closed convex subsets not contained in $\mathbf{P_U}$, although, in a way made precise in that section, this is an unimportant subtlety that does not occur for the $\mathbf{U}$ that will correspond to nontrivial real trees in our eventual real cubing structure. We merely emphasise that we work with parallel copies of $\mathbf{F_U}$ lying in $\mathbf{P_U}$, and that we will often be concerned only with the set of walls crossing $\mathbf{F_U}$, which is in any case independent of the parallel copy by Lemma 2.15.

25. Hierarchy paths become D–geodesics

We need to discuss geodesics in $\mathrm{Cone}^\omega(\mathcal{X})$. Let $\mathbf{x} \in \mathrm{Cone}^\omega(\mathcal{X})$ and let $(x_n)_n$ be an admissible sequence in $\mathcal{X}$ representing it. For each n, let $\gamma_n : [0, L_n] \to \mathcal{X}$ be a (D, D)–hierarchy path joining o_n to x_n, whose existence is guaranteed by Theorem 10.7.

Since γ_n is a (D, D)–quasigeodesic and (x_n) is admissible, we have $\lim_\omega L_n/j_n = L < \infty$, and we can thus take ultralimits to obtain an embedded path $\boldsymbol{\gamma} : [0, L] \to \mathrm{Cone}^\omega(\mathcal{X})$. By the definition of a hierarchy path, we have for all $a \in [0, L_n]$ that $m(o_n, x_n, \gamma_n(a))$ lies uniformly close to $\gamma_n(a)$. Hence, for all $t \in [0, L]$, we have $\boldsymbol{\mu}(\boldsymbol{\gamma}(t), \mathbf{o}, \mathbf{x}) = \boldsymbol{\gamma}(t)$.

Lemma 25.1. *The path* $\boldsymbol{\gamma} : [0, L] \to \mathrm{Cone}^\omega(\mathcal{X})$ *is a geodesic with respect to the metric* $\mathbf{D}$*.*

Proof. Since each γ_n is a uniform hierarchy path, we have for ω–a.e. n that whenever $0 \leqslant s \leqslant t \leqslant u \leqslant L_n$, the coarse median of $\gamma_n(s), \gamma_n(t), \gamma_n(u)$ is uniformly close to $\gamma_n(t)$.

Hence, whenever $0 \leqslant s \leqslant t \leqslant u \leqslant L$, we have

$$\boldsymbol{\mu}(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(t), \boldsymbol{\gamma}(u)) = \boldsymbol{\gamma}(t).$$

Since $(\mathrm{Cone}^\omega(\mathcal{X}), \mathbf{D}, \boldsymbol{\mu})$ is a median metric space, we deduce $\mathbf{D}(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(u)) = \mathbf{D}(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(t)) + \mathbf{D}(\boldsymbol{\gamma}(t), \boldsymbol{\gamma}(u))$. More generally, if $0 = s_0 < s_1 < \ldots < s_k = L$, then $\boldsymbol{\mu}(\boldsymbol{\gamma}(s_i), \boldsymbol{\gamma}(s_j), \boldsymbol{\gamma}(s_k)) = \boldsymbol{\gamma}(s_j)$ whenever $i \leqslant j \leqslant k$. Hence

$$\mathbf{D}(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{k-1} \mathbf{D}(\boldsymbol{\gamma}(s_i), \boldsymbol{\gamma}(s_{i+1})),$$

so $\mathbf{D}(\mathbf{x}, \mathbf{y}) = |\boldsymbol{\gamma}|$. Hence (after reparameterising by arc length), $\boldsymbol{\gamma}$ is a $\mathbf{D}$–geodesic. □

In the preceding lemma, there was nothing special about $(o_n)_n$; it could have been replaced with any other admissible sequence to show that any limit of hierarchy paths (not just one starting at the basepoint in $\mathrm{Cone}^\omega(\mathcal{X})$) is a $\mathbf{D}$–geodesic.

Remark 25.2. In the case where $\mathcal{X}$ is the Cayley graph of the mapping class group of a finite-type hyperbolic surface, the "hierarchy paths" part of Theorem 1.1 [BDS11b] corresponds to the preceding lemma. □

26. Statement of Theorem 26.3

In this section, we fix an HHS $(\mathcal{X}, \mathfrak{F})$, and let E be the associated constant (in particular, each π_U is E–coarsely surjective).

Fix a non-principal ultrafilter ω on $\mathbb{N}$, a scaling sequence (j_n), and an observation point (o_n). Let $\mathrm{Cone}^\omega(\mathcal{X})$ be the associated asymptotic cone of $\mathcal{X}$, and let $\mathbf{o}$ be the point represented by the admissible sequence $(o_n)_n$.

We let $\boldsymbol{\mu}$ denote the median on $\mathrm{Cone}^\omega(\mathcal{X})$ from Section 23.1 and we let $\mathbf{D}$ denote the median metric from Lemma 23.2 (and we let K_0 be the bilipschitz constant from the same lemma, so that $\mathbf{D}$ is K_0–bilipschitz to the original asymptotic cone metric).

In most of our applications, the HHS $(\mathcal{X}, \mathfrak{F})$ satisfies two additional "combinatorial" conditions, namely that $\mathfrak{F}$ has *wedges* and *clean containers*. Later we will show that these conditions persist in the real cubing structure on the asymptotic cone.

Remark 26.1 (Wedges)**.** We say that $\mathfrak{F}$ has the *wedge property*, or *has wedges* to mean the following. Let $U, V \in \mathfrak{F}$ and suppose there exists $W \in \mathfrak{F}$ with $W \sqsubseteq U$ and $W \sqsubseteq V$. Then there is a unique $\sqsubseteq$–maximal such W, denoted $U \wedge V$. We will verify later that this holds for our desired examples, like cube complexes and mapping class groups. □

Remark 26.2 (Clean containers)**.** We say that $\mathfrak{F}$ has *clean containers*, a property we elaborate in Section 35 and verify for the main examples in Section 36, if the following holds.

Let $W \in \mathfrak{F}$ and let $U \sqsubseteq W$. Suppose that there exists $V \sqsubseteq W$ with $U \perp V$. Then there exists $U^\perp \sqsubsetneq W$ such that $U^\perp \perp U$ and, if $V \sqsubsetneq W$ satisfies $V \perp U$, then $V \sqsubseteq U^\perp$. Note that $U^\perp$ is unique among elements properly nested in W with the given two properties. This property strengthens Definition 10.1.(3). □

The primary goal of this section is to prove the following theorem.

Theorem 26.3. *Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space. Then any asymptotic cone $\mathrm{Cone}^\omega(\mathcal{X})$ of $\mathcal{X}$ is bilipschitz homeomorphic to an $\mathbb{R}$–cubing $(\mathrm{Cone}^\omega(\mathcal{X}), \mathfrak{F}^\infty)$ with nonempty products. Furthermore, if the HHS index set $\mathfrak{F}$ has the wedge property and clean containers, then so does the real cubing index set $\mathfrak{F}^\infty$.*

The construction of the $\mathbb{R}$–cubing in Theorem 26.3 has several parts, carried out over the next several subsections. Much of the work has already been done, since the crux of the proof is an application of Theorem 5.1 (recall that that is the theorem that turns finite-depth tangible poset-colourings into real cubing structures).

27. The index set $\mathfrak{F}^\infty$ and its relations

Let $\chi^{\sqsubseteq}$ be the maximal length of $\sqsubseteq$–chains in $\mathfrak{F}$ and let $\chi^{\perp}$ be the cardinality of a largest subset of $\mathfrak{F}$ whose elements are pairwise orthogonal; these constants exist by Definition 10.1 and Lemma 11.2; we can take $\chi^{\sqsubseteq}, \chi^{\perp} \leqslant \chi$.

Let $(U_n)_{n \in \mathbb{N}}$ be a sequence with each $U_n \in \mathfrak{F}$. Recall that $(U_n)_n$ is *legal* if $\lim_{\omega,n} \mathsf{d}_{\mathcal{X}}(o_n, \mathfrak{g}_{P_{U_n}}(o_n)) < \infty$, where $P_{U_n} \subset \mathcal{X}$ is the standard product region for U_n.

Recall that the legal sequences $(U_n)_n, (V_n)_n$ are *equivalent* if $U_n = V_n$ for ω–a.e. n, and that $\mathfrak{F}^\infty$ denotes the set of equivalence classes of legal sequences.

Definition 27.1 (Nesting, orthogonality, transversality in $\mathfrak{F}^\infty$)**.** Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty$ be represented by legal sequences $(U_n)_n, (V_n)_n$ respectively. Then one of the following holds:

- $U_n \sqsubsetneq V_n$ for ω–a.e. n, in which case we say $\mathbf{U} \sqsubsetneq \mathbf{V}$.

- $V_n \sqsubsetneq U_n$ for ω–a.e. n, in which case we say $\mathbf{V} \sqsubsetneq \mathbf{U}$.
- $U_n \perp V_n$ for ω–a.e. n, in which case we say $\mathbf{U} \perp \mathbf{V}$.
- $U_n \pitchfork V_n$ for ω–a.e. n, in which case we say $\mathbf{U} \pitchfork \mathbf{V}$.
- $U_n = V_n$ for ω–a.e. n, in which case $\mathbf{U} = \mathbf{V}$.

These are the relations we will use for making $\mathfrak{F}^\infty$ a real cubing structure on the metric space $(\mathrm{Cone}^\omega(\mathcal{X}), \mathbf{D})$. □

In view of the above definitions and Definition 10.1, we have the following. First, $\perp$ is an anti-reflexive, symmetric relation on $\mathfrak{F}^\infty$, the relations $\sqsubseteq$ and $\perp$ are mutually exclusive, and $\mathbf{U} \sqsubsetneq \mathbf{V}$ and $\mathbf{V} \perp \mathbf{W}$ imply $\mathbf{U} \perp \mathbf{W}$. Moreover, any collection of pairwise orthogonal elements of $\mathfrak{F}^\infty$ has cardinality at most $\chi^\perp$. Also, $\sqsubseteq$ is a partial order on $\mathfrak{F}^\infty$, and any $\sqsubseteq$-chain has length at most $\chi^\sqsubseteq$. By Definition 10.1, $\mathfrak{F}$ has a unique $\sqsubseteq$–maximal element, denoted S, and $P_S = \mathcal{X}$. So the constant sequence (S) is legal and represents an element $\mathbf{S} \in \mathfrak{F}^\infty$ which is the unique $\sqsubseteq$–maximal element of $\mathfrak{F}^\infty$.

Remark 27.2. Thus far, we have verified that $\mathfrak{F}^\infty$, with the relations $\sqsubseteq, \perp, \pitchfork$, satisfies all of the properties from Definition 4.2 that do not involve the ℝ–trees, projections, etc. □

If the HHS $(\mathcal{X}, \mathfrak{F})$ has the wedge property and clean containers, so does the index set $\mathfrak{F}^\infty$:

Lemma 27.3 (Wedges and clean containers in $\mathfrak{F}^\infty$)**.** *Suppose that $(\mathcal{X}, \mathfrak{F})$ has the wedge property and clean containers. Let $\mathbf{U}, \mathbf{V}, \mathbf{W} \in \mathfrak{F}^\infty$. The following hold:*

1. *Suppose that $\mathbf{U} \pitchfork \mathbf{V}$ and the set of $\mathbf{T} \in \mathfrak{F}^\infty$ such that $\mathbf{T} \sqsubseteq \mathbf{U}$ and $\mathbf{T} \sqsubseteq \mathbf{V}$ is nonempty. Then there exists a unique $\sqsubseteq$–maximal $\mathbf{T} \in \mathfrak{F}^\infty$ with $\mathbf{T} \sqsubseteq \mathbf{U}, \mathbf{V}$.*
   If no such $\mathbf{T}$ exists, then $\lim_{\omega,n} \mathsf{diam}(\mathfrak{g}_{F_{U_n}}(F_{V_n}))/j_n = 0$, where $(U_n)_n, (V_n)_n$ are legal sequences representing $\mathbf{U}, \mathbf{V}$.
   In the former case, $\mathbf{T}$ is represented by a legal sequence $(T_n)_n$ such that $\mathfrak{g}_{F_{U_n}}(F_{V_n})$ coarsely coincides with a parallel copy of F_{T_n} for ω–a.e. n. Hence $\mathfrak{h}_{\mathbf{U}}(\mathbf{F_V})$ is a parallel copy of $\mathbf{F_T}$, where $\mathfrak{h}_{\mathbf{U}}$ is the gate map to $\mathbf{F_U}$.
2. *Suppose that $\mathbf{U}, \mathbf{V} \sqsubseteq \mathbf{W}$ and $\mathbf{U} \perp \mathbf{V}$. Then there exists a unique $\mathbf{U}^\perp \sqsubsetneq \mathbf{W}$ such that $\mathbf{U} \perp \mathbf{U}^\perp$ and $\mathbf{T} \sqsubseteq \mathbf{U}^\perp$ for all $\mathbf{T} \sqsubseteq \mathbf{W}$ with $\mathbf{T} \perp \mathbf{U}$.*

Proof. Suppose that $\mathbf{U} \pitchfork \mathbf{V}$ and that the set of $\mathbf{T} \in \mathfrak{F}^\infty$ nested into $\mathbf{U}$ and $\mathbf{V}$ is nonempty. Let $(U_n)_n, (V_n)_n$ be legal sequences representing $\mathbf{U}, \mathbf{V}$. Choose parallel copies F_{U_n}, F_{V_n} such that $\mathsf{d}_{\mathcal{X}}(o_n, \mathfrak{g}_{F_{U_n}}(o_n))/j_n$ is bounded ω–a.e. and the same holds with V_n replacing U_n.

By the definitions of standard product regions and gates and the wedges assumption, $\mathfrak{g}_{F_{U_n}}(F_{V_n})$ either has uniformly bounded diameter or coarsely coincides with a parallel copy F_{T_n}, where $T_n \in \mathfrak{F}$ is the unique $\sqsubseteq$–maximal element nested in both U_n and V_n and having the property that any $W \sqsubseteq U_n, V_n$ is nested in T_n; the existence of T_n is provided by the wedge property. In the former case, $\lim_{\omega,n} \mathsf{diam}(\mathfrak{g}_{F_{U_n}}(F_{V_n}))/j_n = 0$, as required.

In the latter case, let W_n be such that $W_n \sqsubseteq U_n, V_n$ and $(W_n)_n$ is legal. Then we can choose a parallel copy F_{W_n} that is uniformly coarsely contained in F_{V_n} and has the property that $\mathsf{d}_{\mathcal{X}}(o_n, F_{W_n})/j_n$ is ω–a.e. bounded. Then $\mathfrak{g}_{F_{U_n}}(F_{W_n})$ coarsely coincides with a parallel copy of F_{W_n} that lies uniformly close to the parallel copy of F_{T_n} described above. Similarly, we can choose a parallel copy F'_{W_n} lying uniformly close to F_{U_n} and having the property that $\mathsf{d}_{\mathcal{X}}(o_n, F_{W_n})/j_n$ is ω–a.e. bounded. Then $\mathfrak{g}_{F_{V_n}}(F_{W_n})$ is coarsely contained in a parallel copy of F_{T_n} coarsely equal to $\mathfrak{g}_{F_{V_n}}(F_{U_n})$. Now, $\mathsf{d}_{\mathcal{X}}(F_{W_n}, F'_{W_n})/j_n$ is ω–a.e. bounded, by the triangle inequality. So, by Lemma 1.19 of [BHS17c], $\mathsf{d}_{\mathcal{X}}(F'_{W_n}, \mathfrak{g}_{F_{U_n}}(F_{W_n}))/j_n$, and hence $\mathsf{d}_{\mathcal{X}}(o_n, F_{T_n})/j_n$, is ω–a.e. bounded. Thus $(T_n)_n$ is a legal sequence representing some $\mathbf{T} \in \mathfrak{F}^\infty$ with the property stated in assertion (1).

So, for ω–a.e. n, we have parallel copies $F_{U_n}, F_{V_n}, F_{T_n}$ such that

$$\mathfrak{g}_{F_{U_n}}(F_{V_n}) \asymp F_{T_n},$$

where $\asymp$ denotes coarse equality. Taking ultralimits and recalling that $\mathfrak{h}_{\mathbf{U}}$ — the gate map to the parallel copy $\mathbf{F}_{\mathbf{U}} = \lim_\omega F_{U_n}$ — is the ultralimit of the maps $\mathfrak{g}_{F_{U_n}}$ yields $\mathfrak{h}_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}}) = \mathbf{F}_{\mathbf{T}}$.

Next, suppose that $\mathbf{U}, \mathbf{V} \sqsubseteq \mathbf{W}$ and $\mathbf{U} \perp \mathbf{V}$. Then for ω–a.e. n, we have $U_n, V_n \sqsubseteq W_n$ and $U_n \perp V_n$, so there exists $U_n^\perp \sqsubseteq W_n$ such that $U_n^\perp \perp U_n$ and every $T \sqsubseteq W_n$ orthogonal to U_n is nested in $U_n^\perp$. Thus each parallel copy of E_{U_n} coarsely coincides with some parallel copy of $F_{U_n^\perp}$, and in particular P_{U_n} is coarsely contained in $P_{U_n^\perp}$. Thus $(U_n^\perp)_n$ is a legal sequence, and thus represents some $\mathbf{U}^\perp \in \mathfrak{F}^\infty$ with the property stated as assertion (2). □

When it is clear from context, and we have clean containers, given $\mathbf{U} \in \mathfrak{F}^\infty$ orthogonal to at least one other element of $\mathfrak{F}^\infty$, we let $\mathbf{U}^\perp$ be the unique maximal element orthogonal to $\mathbf{U}$ (i.e. the element provided by the second item in the preceding lemma, taking $\mathbf{W} = \mathbf{S}$).

The *level* in $\mathfrak{F}^\infty$ is defined similarly to the level in the HHS structure:

Definition 27.4 (Level)**.** Let $\mathbf{U} \in \mathfrak{F}^\infty$. We define the *level* $Level(\mathbf{U})$ inductively as follows. If there does not exist $\mathbf{V} \in \mathfrak{F}^\infty$ with $\mathbf{V} \sqsubsetneq \mathbf{U}$, then $Level(\mathbf{U}) = 1$. If there exists $\mathbf{V} \in \mathfrak{F}^\infty$ such that $Level(\mathbf{V}) = n - 1 \geqslant 1$ and $\mathbf{V} \sqsubsetneq \mathbf{U}$, then $Level(\mathbf{U}) \geqslant n$. We say that $Level(\mathbf{U}) = \ell$ if ℓ is the minimal n such that $Level(\mathbf{U}) \geqslant n$. □

Remark 27.5 (Complexity drop)**.** Note that $Level(\mathbf{S}) \leqslant \chi^{\sqsubseteq}$. Without a cobounded group action on $\mathcal{X}$ by HHS automorphisms, it may be the case that certain elements of $\mathfrak{F}$ do not survive in $\mathfrak{F}^\infty$ because the corresponding sequences of product regions are not legal, so we may have $\chi^{\sqsubseteq} < \chi$. This will not happen in our applications to HHGs, but it also does not matter. It does show that one can construct HHSes whose asymptotic cones with different rescaling sequences have different dimensions (and essentially different ℝ–cubing structures), which explains why our later uniqueness results are stated for HHGs only. □

The basic data for an ℝ–cubing involves an index set and some relations, which we have just constructed. We now construct a poset-colouring, on our way to making ℝ–trees and projections.

28. Poset-colouring using $\mathfrak{F}^\infty$

Let $\mathcal{W}$ be the set of walls in the median metric space $(\mathrm{Cone}^\omega(\mathcal{X}), \mathbf{D}, \boldsymbol{\mu})$. Let fio be the measure from Section 2.4.

Recall that for each $\mathbf{U} \in \mathfrak{F}^\infty$, we have a product region $\mathbf{P}_{\mathbf{U}} = \mathbf{F}_{\mathbf{U}} \times \mathbf{E}_{\mathbf{U}}$. In particular, the various $\mathbf{F}_{\mathbf{U}} \times \{\mathbf{e}\}$ are parallel in the sense of Definition 2.14, since the gate map to any parallel copy restricts to an isometry on any other parallel copy, by Lemma 24.5. In particular, $\mathcal{W}(\mathbf{F}_{\mathbf{U}} \times \{\mathbf{e}\})$ is independent of the choice of $\mathbf{e} \in \mathbf{E}_{\mathbf{U}}$, by Lemma 2.15.

Definition 28.1 (Relevant sequences, maximal relevant sequences)**.** Let $(U_n)_n$ be a legal sequence in $\mathfrak{F}$ and let $(x_n)_n, (y_n)_n$ be admissible sequences of elements in $\mathcal{X}$.

We say that (the ω–equivalence class of) $(U_n)_n$ is *relevant* for the sequences $(x_n)_n$, $(y_n)_n$ if $\lim_\omega \mathsf{d}_{U_n}(x_n, y_n) = \infty$. Let $\mathrm{Rel}((X_n), (y_n))$ be the set of relevant ω–equivalence classes of sequences (U_n).

Let $\max \mathrm{Rel}((x_n)_n, (y_n)_n)$ be the set of ω-classes of admissible sequences $(U_n)_n$ such that:

- $(U_n)_n$ is relevant for $(x_n)_n, (y_n)_n$;
- if $(V_n)_n$ is an admissible sequence in $\mathfrak{F}$ such that $U_n \sqsubsetneq V_n$ for ω-a.e n, then $(V_n)_n$ is not relevant for $(x_n)_n, (y_n)_n$.

We call $\max \mathrm{Rel}((x_n)_n, (y_n)_n)$ the *max-relevant set* for $(x_n)_n, (y_n)_n$. □

Remark 28.2. We warn the reader that $(U_n)_n \in \mathrm{Rel}((x_n), (y_n))$ *need not* have the property that $\mathsf{d}_{U_n}(x_n, y_n)/j_n$ is unbounded. We really mean the un-rescaled ultralimit is infinite. □

We remark that the max-relevant sets for $(x_n)_n, (y_n)_n$ and $(x'_n)_n, (y'_n)_n$ might be different even if $\lim_\omega x_n = \lim_\omega x'_n$ and $\lim_\omega y_n = \lim_\omega y'_n$. So in the lemmas below, we have to be careful to distinguish between admissible sequences and the points in the asymptotic cone that they represent.

First, the max-relevant set defined above is finite and nonempty for any pair of admissible sequences representing distinct points in the cone:

Lemma 28.3. *Let $(x_n)_n$ and $(y_n)_n$ be admissible. Then $|\max \mathrm{Rel}((x_n)_n, (y_n)_n)| < \infty$. Furthermore, if the sequences represent different points in the cone, then*

$$\max \mathrm{Rel}((x_n)_n, (y_n)_n) \neq \varnothing.$$

Proof. We first check finiteness, using Definition 10.1, and then we check nonemptiness.

Finiteness: Given an ω–class (V_n) of sequences in $\mathfrak{F}$, let $\mathrm{Level}((V_n))$ be the integer c such that V_n has level c for ω–a.e. n, which exists because of the bound on complexity. (Recall that the level of $V \in \mathfrak{F}$ is the length of a longest $\sqsubseteq$–chain with maximal element V.)

Let $\max \mathrm{Rel}^{(V_n)}((x_n)_n, (y_n)_n)$ be the set of $(U_n) \in \max \mathrm{Rel}((x_n)_n, (y_n)_n)$ such that $U_n \sqsubseteq V_n$ for ω–a.e. n.

Let $S \in \mathfrak{F}$ be the $\sqsubseteq$–maximal element. Note that if $\lim_\omega \mathsf{d}_S(x_n, y_n) = \infty$, then $\max \mathrm{Rel}((x_n)_n, (y_n)_n)$ has a single element (and in particular the set is nonempty), represented by the constant sequence $(S)_n$. In this case, we are done, so suppose that $\lim_\omega \mathsf{d}_S(x_n, y_n) < \infty$. Then there exists M such that for ω–a.e. n, we have $V_n^1, \ldots, V_n^M \in \mathfrak{F} - \{S\}$ such that any $U \in \mathfrak{F}$ with $\mathsf{d}_U(x_n, y_n) > E$ satisfies $U \sqsubseteq V_n^i$ for some i, by the large link axiom (Definition 10.1.(6)).

Hence there exist sequences $(V_n^1)_n, \ldots, (V_n^M)_n$ such that the following holds. If $(U_n)_n$ is a sequence with $\lim_\omega \mathsf{d}_{U_n}(x_n, y_n) = \infty$, then there exists $i \leqslant M$ such that $U_n \sqsubseteq V_n^i$ for ω–a.e. n. In particular,

$$\max \mathrm{Rel}((x_n)_n, (y_n)_n) = \bigcup_{i=1}^{M} \max \mathrm{Rel}^{(V_n^i)_n}((x_n)_n, (y_n)_n).$$

So, it suffices to show that each $\max \mathrm{Rel}^{(V_n^i)_n}((x_n)_n, (y_n)_n)$ is finite.

Fix i and let $V_n = V_n^i$ (for convenience). We argue by induction on $\mathrm{Level}((V_n))$ that $\max \mathrm{Rel}^{(V_n)}((x_n)_n, (y_n)_n)$ is finite.

If $\mathrm{Level}((V_n)) = 1$, then V_n is $\sqsubseteq$–minimal for ω–a.e. n, so any sequence that is ω–a.e. nested in (V_n) is ω–equivalent to (V_n). In particular, $\max \mathrm{Rel}^{(V_n)}((x_n)_n, (y_n)_n)$ has cardinality at most 1. This handles the base case.

The inductive step is almost the same as the preceding discussion of S. Specifically, if $\lim_\omega \mathsf{d}_{V_n}(x_n, y_n) = \infty$, then $\max \mathrm{Rel}^{(V_n)}((x_n)_n, (y_n)_n)$ has cardinality 1, because it contains only the sequence (V_n), and we are done. Hence suppose that $\lim_\omega \mathsf{d}_{V_n}(x_n, y_n) < \infty$.

Then there exists M such that, for ω–a.e. n, we have elements $U_n^1, \ldots, U_n^M \subsetneq V_n$ such that any $U \in \mathfrak{F}$ with $\mathsf{d}_U(x_n, y_n) > E$, we have $U \sqsubseteq U_n^i$ for some i (this is an application of the large link axiom).

In particular, if $(U_n)_n \in \max \mathrm{Rel}^{(V_n)}((x_n)_n, (y_n)_n)$, then for ω–a.e. n, we have that:

- $U_n \subsetneq V_n$, and
- $\mathsf{d}_{U_n}(x_n, y_n) > E$, so $U_n \sqsubseteq U_n^i$ for some i.

Hence there exists i such that $U_n \sqsubseteq U_n^i$ for ω–a.e. n. In particular,

$$\max \mathrm{Rel}^{(V_n)}((x_n)_n, (y_n)_n) = \bigcup_{i=1}^{M} \max \mathrm{Rel}^{(U_n^i)}((x_n)_n, (y_n)_n).$$

So, by induction on level, $\max \mathrm{Rel}^{(V_n)}((x_n)_n, (y_n)_n)$ is a finite union of finite sets.

Nonemptiness: Suppose that $(x_n)_n, (y_n)_n$ represent distinct points in $\mathrm{Cone}^\omega(\mathcal{X})$, and let $\epsilon > 0$ be less than the distance between these two points (in the original asymptotic cone metric).

Recall from Definition 10.1.(9) (uniqueness axiom) that there is a function $\theta_u : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $a, b \in \mathcal{X}$ and all $\kappa \geqslant 0$, we have that $\mathsf{d}_{\mathcal{X}}(a,b) > \theta_u(\kappa)$ implies that $\mathsf{d}_V(a,b) \geqslant \kappa$ for some $V \in \mathfrak{F}$.

For ω–a.e. n, we have $\mathsf{d}_{\mathcal{X}}(x_n, y_n) \geqslant \epsilon j_n$. For each n, let V_n be such that $\mathsf{d}_{V_n}(x_n, y_n)$ is as large as possible for the fixed x_n, y_n, as V_n varies in $\mathfrak{F}$. This is possible since we can assume the maximal such projection distance is more than E and apply Lemma 11.4. We claim that

- (V_n) is legal, and
- $\lim_\omega \mathsf{d}_{V_n}(x_n, y_n) = \infty$.

For any $N \in \mathbb{N}$, we have that $\mathsf{d}_{V_n}(x_n, y_n) < N$ implies that $\mathsf{d}(x_n, y_n) < \theta_u(N)$, which is less than ϵj_n for ω–a.e. n since ϵj_n is unbounded. So the set of n such that $\mathsf{d}_{V_n}(x_n, y_n) < N$ is not in ω, i.e. $\mathsf{d}_{V_n}(x_n, y_n) > N$ for ω–a.e. n. This proves the second bullet point. On the other hand, for ω–a.e. n, we have $\mathsf{d}_{V_n}(x_n, y_n) > 200DE$, where D is the constant from Proposition 18.1. Hence x_n, y_n are joined by a (D, D)–hierarchy path that comes within a uniformly bounded distance of P_{V_n}. Thus $\mathsf{d}(o_n, P_{V_n})$ can be bounded in terms of $\mathsf{d}(o_n, x_n)$ and $\mathsf{d}(x_n, y_n)$, so (V_n) is legal. Thus $\mathrm{Rel}((x_n)_n, (y_n)_n)$, and hence $\max \mathrm{Rel}((x_n)_n, (y_n)_n))$, is nonempty. This completes the proof. □

The next lemma is important because it relates the max-relevant set for a pair of admissible sequences to the set of walls separating the corresponding points in $\mathrm{Cone}^\omega(\mathcal{X})$.

Lemma 28.4. *Let $\mathbf{U} \in \mathfrak{F}^\infty$, fix a parallel copy $\mathbf{F_U} \subset \mathbf{P_U}$, and let $\mathbf{x}, \mathbf{y} \in \mathbf{F_U}$.*

Let $(U_n)_n$ be an admissible sequence of elements from $\mathfrak{F}$ that represents $\mathbf{U}$.

Then there exist $(x_n)_n, (y_n)_n$, admissible sequences that represent $\mathbf{x}$ and $\mathbf{y}$ respectively, such that

- *$x_n, y_n \in F_{U_n}$, where $\lim_\omega F_{U_n} = \mathbf{F_U}$;*
- *$\max \mathrm{Rel}((x_n)_n, (y_n)_n) = \max \mathrm{Rel}^{(U_n)_n}((x_n)_n, (y_n)_n) = \{(U_n^1)_n, \ldots, (U_n^r)_n\}$; in particular, for $1 \leqslant i \leqslant r$, we have $U_n^i \sqsubseteq U_n$ for ω-a.e. n;*
- *$\mathcal{W}(\mathbf{x}, \mathbf{y}) \subseteq \bigcup_{1 \leqslant i \leqslant r} \mathcal{W}(\mathbf{F}_{(U_n^i)_n})$.*

Moreover, the second two bullet points hold for any sequences $(x_n), (y_n)$ that represent $\mathbf{x}, \mathbf{y}$ and satisfy the first bullet point.

Remark 28.5. The set $\max \mathrm{Rel}((x_n)_n, (y_n)_n)$ depends on the choice of sequences, not just on $\mathbf{x}, \mathbf{y}$. But suppose that $(x'_n)_n$ also represents $\mathbf{x}$. Then any (V_n) which is relevant for $(x_n)_n, (y_n)_n$ but not $(x'_n)_n, (y_n)_n$ has the property that $\mathsf{d}_{F_{V_n}}(\mathfrak{g}_{F_{V_n}}(x_n), \mathfrak{g}_{F_{V_n}}(x'_n))$ is sublinear in (j_n), and thus no wall separating $\mathbf{x}, \mathbf{y}$ crosses $\mathbf{F_V}$, where (V_n) represents $\mathbf{V}$. So the (U_n^i) from the statement that actually matter for the purposes of the third bullet point are uniquely determined just by $\mathbf{x}, \mathbf{y}$. However, we won't need this. □

Proof of Lemma 28.4. Let $(x'_n)_n, (y'_n)_n$ be admissible sequences representing $\mathbf{x}$ and $\mathbf{y}$. By Definition 24.3, $\mathbf{P_U} = \lim_w P_{U_n}$ and so there exist parallel copies of F_{U_n} such that $\mathbf{F_U} = \lim_w F_{U_n}$. Define $x_n = \mathfrak{g}_{F_{U_n}}(x'_n)$ and $y_n = \mathfrak{g}_{F_{U_n}}(y'_n)$.

Since $\mathbf{x}, \mathbf{y} \in \mathbf{F_U}$, it follows that $(x'_n)_n$ is at distance $o(j_n)$ (sublinear growth with respect to the rescaling constants) from F_{U_n} and so at sublinear distance from x_n (see e.g. [BHS17c, Lemma 1.27]).

Therefore, $(x'_n)_n$ and $(x_n)_n$ (resp. $(y_n)_n$ and $(y'_n)_n$) represent the same point in $\mathrm{Cone}^\omega(\mathcal{X})$. The sequences $(x_n)_n$ and $(y_n)_n$ satisfy the condition of the first item.

Since $x_n, y_n \in F_{U_n}$, it follows that if $d_{V_n}(x_n, y_n)$ is sufficiently large in terms of the HHS constants (independently of n), then $V_n \sqsubseteq U_n$. Therefore, $\max \mathrm{Rel}((x_n)_n, (y_n)_n) =$

$\max \mathrm{Rel}^{(U_n)_n}((x_n)_n, (y_n)_n)$ has finitely many elements $\{(U_n^1)_n, \ldots, (U_n^r)_n\}$ (by Lemma 41.4) and $U_n^i \sqsubseteq U_n$ for ω-a.e. n and for all $i = 1, \ldots, r$. This proves the second item of the statement.

We now prove the third item, which will involve some sub-claims.

Proof of the third item: If $\mathbf{x} = \mathbf{y}$, then $\mathcal{W}(\mathbf{x}, \mathbf{y})$ is empty and so the third item holds trivially. So assume that $\mathbf{x} \neq \mathbf{y}$. Then $\mathcal{W}(\mathbf{x}, \mathbf{y})$ is nonempty and from Lemma 41.4 the set $\max \mathrm{Rel}((x_n)_n, (y_n)_n)$ is also nonempty. Let $\{(U_n^i)_n\}_{i=1}^r$ be the max-relevant set, which is finite by Lemma 41.4. For each i, let $\mathbf{U}^i$ be represented by the sequence $(U_n^i)_n$.

Let $\hat{w} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$ and suppose that $\mathbf{x} \in w$ and $\mathbf{y} \in w^*$. We need to prove that $\hat{w}$ crosses some $\mathbf{F}_{\mathbf{U}^i}$ (recall from Lemma 2.15 that whether or not $\hat{w}$ crosses $\mathbf{F}_{\mathbf{U}_i}$ is independent of the choice of parallel copy).

Claim 16. For $1 \leqslant i \leqslant r$, there exists a parallel copy $\mathbf{F}_{\mathbf{U}^i}$ such that $I(\mathbf{x}, \mathbf{y}) \cap \mathbf{F}_{\mathbf{U}^i} \neq \varnothing$, and that intersection is closed and convex.

Proof of Claim 16. Since the median $\boldsymbol{\mu}$ on $\mathrm{Cone}^\omega(\mathcal{X})$ is defined to be the limit of the coarse median μ on $\mathcal{X}$, we have that $I(\mathbf{x}, \mathbf{y}) = \lim_\omega H_\theta(x_n, y_n)$ (see e.g. [BHS17c, Lemma 3.2]).

For each $i \leqslant r$, we have that $\mathsf{d}_{(U_n^i)}(x_n, y_n) > 200DE$ for ω–a.e. n, so by Proposition 18.1 and hierarchical quasiconvexity of $H_\theta(x_n, y_n)$, we have that $H_\theta(x_n, y_n)$ contains points uniformly close to $P_{U_n^i}$. So $\mathbf{P}_{\mathbf{U}^i} \cap I(\mathbf{x}, \mathbf{y}) \neq \varnothing$, whence we can choose $\mathbf{F}_{\mathbf{U}^i}$ in its parallelism class so that $I(\mathbf{x}, \mathbf{y}) \cap \mathbf{F}_{\mathbf{U}^i} \neq \varnothing$. As an intersection of closed convex sets, this intersection is closed and convex in $I(\mathbf{x}, \mathbf{y})$. $\square$

For each $i \leqslant r$, let $\mathbf{A}^i = \mathbf{F}_{\mathbf{U}^i} \cap I(\mathbf{x}, \mathbf{y})$. By the preceding claim, $\mathbf{A}^i$ is nonempty, closed, and convex.

Suppose towards a contradiction that $\hat{w} \notin \mathcal{W}(\mathbf{F}_{\mathbf{U}^i})$ for all $i \in \{1, \ldots, r\}$. Then for each i, we have $\mathbf{A}^i \subset w$ or $\mathbf{A}^i \subset w^*$.

Let L be the convex hull of the set $\{\mathbf{x}\} \cup \{\mathbf{A}^i \mid \mathbf{A}^i \subset w\}$ and let R be the convex hull of the set $\{\mathbf{y}\} \cup \{\mathbf{A}^i \mid \mathbf{A}^i \subset w^*\}$. (By the convex hull of a set, we mean the intersection of all closed convex subsets containing that set. So, L and R are closed convex subsets of $I(\mathbf{x}, \mathbf{y})$.)

Claim 17. There exist $\mathbf{w} \in w \cap I(\mathbf{x}, \mathbf{y})$ and $\mathbf{z} \in w^* \cap I(\mathbf{x}, \mathbf{y})$ such that $\mathfrak{h}_{\mathbf{U}^i}(\mathbf{w}) = \mathfrak{h}_{\mathbf{U}^i}(\mathbf{z})$ for $1 \leqslant i \leqslant r$.

Proof of Claim 17. It suffices to find $\mathbf{w} \in w \cap I(\mathbf{x}, \mathbf{y})$ and $\mathbf{z} \in w^* I(\mathbf{x}, \mathbf{y})$ such that the gates of $\mathbf{w}, \mathbf{z}$ on $\mathbf{A}^i$ are equal for all i. (These gates are well-defined since $\mathbf{A}^i$ is closed and convex.)

By choosing a closest pair $\mathbf{w} \in L, \mathbf{z} \in R$, we have that the gates of both $\mathbf{w}, \mathbf{z}$ in L coincide with $\mathbf{z}$, and the gates of both $\mathbf{w}, \mathbf{z}$ in R coincide with $\mathbf{z}$ (see e.g. [Fio20, Lemma 2.4]). So, since each $\mathbf{A}^i$ belongs to L or R, the gates of $\mathbf{w}, \mathbf{z}$ in each $\mathbf{A}^i$ coincide.

So to conclude, it suffices to prove that $L \subset w$ and $R \subset w^*$. (The worry is that, although each $\mathbf{A}^i$ in L belongs to w, the set L itself might only belong to the closure of w, a priori.)

Since $\mathrm{Cone}^\omega(\mathcal{X})$ is a finite-rank, complete median space, we may assume in view of [Fio20, Corollary 2.23] that w^* is closed and w is open. So $R \subset w^*$, and we just need to prove that $L \subset w$.

Up to relabelling, there exists $s \leqslant r$ such that $i \leqslant s$ if and only if $\mathbf{A}^i \subset w$. (If $s = 0$, then $L = \{\mathbf{x}\}$ and we are done, otherwise $s \geqslant 1$.)

We will construct a closed, median-convex susbet B such that

- $\mathbf{x} \in B$;
- $\mathbf{A}^i \subset B$ for $i \leqslant s$;
- $B \subset w$.

By the definition of L, it will then follow that $L \subset B$ and hence $L \subset w$, as required.

Since $I(\mathbf{x},\mathbf{y})$ is a complete, finite rank median metric space, it admits a metric σ, bilipschitz equivalent to the restriction of $\mathbf{D}$ to $I(\mathbf{x},\mathbf{y})$, such that closed balls in the σ metric are median convex. (This follows by combining [Bow20, Lemma 3.2] with the results from [Bow20, Section 6].)

For each $i \leqslant s$, let $\mathbf{y}^i$ be the gate of $\mathbf{y}$ on $\mathbf{A}^i$. Note that $\mathbf{A}^i \subset I(\mathbf{x},\mathbf{y}^i)$ since $\mathbf{A}^i \subset I(\mathbf{x},\mathbf{y})$. (More specifically, for any $\mathbf{a} \in \mathbf{A}^i$, we have $\boldsymbol{\mu}(\mathbf{x},\mathbf{y},\mathbf{a}) = \mathbf{a}$, and we also have $\boldsymbol{\mu}(\mathbf{a},\mathbf{y},\mathbf{y}^i) = \mathbf{y}^i$. Combining these with a standard identity for median algebras gives

$$\boldsymbol{\mu}(\mathbf{a},\mathbf{x},\mathbf{y}^i) = \boldsymbol{\mu}(\mathbf{a},\mathbf{x},\boldsymbol{\mu}(\mathbf{a},\mathbf{y},\mathbf{y}^i)) = \boldsymbol{\mu}(\mathbf{a},\mathbf{y}^i,\boldsymbol{\mu}(\mathbf{a},\mathbf{x},\mathbf{y})) = \boldsymbol{\mu}(\mathbf{a},\mathbf{y}^i,\mathbf{a}) = \mathbf{a}.$$

So, $\mathbf{a} \in I(\mathbf{x},\mathbf{y}^i)$.)

Since each $\mathbf{y}^i$ lies at positive $\mathbf{D}$–distance from w^*, the same is true for the σ–distance. Hence there is a closed σ–ball B, centred at $\mathbf{x}$, such that $B \subset w$ and $\mathbf{y}^i \in B$ for $i \leqslant s$. By convexity of σ–balls, we have $I(\mathbf{x},\mathbf{y}^i) \subset B$ for $i \leqslant s$, and hence $\mathbf{A}^i \subset B$, as required. □

At this point, we have $\mathbf{w} \in w, \mathbf{z} \in w^*$, so $\mathbf{w} \neq \mathbf{z}$, but $\mathbf{w},\mathbf{z}$ have the same gate on $\mathbf{A}_i$, and hence on $\mathbf{F}_{\mathbf{U}_i}$, for $1 \leqslant i \leqslant r$. Using the points $\mathbf{w},\mathbf{z}$, we derive a contradiction as follows.

Recall that $I(\mathbf{x},\mathbf{y}) = \lim_\omega H_\theta(x_n, y_n)$, so we can choose admissible sequences $(w_n)_n, (z_n)_n$, respectively representing $\mathbf{w},\mathbf{z}$, with $w_n, z_n \in H_\theta(x_n,y_n)$.

Let M be sufficiently large, as prescribed by Proposition 16.1.

For each n, apply the cubical approximation theorem (Proposition 16.1). This yields a constant C, independent of n but depending on M, and C–quasimedian maps $f_n^M : \mathbf{Y}_n \to \mathcal{X}$, where $\mathbf{Y}_n$ is a finite CAT(0) cube complex and f_n^M is a (C,C)–quasi-isometry onto $H_\theta(x_n,y_n)$. Moreover, the labels of the hyperplanes in $\mathbf{Y}_n$ are precisely those $U \in \mathfrak{F}$ for which $\mathsf{d}_U(x_n,y_n) > M$.

Choose $w_n', z_n' \in \mathbf{Y}_n$ such that $\mathsf{d}(f_n^M(w_n'), w_n) \leqslant C$ and the same is true for z_n', z_n. Now, since $\mathbf{w},\mathbf{z}$ are distinct, Lemma 41.4 gives $\max \mathrm{Rel}((w_n),(z_n)) \neq \varnothing$.

Let (W_n) be an admissible sequence such that $\lim_\omega \mathsf{d}_{W_n}(w_n,z_n) = \infty$.

Applying Proposition 16.4.(IV), we have that for ω–a.e. n, the element W_n appears as a label of a hyperplane in $\mathbf{Y}_n$ separating w_n', z_n'.

Moreover, let α be a combinatorial geodesic in $\mathbf{Y}_n$ joining vertices mapping to x_n, y_n and passing through w_n'. Then since $\mathbf{Y}_n$ is the convex hull of the endpoints of α (Proposition 16.1) and $f_n^M \circ \alpha$ is a hierarchy path with constants depending only on the HHS structure and the fixed constants M, C (again by Proposition 16.1), we see that $\pi_{W_n}(w_n)$ lies uniformly close to a $\mathcal{C}W_n$–geodesic from $\pi_{W_n}(x_n)$ to $\pi_{W_n}(y_n)$. The same is true for $\pi_{W_n}(z_n)$. Hence $\mathsf{d}_{W_n}(x_n,y_n)$ is unbounded, since $\mathsf{d}_{W_n}(w_n,z_n)$ is. Thus there exists $i \leqslant r$ such that $W_n \sqsubseteq U_n^i$ for ω–a.e. n.

Observe that there exists R, independent of n, such that

$$\sup_{V \in \mathfrak{F} - \cup_i \mathfrak{F}_{U_n^i}} \mathsf{d}_V(w_n,z_n) \leqslant R$$

for ω–a.e. n.

If not, then for each $R \geqslant 0$, we can choose for ω–a.e. n some $V_{R,n}$ not nested into any U_n^i and satisfying $\mathsf{d}_{V_{R,n}}(w_n,z_n) > R$. For each n, choose V_n so that $\mathsf{d}_{V_n}(w_n,z_n)$ is maximal over V_n not nested in any U_n^i, which is possible for ω–a.e. n in view of Lemma 11.4, taking $R \geqslant E$. So for any R and ω–a.e. n, we have $\mathsf{d}_{V_n}(w_n,z_n) \geqslant \mathsf{d}_{V_{R,n}}(w_n,z_n) > R$. Thus $(V_n) \in \mathrm{Rel}((w_n),(z_n))$ but $V_n \not\sqsubseteq U_n^i$, which contradicts the preceding discussion.

By two applications of the distance formula (Theorem 10.7), with threshold R, and the definition of coarse gate maps in $\mathcal{X}$, we thus have a constant A, independent of n, such that for ω–a.a. n,

$$\mathsf{d}(w_n, z_n) \leqslant A \sum_{i=1}^{r} \mathsf{d}(\mathfrak{g}_{F_{U_n^i}}(w_n), \mathfrak{g}_{F_{U_n^i}}(z_n)) + A.$$

The assumption that $\mathbf{w}, \mathbf{z}$ have the same gate on each $\mathbf{F}_{\mathbf{U}^i}$ implies that the right hand side is sublinear in (j_n), whence $\mathsf{d}(\mathbf{w}, \mathbf{z}) = 0$, a contradiction.

Thus the wall $\hat{w}$ crosses some $\mathbf{F}_{\mathbf{U}^i}$, as required by the third item in the lemma. □

Lemma 28.6. *Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty$ be such that $\mathfrak{h}_{\mathbf{F_U}}(\mathbf{F_V})$ is not trivial, i.e. there exists a wall crossing $\mathfrak{h}_{\mathbf{F_U}}(\mathbf{F_V})$. Let $\mathbf{x}, \mathbf{y} \in \mathfrak{h}_{\mathbf{F_U}}(\mathbf{F_V})$. Then there exist $\mathbf{x}', \mathbf{y}' \in \mathbf{F_V}$ and representative sequences $(x_n)_n, (y_n)_n, (x'_n)_n, (y'_n)_n$ of $\mathbf{x}, \mathbf{y}, \mathbf{x}'$, and $\mathbf{y}'$ respectively, such that $\max \mathrm{Rel}((x_n)_n, (y_n)_n) = \max \mathrm{Rel}((x'_n)_n, (y'_n)_n)$ and $\mathrm{Rel}((x_n), (y_n)) = \mathrm{Rel}((x'_n), (y'_n))$.*

Proof. Let $(U_n)_n$ and $(V_n)_n$ be sequences representing $\mathbf{U}$ and $\mathbf{V}$ respectively.

Since by assumption $\mathbf{x}, \mathbf{y} \in \mathfrak{h}_{\mathbf{F_U}}(\mathbf{F_V})$, $\mathbf{x}$ and $\mathbf{y}$ can be represented by sequences $(x_n)_n, (y_n)_n$ such that $x_n, y_n \in \mathfrak{g}_{F_{U_n}}(F_{V_n})$. Define $x'_n = \mathfrak{g}_{V_n}(x_n), y'_n = \mathfrak{g}_{F_{V_n}}(y_n)$. Then $(x'_n)_n, (y'_n)_n$ represent points $\mathbf{x}', \mathbf{y}'$ in $\mathfrak{h}_{\mathbf{U}}(\mathbf{F_V})$.

By [BHS17c, Lemma 1.20], we have for any sufficiently large K (in terms of the HHS structure) that $\mathrm{Rel}_K(x_n, y_n) = \mathrm{Rel}_K(x'_n, y'_n)$ for ω–a.e. n. The lemma follows. □

Lemma 28.7. *Let $\hat{w} \in \mathcal{W}$, and let w, w^* be the associated halfspaces. If $\hat{w}$ crosses $\mathbf{F_U}$, i.e. $w \cap \mathbf{F_U}$ and $w^* \cap \mathbf{F_U}$ are both nonempty, and $\hat{w}$ crosses $\mathbf{F_V}$, then there exists $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$ such that $\hat{w}$ crosses $\mathbf{F_W}$. In particular, there exists a unique $\sqsubseteq$–minimal $\mathbf{U} \in \mathfrak{F}^\infty$ such that $\hat{w}$ crosses $\mathbf{F_U}$.*

Proof. Let $(U_n)_n, (V_n)_n$ be legal sequences representing $\mathbf{U}, \mathbf{V}$. By assumption, $\mathbf{F_U}$ and $\mathbf{F_V}$ are crossed by $\hat{w}$. Our goal is to show that there exists $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$ such that $\hat{w}$ crosses $\mathbf{F_W}$. Since $\hat{w}$ crosses both $\mathbf{F_U}$ and $\mathbf{F_V}$, we also have that $\hat{w}$ crosses $\mathfrak{h}_{\mathbf{U}}(\mathbf{F_V})$, which is therefore non-trivial (since it intersects both halfspaces w, w^*).

Let $\mathbf{x}, \mathbf{y} \in \mathfrak{h}_{\mathbf{U}}(\mathbf{F_V})$ be such that $\hat{w} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$ and let $(x_n)_n, (y_n)_n$ be two admissible sequences that represent $\mathbf{x}, \mathbf{y}$ respectively and such that $x_n, y_n \in \mathfrak{g}_{U_n}(F_{V_n})$ for ω-a.e. n. Let $(x'_n)_n = (\mathfrak{g}_{V_n}(x_n), (y'_n)_n = \mathfrak{g}_{V_n}(y_n) \in F_{V_n}$. From Lemma 28.6, it follows that $\max \mathrm{Rel}((x_n)_n, (y_n)_n) = \max \mathrm{Rel}((x'_n)_n, (y'_n)_n)$.

By Lemma 28.4, we have that there exists $(W_n)_n \in \max \mathrm{Rel}((x_n)_n, (y_n)_n) = \max \mathrm{Rel}((x'_n)_n, (y'_n)_n)$ such that $W_n \sqsubseteq U_n$, $W_n \sqsubseteq V_n$ for ω-a.e. n (and so $(W_n)_n \sqsubseteq (U_n)_n, (V_n)_n$) and $\hat{w} \in \mathcal{W}_{\mathbf{F}_{(W_n)_n}}$. The element $\mathbf{W} \in \mathfrak{F}^\infty$ represented by $(W_n)_n$ satisfies the desired properties.

Let us now show that the existence of a $\sqsubseteq$-minimal element crossed by a given wall $\hat{w}$. First, we have $\mathrm{Cone}^\omega(\mathcal{X}) = \mathbf{F_S}$ where $\mathbf{S}$ is the unique $\sqsubseteq$–maximal element of $\mathfrak{F}^\infty$. So, by finite complexity, there exists $\mathbf{U}$ that is $\sqsubseteq$–minimal with the property that $\hat{w}$ crosses some (hence any) parallel copy $\mathbf{F_U}$. However, if $\mathbf{V}$ is some other element of $\mathfrak{F}^\infty$ such that $\hat{w}$ crosses $\mathbf{F_V}$, then, as shown above, there exists $\mathbf{W}$ such that $\mathbf{W} \sqsubseteq \mathbf{U}, \mathbf{V}$ and $\hat{w}$ crosses $\mathbf{F_W}$. By minimality of $\mathbf{U}$, we thus have $\mathbf{U} = \mathbf{W}$, so either $\mathbf{V} = \mathbf{U}$ or $\mathbf{V}$ wasn't $\sqsubseteq$–minimal. Hence there exists a unique such $\mathbf{U}$ with the desired properties. □

Define a map $Col : \mathcal{W} \to \mathfrak{F}^\infty$ by declaring $Col(\hat{w})$ to be the unique $\sqsubseteq$–minimal element $\mathbf{U} \in \mathfrak{F}^\infty$ such that $\hat{w}$ crosses $\mathbf{F_U}$; this element is provided by the preceding lemma.

Lemma 28.8. *The map $Col : \mathcal{W} \to (\mathfrak{F}^\infty, \sqsubseteq)$ is a finite-depth poset-colouring.*

Proof. We verify the conditions from Definition 3.1. First, as explained above, $\mathfrak{F}^\infty$ has a unique $\sqsubseteq$–maximal element $\mathbf{S}$. Finite depth follows since $\sqsubseteq$–chains have length at most $\chi^{\sqsubseteq}$.

It remains to check the enumerated parts of Definition 3.1. Recall that for each $\mathbf{U} \in \mathfrak{F}^\infty$, the set $\mathcal{W}_{\mathbf{U}}$ is the set of walls $\hat{w}$ with $Col(\hat{w}) \sqsubseteq \mathbf{U}$, and $\mathcal{H}_{\mathbf{U}}$ is the set of associated halfspaces.

Item (I): We must show that $\mathcal{W}_{\mathbf{U}}$ is inseparable. Let $\hat{h},\hat{v}\in\mathcal{W}_{\mathbf{U}}$. Since $\hat{h}$ crosses $\mathbf{F}_{Col(\hat{h})}$ and $Col(\hat{h})\sqsubseteq\mathbf{U}$ (so $\mathbf{F}_{Col(\hat{h})}\subset\mathbf{F}_{\mathbf{U}}$, up to choosing parallel copies), we have that $\hat{h}$ crosses $\mathbf{F}_{\mathbf{U}}$. Similarly, $\hat{v}$ crosses $\mathbf{F}_{\mathbf{U}}$. Hence, by convexity of $\mathbf{F}_{\mathbf{U}}$, if $\hat{u}$ is a wall separating $\hat{h},\hat{v}$, then $\hat{u}$ crosses $\mathbf{F}_{\mathbf{U}}$. Let $\mathbf{V}=Col(\hat{u})$. Then by definition $\hat{u}$ crosses $\mathbf{F}_{\mathbf{V}}$. Since $\hat{u}$ crosses both $\mathbf{F}_{\mathbf{U}}$ and $\mathbf{F}_{\mathbf{V}}$, it follows from Lemma 28.7, that there exist $\mathbf{W}\sqsubseteq\mathbf{U},\mathbf{V}$ such that $\hat{u}$ crosses $\mathbf{F}_{\mathbf{W}}$. From the minimality of $\mathbf{V}$, it follows that $\mathbf{W}=\mathbf{V}$ and so $Col(\hat{u})=\mathbf{V}\sqsubseteq\mathbf{U}$. We conclude that $\hat{u}\in\mathcal{W}_{\mathbf{U}}$, as required.

Item (II): Let $\mathbf{U},\mathbf{V}\in\mathfrak{F}^{\infty}$. Recall that $\mathcal{H}_{\mathbf{U}},\mathcal{H}_{\mathbf{V}}$ denote the sets of halfspaces associated to walls in $\mathcal{W}_{\mathbf{U}},\mathcal{W}_{\mathbf{V}}$. Suppose that we have nonempty $\mathcal{A}\subset\mathcal{W}_{\mathbf{U}}$ and $\mathcal{B}\subset\mathcal{W}_{\mathbf{V}}$. Recall that $\mathcal{H}_{\mathcal{A}}$ denotes the set of halfspaces associated to walls in $\mathcal{A}$.

Suppose that $\text{fio}(\mathcal{H}_{\mathbf{U}}-\mathcal{H}_{\mathcal{A}})=0$, and the same holds with $\mathbf{V}$ replacing $\mathbf{U}$ and $\mathcal{B}$ replacing $\mathcal{A}$. Suppose moreover that every wall in $\mathcal{A}$ crosses every wall in $\mathcal{B}$.

Let $A=\mathbf{F}_{\mathbf{U}}$, $B=\mathbf{F}_{\mathbf{V}}$. Up to measure–0 sets, we have $\mathcal{H}_{A}=\mathcal{H}_{\mathcal{A}}\cap\mathcal{H}(\mathbf{F}_{\mathbf{U}})$ and $\mathcal{H}_{B}=\mathcal{H}_{\mathcal{B}}\cap\mathcal{H}(\mathbf{F}_{\mathbf{V}})$. Applying Proposition 2.22, we have that (up to parallel copies) there exists an isometric embedding $\mathbf{F}_{\mathbf{U}}\times\mathbf{F}_{\mathbf{V}}\to\text{Cone}^{\omega}(\mathcal{X})$ with median-convex image. Since $\mathbf{F}_{\mathbf{U}},\mathbf{F}_{\mathbf{V}}$ are nontrivial, it follows that neither contains the other (even up to parallelism), whence $\mathbf{F}_{\mathbf{U}},\mathbf{F}_{\mathbf{V}}$ are unrelated by containment up to parallelism. Hence $\mathbf{U},\mathbf{V}$ are $\sqsubseteq$–incomparable, as required.

Item (III): Let $\mathbf{U},\mathbf{V}\in\mathfrak{F}^{\infty}$. Suppose that $\mathcal{A}$ is an inseparable set of walls with $Col(\mathcal{A})\sqsubseteq\mathbf{U},\mathbf{V}$ and suppose that $\text{fio}(\mathcal{H}_{\mathcal{A}})>0$. Then each $\hat{a}\in\mathcal{A}$ crosses $\mathbf{F}_{\mathbf{U}}$ and $\mathbf{F}_{\mathbf{V}}$.

Fix a parallel copy $\mathbf{F}_{\mathbf{U}}$ and a parallel copy $\mathbf{F}_{\mathbf{V}}$, and let $H=\mathfrak{h}_{\mathbf{U}}(\mathbf{F}_{\mathbf{V}})$. So, each $\hat{a}\in\mathcal{A}$ crosses H.

Let P be the set of pairs $\{\mathbf{x},\mathbf{y}\}\subset H$ such that $\text{fio}(\mathcal{H}(\mathbf{x},\mathbf{y})\cap\mathcal{H}_{\mathcal{A}})>0$.

For each $\{\mathbf{x},\mathbf{y}\}\in H$, we can choose a finite collection $R(\mathbf{x},\mathbf{y})$ of elements $\mathbf{W}\in\mathfrak{F}^{\infty}$ such that

- $\mathbf{W}\sqsubseteq\mathbf{U},\mathbf{V}$;
- $\mathcal{W}(\mathbf{x},\mathbf{y})=\bigcup_{\mathbf{W}\in R(\mathbf{x},\mathbf{y})}\mathcal{W}(\mathbf{F}_{\mathbf{W}})\cap\mathcal{W}(\mathbf{x},\mathbf{y})$;
- $\text{fio}(\mathcal{H}(\mathbf{x},\mathbf{y})\cap\mathcal{H}_{\mathcal{A}}\cap\mathcal{H}_{\mathbf{W}})>0$.

The set $R(\mathbf{x},\mathbf{y})$ is provided by Lemma 28.4 and Lemma 28.6.

Now let $R=\bigcup_{\{\mathbf{x},\mathbf{y}\}\in P}R(\mathbf{x},\mathbf{y})$. Then, by construction,

- $\mathbf{W}\sqsubseteq\mathbf{U},\mathbf{V}$ for all $\mathbf{W}\in R$;
- $\text{fio}(\mathcal{H}_{\mathcal{A}}\cap\mathcal{H}_{\mathbf{W}})>0$ for all $\mathbf{W}\in R$.

We are left to show that $\text{fio}(\mathcal{H}_{\mathcal{A}}-(\bigcup\limits_{\mathbf{W}\in R'}\mathcal{H}_{\mathcal{A}}\cap\mathcal{H}_{\mathbf{F_W}}))=0$. By the definition of fio (see [Fio20, Section 3.1]), it suffices to show that, for any $\mathbf{x},\mathbf{y}\in\text{Cone}^{\omega}(\mathcal{X})$, the set

$$\mathcal{H}(\mathbf{x},\mathbf{y})\cap\mathcal{H}_{\mathcal{A}}-(\bigcup_{\mathbf{W}\in R}\mathcal{H}_{\mathcal{A}}\cap\mathcal{H}_{\mathbf{F_W}}\cap\mathcal{H}(\mathbf{x},\mathbf{y}))$$

has measure 0 for all $\{\mathbf{x},\mathbf{y}\}\in P$. But this is immediate since

$$\mathcal{W}(\mathbf{x},\mathbf{y})=\bigcup_{\mathbf{W}\in R(\mathbf{x},\mathbf{y})}\mathcal{W}(\mathbf{F}_{\mathbf{W}})\cap\mathcal{W}(\mathbf{x},\mathbf{y}).$$

Item (IV): Suppose that $\mathbf{U},\mathbf{V}\in\mathfrak{F}^{\infty}$ (we allow equality). Suppose that $\mathcal{A},\mathcal{B}$ are sets of walls such that

- every element of $\mathcal{A}$ crosses every element of $\mathcal{B}$;
- every element of $\mathcal{A}$ has colour nested in $\mathbf{U}$;
- every element of $\mathcal{B}$ has colour nested in $\mathbf{V}$;
- the sets of halfspaces associated to $\mathcal{A},\mathcal{B}$ have positive fio–measure.

Consider sets $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_j\}$ satisfying Definition 3.1.(III) for $\mathcal{A}, \mathbf{U}$ and $\mathcal{B}, \mathbf{V}$ respectively. In other words, suppose we have:

- $\mathbf{U}_i \sqsubseteq \mathbf{U}$ for all i, and
- $\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i}) > 0$ for all i, and
- $\mathrm{fio}(\mathcal{H}_A - \bigcup_i \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}_{\mathcal{A}}) = 0$,

and the same holds with $\{\mathbf{V}_j\}$ replacing $\{\mathbf{U}_i\}$ and $\mathcal{B}$ replacing $\mathcal{A}$ and $\mathbf{V}$ replacing $\mathbf{U}$.

Among sets $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_j\}$ with the given properties, choose these sets so that the maximum levels of the $\mathbf{U}_i, \mathbf{V}_j$ are as small as possible. (Recall that the *level* of $\mathbf{U}_i$ is the length of a longest $\sqsubseteq$–chain with maximal element $\mathbf{U}_i$.) More precisely, assume that for each i, there is no family $\{\mathbf{U}'_j\}$ such that each $\mathbf{U}'_j \sqsubsetneq \mathbf{U}_i$ and the family $\{\mathbf{U}_{i'}\}_{i' \neq i} \cup \{\mathbf{U}'_j\}$ has the properties enumerated above. This is possible by finite complexity.

Note that the first, second, fourth, and fifth bullet points from Definition 3.1.(IV) are satisfied by these sets.

Claim 18. The families $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_j\}$ have the property that $\mathbf{U}_i \perp \mathbf{V}_j$ for all i, j.

Proof of Claim 18. Fix $i \in I, j \in J$. Since $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i}$ has positive fio–measure, there exist $\mathbf{x}, \mathbf{y} \in \mathbf{F}_{\mathbf{U}_i}$ such that

$$\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})) > 0.$$

Indeed, as explained in [Fio20, Section 3], a set of halfspaces has positive measure only if it has positive-measure intersection with $\mathcal{H}(\mathbf{x}, \mathbf{y})$ for some $\mathbf{x}, \mathbf{y}$.

Likewise, we can choose $\mathbf{w}, \mathbf{z} \in \mathbf{F}_{\mathbf{V}_j}$ such that

$$\mathrm{fio}(\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j} \cap \mathcal{H}(\mathbf{w}, \mathbf{z})) > 0.$$

Since all walls in $\mathcal{A}$ cross all walls in $\mathcal{B}$, we thus have $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbf{X}$ such that

- $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, in that order, form a median rectangle in $\mathbf{X}$, and
- $\mathcal{H}(\{\mathbf{a}, \mathbf{d}\}, \{\mathbf{b}, \mathbf{c}\})$ contains $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ and $\mathbf{U}_i \in \mathit{Rel}(\mathbf{a}, \mathbf{b})$, and
- $\mathcal{H}(\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{c}, \mathbf{d}\})$ contains $\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j} \cap \mathcal{H}(\mathbf{w}, \mathbf{z})$ and $\mathbf{V}_i \in \mathit{Rel}(\mathbf{a}, \mathbf{d})$.

Indeed, we can start with $\mathbf{x} = \mathbf{a}, \mathbf{y} = \mathbf{b}$, and then obtain $\mathbf{d}, \mathbf{c}$ by moving $\mathbf{a}, \mathbf{b}$, respectively, across the walls in $\mathcal{H}_{\mathcal{B}} \cap \mathcal{H}_{\mathbf{V}_j} \cap \mathcal{H}(\mathbf{w}, \mathbf{z})$. Then use use [CDH10, Remark 2.23.(2)] to tighten to a median rectangle. See Figure 12.

We call the ordered 4–tuple $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ a *test rectangle* for $\mathbf{U}_i, \mathbf{V}_j, \mathcal{A}, \mathcal{B}$.

Let $(a_n)_n, (b_n)_n, (c_n)_n, (d_n)_n$ be sequences in $\mathcal{X}$ converging to $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, with the property that b_n uniformly coarsely coincides with the coarse median of a_n, b_n, c_n for ω–a.e. n, and the analogous property holds for each of the triples b_n, c_n, d_n and c_n, d_n, a_n and d_n, a_n, b_n (i.e. for ω–a.e. n, the points a_n, b_n, c_n, d_n form a coarse median rectangle).

To arrange this, first choose $(a_n), (b_n), (c_n), (d_n)$ representing $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ respectively. For each n, replace b_n, c_n by their images under the gate map to $H_\theta(\{a_n, c_n\})$. This moves each of b_n, c_n a distance bounded by a sublinear function of j_n. Then replace a_n, c_n by their images under the gate map to $H_\theta(\{b_n, c_n\})$.

Now apply Proposition 16.1 and consider the cubical approximation $f_n : \mathbf{Y}_n \to \mathcal{X}$ of the hierarchically quasiconvex hull of $\{a_n, b_n, c_n, d_n\}$. Since this map is quasimedian, the vertices a'_n, b'_n, c'_n, d'_n mapping to a_n, b_n, c_n, d_n can be chosen to form a median rectangle in $\mathbf{Y}_n$. So the set of hyperplanes is partitioned into two subsets $\mathcal{U}$ (separating $\{a, b\}$ from $\{c, d\}$) and $\mathcal{V}$ (separating $\{a, d\}$ from $\{b, c\}$) such that every hyperplane in $\mathcal{U}$ crosses every hyperplane in $\mathcal{V}$. By Proposition 16.1, we can thus write (for some sufficiently large constant M independent of n) the set of $W \in \mathfrak{F}$ such that some pair of the points in $\{a_n, b_n, c_n, d_n\}$ project at least M–far apart in $\mathcal{C}W$ as

$$\mathrm{Rel}_M(a_n, b_n) \sqcup \mathrm{Rel}_M(a_n, d_n),$$

with every element of the first factor of the disjoint union orthogonal to every element of the second.

Conclusion: To complete the proof of the claim, i.e. to show that $\mathbf{U}_i \perp \mathbf{V}_j$, we will use the nest-minimality assumption on $\mathbf{U}_i, \mathbf{V}_j$ to show that the test rectangle $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ and the representative sequences $(a_n), (b_n), (c_n), (d_n)$ as above can be chosen so that there are legal sequences $(U_n), (V_n)$, representing $\mathbf{U}_i, \mathbf{V}_j$, for which $U_n \in \mathrm{Rel}_M(a_n, b_n)$ and $V_n \in \mathrm{Rel}_M(a_n, d_n)$ for ω–a.e. n. Then the above will show that $U_n \perp V_n$ for ω–a.e. n, i.e. $\mathbf{U}_i \perp \mathbf{V}_j$.

Next, note that:

$$\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})) = \mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{a}, \mathbf{b})) > 0.$$

The same holds with $\mathcal{B}$ replacing $\mathcal{A}$ and $\mathbf{V}_j$ replacing $\mathbf{U}_i$ and $\{\mathbf{w}, \mathbf{z}\}$ replacing $\{\mathbf{x}, \mathbf{y}\}$ and $\{\mathbf{a}, \mathbf{d}\}$ replacing $\{\mathbf{a}, \mathbf{b}\}$.

Moreover, without changing the measures of any of the preceding intersections of sets of halfspaces, we can assume, by taking gates, that $\mathbf{x}, \mathbf{y}$ are respectively the gates of $\mathbf{a}, \mathbf{b}$ in $\mathbf{F}_{\mathbf{U}_i}$ and $\mathbf{w}, \mathbf{z}$ are respectively the gates of $\mathbf{a}, \mathbf{d}$ in $\mathbf{F}_{\mathbf{V}_j}$, as in Figure 23.

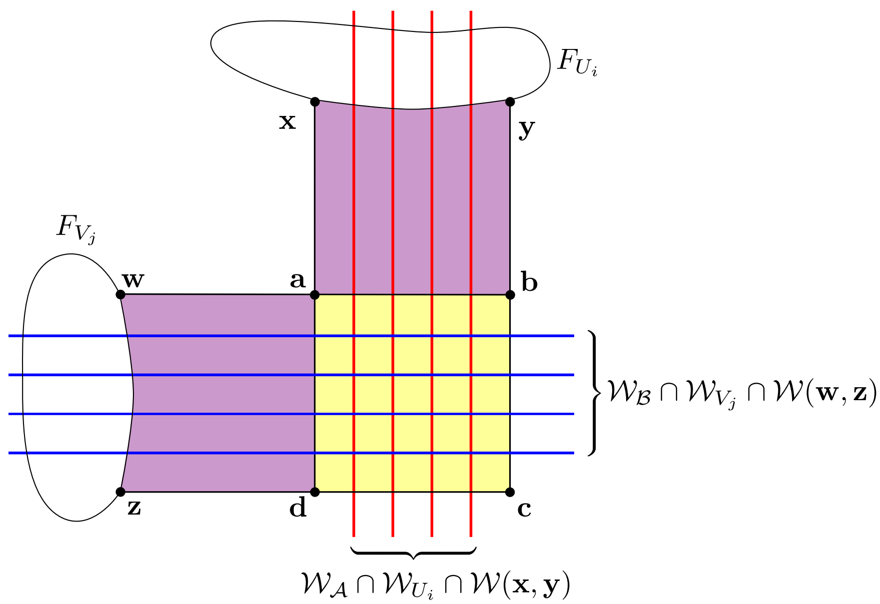

FIGURE 23. A test rectangle in the conclusion of the proof of Claim 18.

Now let $(x_n), (y_n)$ be sequences such that x_n is the (coarse) gate of a_n in F_{U_n} and y_n is the gate of b_n in F_{U_n}. Similarly, choose $(w_n), (z_n)$ to be sequences in F_{V_n} arising as gates of $(a_n), (d_n)$. So $(x_n), (y_n), (w_n), (z_n)$ respectively converge to $\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}$. Note that the coarse gates of x_n, y_n on the coarse median interval in $\mathcal{X}$ between a_n and c_n coarsely coincide with a_n and b_n.

Hence (see [BHS17c, Lemma 1.20]) we have $\max \mathrm{Rel}((a_n), (b_n)) = \max \mathrm{Rel}((x_n), (y_n))$. Now, if $(U_n) \in \max \mathrm{Rel}((x_n), (y_n))$, then $U_n \in \mathrm{Rel}_M(a_n, b_n)$ for ω–a.e. n, as desired. Otherwise, Lemma 28.4 implies that there exist $\mathbf{W}_1, \ldots, \mathbf{W}_k \subsetneq \mathbf{U}_i$ such that $\mathcal{W}(\mathbf{x}, \mathbf{y}) \subset \cup_{\ell=1}^{k} \mathcal{W}(\mathbf{F}_{\mathbf{W}_\ell})$.

Similarly, either $V_n \in \mathrm{Rel}_M(a_n, d_n)$ for ω–a.e. n, or there exist $\mathbf{T}_1, \ldots, \mathbf{T}_k \subsetneq \mathbf{V}_j$ such that $\mathcal{W}(\mathbf{w}, \mathbf{z}) \subset \cup_{\ell=1}^{k} \mathcal{W}(\mathbf{F}_{\mathbf{T}_\ell})$.

Hence we have the following. Either we can choose our test rectangle $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ in such a way that $U_n \in \mathrm{Rel}_M(a_n, b_n)$ and $V_n \in \mathrm{Rel}_M(a_n, d_n)$ for ω–a.e. n — and hence $\mathbf{U}_i \perp \mathbf{V}_j$ — or we can assume the following holds (up to replacing $\mathbf{U}_i$ with $\mathbf{V}_j$ etc.): for each $\mathbf{x}, \mathbf{y}$ such that

$$\operatorname{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})) > 0,$$

there exist $\mathbf{W}_1, \ldots, \mathbf{W}_k \subsetneq \mathbf{U}_i$ such that $\mathcal{W}(\mathbf{x}, \mathbf{y}) \subset \cup_{\ell=1}^{k} \mathcal{W}(\mathbf{F}_{\mathbf{W}_\ell})$. In this case, we can modify the collection $\{\mathbf{U}_{i'}\}_{i' \in I}$ by replacing $\mathbf{U}_i$ with the collection of all such $\mathbf{W}_\ell$, as $\mathbf{x}, \mathbf{y}$ vary. This contradicts our nest–minimal choice of $\{\mathbf{U}_{i'}\}_{i' \in I}$. This proves the claim. □

From Claim 18, we have that $\mathbf{U}_i \perp \mathbf{V}_j$ for all i, j. This then provides a product region $\mathbf{F}_{\mathbf{U}_i} \times \mathbf{F}_{\mathbf{U}_j}$, which in turn implies that all walls crossing the first factor — i.e. all walls in $\mathcal{W}_{\mathbf{U}_i}$ — cross all walls crossing the second factor — i.e. all walls in $\mathcal{W}_{\mathbf{V}_j}$. This completes the verification of Definition 3.1.(IV), and hence the proof that *Col* is a poset-colouring. □

Remark 28.9. In our later applications, the HHS $(\mathcal{X}, \mathfrak{F})$ has wedges and clean containers. Under these hypotheses, $\mathfrak{F}^\infty$ does, also, by Lemma 27.3. This simplifies considerably the proof that *Col* is well-defined and a colouring. Specifically, it allows us to choose the families in the third and fourth parts of the definition of a colouring to be single elements (arising as appropriate wedges and orthogonal complements), and removes the need to use max-relevant sets for sequences. □

We now need to check tangibility. The following statement will be useful for this, and again later.

Corollary 28.10. *Let $\mathbf{U} \in \mathfrak{F}^\infty$. Then $\mathcal{W}_{\mathbf{U}} = \mathcal{W}(\mathbf{F}_{\mathbf{U}})$. In particular, $\mathcal{W}_{\mathbf{U}} = \varnothing$ if and only if $\mathbf{F}_{\mathbf{U}}$ is a single point.*

Proof. Let $\hat{w} \in \mathcal{W}_{\mathbf{U}}$. By definition, this means that $Col(\hat{w}) \sqsubseteq \mathbf{U}$, so up to parallelism, $\mathbf{F}_{Col(\hat{w})} \subset \mathbf{F}_{\mathbf{U}}$. Since $\hat{w}$ crosses $\mathbf{F}_{Col(\hat{w})}$, we see that $\hat{w}$ also crosses $\mathbf{F}_{\mathbf{U}}$, i.e. $\hat{w} \in \mathcal{W}(\mathbf{F}_{\mathbf{U}})$.

On the other hand, suppose that $\hat{w} \in \mathcal{W}(\mathbf{F}_{\mathbf{U}})$, i.e. $\hat{w}$ crosses $\mathbf{F}_{\mathbf{U}}$. Now, by the definition of the colouring, $\hat{w}$ also crosses $\mathbf{F}_{Col(\hat{w})}$. So, applying Lemma 28.7, there exists $\mathbf{W}$ such that $\hat{w}$ crosses $\mathbf{F}_{\mathbf{W}}$ and $\mathbf{W} \sqsubseteq \mathbf{U}$ and $\mathbf{W} \sqsubseteq Col(\hat{w})$. But the nest-minimality part of the definition of $Col(\hat{w})$ implies $\mathbf{W} = Col(\hat{w})$, so $Col(\hat{w}) \sqsubseteq \mathbf{U}$, i.e. $\hat{w} \in \mathcal{W}_{\mathbf{U}}$. This completes the proof. □

When $\mathcal{W}_{\mathbf{U}}$ is nonempty, we construct a halfspace-filter $\sigma_{\mathbf{U}}$ as in Definition 3.3.

Lemma 28.11. *For each $\mathbf{U} \in \mathfrak{F}^\infty$ for which $\mathcal{W}_{\mathbf{U}} \neq \varnothing$, the filter $\sigma_{\mathbf{U}}$ is tangible. Hence Col satisfies the tangible filter condition from Definition 2.17.*

Proof. By Corollary 28.10, $\mathcal{W}_{\mathbf{U}} = \mathcal{W}(\mathbf{F}_{\mathbf{U}})$. Recall the construction of $\sigma_{\mathbf{U}}$. Fixing an (arbitrary) basepoint $\mathbf{x}_0 \in \mathrm{Cone}^\omega(\mathcal{X})$, we choose for each wall $\{w, w^*\} = \hat{w}$ not crossing $\mathbf{F}_{\mathbf{U}}$ an associated halfspace $w \in \sigma_{\mathbf{U}}$ as follows:

- if $\hat{w}$ crosses all of the walls in $\mathcal{W}(\mathbf{F}_{\mathbf{U}})$, then $\mathbf{x}_0 \in w$;
- otherwise, w is the halfspace containing a halfspace associated to a wall crossing $\mathbf{F}_{\mathbf{U}}$.

Let $\mathbf{Q}$ be the union of all closed convex subspaces parallel to $\mathbf{F}_{\mathbf{U}}$. Note that $\mathbf{P}_{\mathbf{U}} \subseteq \mathbf{Q}$ (in principle, the containment can be proper). Let $\mathbf{x}_0'$ be the gate of $\mathbf{x}_0$ in $\mathbf{Q}$, and let $\mathbf{x}_0'' = \mathfrak{g}_{\mathbf{P}_{\mathbf{U}}}(\mathbf{x}_0)$, which is also the gate in $\mathbf{P}_{\mathbf{U}}$ of $\mathbf{x}_0'$. So $\mathbf{D}(\mathbf{x}_0, \mathbf{x}_0') \leqslant \mathbf{D}(\mathbf{x}_0, \mathbf{P}_{\mathbf{U}}) < \infty$ since $\mathbf{U}$ is legal.

On the other hand,

$$\operatorname{fio}(\sigma_{\mathbf{U}} \triangle \sigma_{\mathbf{x}_0}) = \mathbf{D}(\mathbf{x}_0, \mathbf{x}_0'),$$

which is thus finite. Hence $\sigma_{\mathbf{U}}$ is tangible. □

29. Proof of Theorem 26.3

By Lemma 28.8 and Lemma 28.11, the map $Col : \mathcal{W} \to \mathfrak{F}^\infty$ is a finite-depth poset-colouring satisfying the tangible filter condition.

Now, in general, the orthogonality relation on $\mathfrak{F}^\infty$ coming from Section 27 might not coincide with the orthogonality relation on $\mathfrak{F}^\infty$ coming from the colouring and used in the proof of Theorem 5.1 (which we are about to apply).

However, this issue is fixed by restricting the codomain of the colouring map Col to the image of Col.

Definition 29.1. Let $\mathfrak{F}^\infty_+ \subset \mathfrak{F}^\infty$ be the image of the poset-colouring map, that is, $\mathfrak{F}^\infty_+ = Col(\mathcal{W})$, where Col is as in Lemma 28.8. □

The advantage of restricting the codomain is the following lemma:

Lemma 29.2. *Let $\mathbf{U} \in \mathfrak{F}^\infty_+$. Then for any parallel copy $\mathbf{F_U} \subset \mathbf{P_U}$, there exist $\mathbf{x}, \mathbf{y} \in \mathbf{F_U}$ and sequences $(U_n), (F_{U_n}), (x_n), (y_n)$ such that for ω–a.e. n, we have $x_n, y_n \in F_{U_n}$, and $\lim_\omega x_n = \mathbf{x}, \lim_\omega y_n = \mathbf{y}, \lim_\omega F_{U_n} = \mathbf{F_U}$, and (U_n) represents $\mathbf{U}$, and*

$$(U_n) \in \max \mathrm{Rel}((x_n), (y_n)).$$

Moreover, for any wall $\hat{w}$ with $Col(\hat{w}) = \mathbf{U}$, we can choose $\mathbf{x}, \mathbf{y}$ as above with the additional property that $\hat{w} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$, and in fact $(x_n), (y_n)$ can be chosen as above for any $\mathbf{x}, \mathbf{y} \in \mathbf{F_U}$ separated by $\hat{w}$.

Proof. Suppose that (U_n) is a legal sequence representing $\mathbf{U}$. Fix a parallel copy of $\mathbf{F_U}$, so that we can choose (F_{U_n}) with $\lim_\omega F_{U_n} = \mathbf{F_U}$.

By hypothesis, there exists a wall $\hat{w} = \{w, w^*\}$ with $Col(\hat{w}) = \mathbf{U}$. In particular, $\hat{w}$ crosses $\mathbf{F_U}$, so we can choose $\mathbf{x} \in w \cap \mathbf{F_U}$ and $\mathbf{y} \in w^* \cap \mathbf{F_U}$.

Choose legal sequences $(x_n), (y_n)$ such that $x_n, y_n \in F_{U_n}$ for ω–a.e. n and $\lim_\omega x_n = \mathbf{x}, \lim_\omega y_n = \mathbf{y}$. Now, since $Col(\hat{w}) = \mathbf{U}$, the wall $\hat{w}$ cannot cross $\mathbf{F_W}$ for any $\mathbf{W} \subsetneq \mathbf{U}$, so by Lemma 28.4 and the fact that $\hat{w} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$, we have $(U_n) \in \max \mathrm{Rel}((x_n), (y_n))$. □

Remark 29.3 (Warning)**.** The converse to the preceding lemma does not hold. Specifically, we can choose $\mathbf{U}$ and points $\mathbf{x}, \mathbf{y} \in \mathbf{F_U}$ such that $\mathbf{U}$ is represented by a legal sequence (U_n) and $(U_n) \in \max \mathrm{Rel}((x_n), (y_n))$, but there are (possibly infinitely many) $\mathbf{W} \subsetneq \mathbf{U}$ such that all but a measure–0 set of the walls separating $\mathbf{x}, \mathbf{y}$ cross some $\mathbf{F_W}$ and hence have colour properly nested in $\mathbf{U}$. □

Now we can compare the different notions of orthogonality:

Lemma 29.4. *Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty_+$. Then the following are equivalent:*

(1) Every wall crossing $\mathbf{F_U}$ crosses every wall crossing $\mathbf{F_V}$;
(2) $\mathbf{F_U}, \mathbf{F_V}$ are respectively parallel to closed convex subspaces $A, B \subset \mathcal{X}$ such that the convex hull of $A \cup B$ splits as a product $A \times B$;
(3) for all walls $\hat{h}, \hat{v}$ respectively crossing $\mathbf{F_U}$ and $\mathbf{F_V}$ we have $Col(\hat{h}) \perp Col(\hat{v})$, where orthogonality is as defined in Section 27;
(4) $\mathbf{U} \perp \mathbf{V}$, in the sense of Section 27.

Proof. We first prove the equivalence of (2) and (1).

That (2) implies (1) is obvious. The converse follows by applying Proposition 2.22.

Next, we prove equivalence of (3) and (4). First assume that (3) holds. By hypothesis, we can choose walls $\hat{h}, \hat{v}$ with colours $\mathbf{U}, \mathbf{V}$ respectively. In particular, $\hat{h}$ crosses $\mathbf{F_U}$ and $\hat{v}$ crosses $\mathbf{F_V}$, so by (3), we have $\mathbf{U} \perp \mathbf{V}$. Next assume that (4) holds. Choose walls $\hat{h}$ crossing $\mathbf{F_U}$ and $\hat{v}$ crossing $\mathbf{F_V}$. By Corollary 28.10, we have that $Col(\hat{h}) \sqsubseteq \mathbf{U}$ and $Col(\hat{v}) \sqsubseteq \mathbf{V}$. So $\mathbf{U} \perp \mathbf{V}$ implies $Col(\hat{h}) \perp Col(\hat{v})$, as required by (3).

To conclude, we will prove (1) implies (4) and (3) implies (2).

First assume (3). Let $\hat{h}, \hat{v}$ be walls with $Col(\hat{h}) = \mathbf{U}, Col(\hat{v}) = \mathbf{V}$, which exist since $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\infty}_{+}$. By (3), we see that $\mathbf{U} \perp \mathbf{V}$. Letting (U_n) and (V_n) be legal sequences representing $\mathbf{U}, \mathbf{V}$, we have $U_n \perp V_n$ for ω–a.e. n. Hence, for ω–a.e. n, there is a hierarchically quasiconvex subspace of $\mathcal{X}$ that is uniformly quasimedian quasi-isometric to $F_{U_n} \times F_{V_n}$ lying uniformly close to P_{U_n} and P_{V_n}. Taking ultralimits gives $\mathbf{F_U} \times \mathbf{F_V} \subset \mathrm{Cone}^{\omega}(\mathcal{X})$, as required by (2) (this product consists of admissible points by essentially the proof of Proposition 24.1).

Finally, assume (1), i.e. every wall in $\mathcal{W}(\mathbf{F_U}) = \mathcal{W}_{\mathbf{U}}$ crosses every wall in $\mathcal{W}(\mathbf{F_V}) = \mathcal{W}_{\mathbf{V}}$ (the equalities are from Corollary 28.10).

Let $(U_n)_n$ and $(V_n)_n$ be legal sequences representing $\mathbf{U}, \mathbf{V}$. We will argue very similarly to the proof of Claim 18 to show that $U_n \perp V_n$ for ω–a.e. n, i.e. $\mathbf{U} \perp \mathbf{V}$, as required.

Choose walls $\hat{h}, \hat{v}$ whose colours are respectively $\mathbf{U}, \mathbf{V}$. By (1), the walls $\hat{h}, \hat{v}$ cross. Choose a median rectangle $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ in $\mathrm{Cone}^{\omega}(\mathcal{X})$ so that $\hat{v}$ separates $\{\mathbf{a}, \mathbf{b}\}$ from $\{\mathbf{c}, \mathbf{d}\}$ and $\hat{h}$ separates $\{\mathbf{a}, \mathbf{d}\}$ from $\{\mathbf{b}, \mathbf{c}\}$. Let $\mathbf{x}, \mathbf{y}$ be the gates of $\mathbf{a}, \mathbf{b}$ on $\mathbf{F_U}$ and let $\mathbf{w}, \mathbf{z}$ be the gates of $\mathbf{a}, \mathbf{d}$ on $\mathbf{F_V}$, and then replace $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ by the gates of $\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}$ on the convex hull of the original median rectangle (so that we have the configuration from Figure 23). Apply Lemma 29.2 to see that $(U_n) \in \max \mathrm{Rel}((x_n), (y_n))$ and hence $(U_n) \in \max \mathrm{Rel}((a_n), (b_n))$ and similarly $(V_n) \in \max \mathrm{Rel}((a_n), (d_n))$. Arguing as in the proof of Claim 18 now shows that $U_n \perp V_n$ for ω–a.e. n, as required. □

Lemma 29.5. *The map $Col : \mathcal{W} \to \mathfrak{F}^{\infty}_{+}$ is a finite-depth poset-colouring satisfying the tangible filter condition.*

Proof. By definition, $Col(\mathcal{W}) = \mathfrak{F}^{\infty}_{+}$. Tangibility and finite depth are inherited from $Col : \mathcal{W} \to \mathfrak{F}^{\infty}$.

We now verify that $Col : \mathcal{W} \to \mathfrak{F}^{\infty}$ satisfies the conditions from Definition 3.1. Condition (I) holds since passing from $\mathfrak{F}^{\infty}$ to $\mathfrak{F}^{\infty}_{+}$ did not change $\mathcal{W}_{\mathbf{U}}$ for any $\mathbf{U} \in \mathfrak{F}^{\infty}_{+}$. The same reasoning verifies condition (II).

Verifying Definition 3.1.(III): Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\infty}_{+}$. Suppose that $\mathcal{A}$ is an inseparable set of walls such that $Col(\mathcal{A}) \sqsubseteq \mathbf{U}, \mathbf{V}$ and $\mathrm{fio}(\mathcal{H}_{\mathcal{A}}) > 0$.

Since $Col : \mathcal{W} \to \mathfrak{F}^{\infty}$ is a poset-colouring, there exists a family $\{\mathbf{W}_i\}_{i \in I}$ of elements of $\mathfrak{F}^{\infty}$ such that

- $\mathbf{W}_i \sqsubseteq \mathbf{U}, \mathbf{V}$ for all $i \in I$;
- $\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}_i}) > 0$ for all $i \in I$;
- up to a measure-0 set, $\mathcal{H}_{\mathcal{A}} \subset \bigcup_{i \in I} \mathcal{H}_{\mathbf{W}_i}$.

We choose the $\mathbf{W}_i$ to be nest–minimal, in the same sense as in the proof of the proof of Lemma 28.8: if, for some $i \in I$, there exists a set $\{\mathbf{W}'_k\}_k$ such that $\mathbf{W}_k \sqsubsetneq \mathbf{W}_i$ for all k and $\{\mathbf{W}_{i'}\}_{i' \in I - \{i\}} \cup \{\mathbf{W}'_k\}$ satisfies the above listed properties, we replace $\mathbf{W}_i$ by $\{\mathbf{W}_k\}$. Finite complexity ensures we can choose $\{\mathbf{W}_i\}$ so that no such replacements are possible.

We claim that each such $\mathbf{W}_i \in \mathfrak{F}^{\infty}_{+}$. Fix $i \in I$. Choose $\mathbf{x}, \mathbf{y} \in \mathrm{Cone}^{\omega}(\mathcal{X})$ such that

$$\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})) > 0.$$

By taking gates, we can assume that $\mathbf{x}, \mathbf{y}$ lie in $\mathbf{F}_{\mathbf{W}_i} \subset \mathbf{P}_{\mathbf{W}_i}$, where $\mathbf{F}_{\mathbf{W}_i} \subset \mathfrak{h}_{\mathbf{U}}(\mathbf{F_V})$.

We say that $\mathbf{x}, \mathbf{y}$ are $\mathbf{W}_i$*–replaceable* if there exist $\{\mathbf{W}'_k\}_k$ such that each $\mathbf{W}'_k \sqsubsetneq \mathbf{W}_i$, and $\mathrm{fio}(\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}'_k} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})) > 0$ for all k, and, up to a measure–0 set, $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ is contained in $\bigcup_k \mathcal{H}_{\mathbf{W}'_k}$.

Our minimality assumption implies that $\mathbf{x}, \mathbf{y}$ can be chosen as above so that they are not $\mathbf{W}_i$–replaceable.

If $\mathbf{x}, \mathbf{y}$ are not $\mathbf{W}_i$–replaceable, then by Lemma 28.4, we have $Col(\hat{h}) \sqsubseteq \mathbf{W}_i$ for each $\hat{h} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$.

For each $\hat{h} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$, let $I_{\hat{h}}$ be the image of $\mathbf{F}_{Col(\hat{h})}$ under the gate map to $I(\mathbf{x}, \mathbf{y}) \subset \mathbf{F}_{\mathbf{W}_i}$. Since $\hat{h}$ crosses $I_{\hat{h}}$, the latter is a nontrivial closed convex subspace.

If some $\hat{w} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$ does not cross $I_{\hat{h}}$ for any $Col(\hat{h}) \subsetneq \mathbf{W}_i$, then $\mathbf{W}_i = Col(\hat{w})$ and so $\mathbf{W}_i \in \mathfrak{F}^{\infty}_{+}$, as required.

Assume that this does not hold, so that every $\hat{w} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$ crosses $I_{\hat{h}}$ for some $Col(\hat{h}) \subsetneq \mathbf{W}_i$. Let $\{\mathbf{W}'_k\}_k$ be the set of elements $Col(\hat{h}) \subsetneq \mathbf{W}_i$ with $\hat{h} \in \mathcal{W}(\mathbf{x}, \mathbf{y})$. We will show that $\{\mathbf{W}'_k\}_k$ witness $\mathbf{W}_i$–replaceability of $\mathbf{x}, \mathbf{y}$, which is a contradiction.

First, we argue that $\{\mathbf{W}'_k\}$ has at most countably many elements. Indeed, fix $\mathbf{W}'_k = Col(\hat{h})$ and let $\bar{\mathbf{x}}, \bar{\mathbf{x}}$ be the gates of $\mathbf{x}, \mathbf{y}$ on $I_{\hat{h}}$.

Since median intervals in $\mathrm{Cone}^{\omega}(\mathcal{X})$ are ultralimits of coarse median intervals in $\mathcal{X}$, we can choose representatives $(x_n), (y_n), (\bar{x}_n), (\bar{y}_n)$ of $\mathbf{x}, \mathbf{y}, \bar{\mathbf{x}}, \bar{\mathbf{y}}$ with $\bar{x}_n, \bar{y}_n$ in the coarse median interval from x_n to y_n for ω–a.e. n. Considering the cubical approximation (Proposition 16.1) of this coarse median interval shows that $\mathrm{Rel}((\bar{x}_n), (\bar{y}_n)) \subset \mathrm{Rel}((x_n), (y_n))$. On the other hand, letting (W'_n) be a legal sequence representing $\mathbf{W}'_k$, we have that $(W'_n) \in \max \mathrm{Rel}((\bar{x}_n), (\bar{y}_n))$ (by taking gates on $F_{W'_n}$ and using Lemma 29.2).

Hence $\mathbf{W}'_k$ is represented by a legal sequence $(W'_n) \in \mathrm{Rel}((x_n), (y_n))$. Now, for any sufficiently large M (in terms of the HHS structure on $\mathcal{X}$) and each n, the set $\mathrm{Rel}_M(x_n, y_n)$ is finite by Lemma 11.4, so there are only countably many possibilities for (W'_n) and hence for $\mathbf{W}'_k$.

Let $\{\mathbf{W}''_\ell\}$ be the set of $\mathbf{W}'_k$ for which $\mathrm{fio}(\mathcal{H}_{\mathbf{W}'_k} \cap \mathcal{H}(\mathbf{x}, \mathbf{y}) \cap \mathcal{H}_{\mathcal{A}}) > 0$. Since $\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y}) \subset \bigcup_k \mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}'_k}$ and the union has only countably many terms, we have, up to a measure–0 set, that

$$\mathcal{H}_{\mathcal{A}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y}) \subset \bigcup_{\ell} \mathcal{H}_{\mathcal{A}} \cap \mathcal{H}_{\mathbf{W}''_\ell},$$

and thus $\mathbf{x}, \mathbf{y}$ is a $\mathbf{W}_i$–replaceable pair, giving our contradiction.

Verification of Definition 3.1.(IV): This follows verbatim from the proof of the corresponding part of the proof of Lemma 28.8, once we observe that, by the previous part of the present proof, the nest–minimal sets $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_j\}$ from Lemma 28.8 lie in $\mathfrak{F}^{\infty}_{+}$. □

Now we make a real cubing and analyse the relations on its index set.

By Theorem 5.1, $\mathrm{Cone}^{\omega}(\mathcal{X})$, with the metric $\mathbf{D}$ and the median $\boldsymbol{\mu}$, is median-preservingly isometric to an ℝ–cubing $(\mathrm{Cone}^{\omega}(\mathcal{X}), \mathfrak{F}^{\infty}_{+})$. By construction, the nesting relation in the real cubing structure is the relation $\sqsubseteq$ on $\mathfrak{F}^{\infty}_{+}$ from Section 27.

We now check that the same holds for orthogonality (and hence transversality):

Lemma 29.6. *Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\infty}_{+}$. Then $\mathbf{U} \perp \mathbf{V}$, in the sense of Section 27, if and only if $\mathbf{U} \perp \mathbf{V}$ in the ℝ–cubing $(\mathrm{Cone}^{\omega}(\mathcal{X}), \mathfrak{F}^{\infty}_{+})$.*

Proof. This lemma is the reason why we restricted the codomain. It follows immediately from Lemma 29.4. Indeed, suppose that $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\infty}_{+}$ are orthogonal in the real cubing structure from Theorem 5.1. Then there exist closed convex subspaces A, B, respectively parallel to $\mathbf{F}_{\mathbf{U}}, \mathbf{F}_{\mathbf{V}}$, such that the convex hull of $A \cup B$ is $A \times B$. The lemma now implies that $\mathbf{U} \perp \mathbf{V}$ in the sense of Section 27 (i.e. in the ultralimit sense). The converse follows from the same lemma. □

Thus the relations $\sqsubseteq, \perp, \pitchfork$ on $\mathfrak{F}^{\infty}$ from Section 27 extend the relations on $\mathfrak{F}^{\infty}_{+}$ from the ℝ–cubing structure.

For each $\mathbf{U} \in \mathfrak{F}^{\infty} - \mathfrak{F}^{\infty}_{+}$, we can associate a real tree $\mathcal{T}^{\bullet}\mathbf{U}$ consisting of a single point. Define $\pi_{\mathbf{U}}$ and $\rho^{\bullet}_{\mathbf{U}}$ in the only possible way for such $\mathbf{U}$.

For $\mathbf{U} \in \mathfrak{F}^{\infty} - \mathfrak{F}^{\infty}_{+}$ and $\mathbf{V} \in \mathfrak{F}^{\infty}_{+}$ with $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \subsetneq \mathbf{V}$, we let $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{F}_{\mathbf{U}})$. The fact that $\pi_{\mathbf{V}}(\mathbf{F}_{\mathbf{U}})$ is a single point follows from Definition 3.1.(III), applied to the colouring

$Col : \mathcal{W} \to \mathfrak{F}^\infty$, and the fact that $\mathbf{U}$ is not in the image of the colouring. Indeed, this shows that for any $\mathbf{x}, \mathbf{y} \in \mathfrak{h}_{\mathbf{V}}(\mathbf{F}_{\mathbf{U}})$, there is a set of elements $\mathbf{W}_i \subsetneq \mathbf{U}, \mathbf{V}$ such that each $\mathcal{H}_{\mathbf{W}_i}$ contributes positive measure to $\mathcal{H}(\mathbf{x}, \mathbf{y})$ and the union of the $\mathcal{H}_{\mathbf{W}_i}$ covers $\mathcal{H}(\mathbf{x}, \mathbf{y})$ (up to a null set). So the construction of $\pi_{\mathbf{V}}$ in the proof of Theorem 5.1 implies that $\pi_{\mathbf{V}}(\mathbf{x}) = \pi_{\mathbf{V}}(\mathbf{y})$, as required.

For $\mathbf{U}, \mathbf{V}$ as above with $\mathbf{V} \subsetneq \mathbf{U}$, we have that $\mathcal{T}^\bullet\mathbf{U}$ is a single point, and the map $\rho^{\mathbf{U}}_{\mathbf{V}}$ can be defined arbitrarily.

Adding the "trivial" elements of $\mathfrak{F}^\infty - \mathfrak{F}^\infty_+$ did not affect finite complexity, since the complexity bounds in $\mathfrak{F}^\infty$ came from those in $\mathfrak{F}$.

Any consistent tuple in $\ell_1(\mathfrak{F}^\infty)$ restricts to a consistent tuple in $\ell_1(\mathfrak{F}^\infty_+)$. Conversely, since $\mathcal{T}^\bullet\mathbf{U}$ is a single point when $\mathbf{U} \notin \mathfrak{F}^\infty_+$, any consistent tuple in $\ell_1(\mathfrak{F}^\infty_+)$ extends uniquely to a consistent tuple in $\ell_1(\mathfrak{F}^\infty)$. The fact that $\mathcal{T}^\bullet\mathbf{U}$ is a point for $\mathbf{U} \notin \mathfrak{F}^\infty_+$ also means that the bounded geodesic image property for consistent tuples in $\ell_1(\mathfrak{F}^\infty_+)$ passes to consistent tuples in $\ell_1(\mathfrak{F}^\infty)$.

We therefore have the following more specific statement of Theorem 26.3:

Corollary 29.7. *Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space. Then any asymptotic cone* $\mathrm{Cone}^\omega(\mathcal{X})$ *admits a $\mathbb{R}$–cubing structure* $(\mathrm{Cone}^\omega\mathcal{X}, \mathfrak{F}^\infty)$*, where*

- *the relations $\sqsubseteq, \perp, \pitchfork$ are as in Section 27;*
- *the $\mathbb{R}$–trees and projections are as provided by Theorem 5.1, or they are single points and trivial projections.*

Moreover, this real cubing has nonempty products.

Furthermore, if $\mathfrak{F}$ has wedges and clean containers, so does $\mathfrak{F}^\infty$, by Lemma 27.3.

Proof. The only thing remaining to be verified is the nonempty products property. For elements $\mathbf{U} \in \mathfrak{F}^\bullet_+$, this holds because of Theorem 5.1. Let $\mathbf{U} \in \mathfrak{F}^\bullet$. Let $\mathbf{F}_{\mathbf{U}}$ be the subspace defined above in terms of ultralimits. Then $\mathbf{F}_{\mathbf{U}}$ is nonempty (since $\mathbf{U}$ is represented by a legal sequence), so fix $\mathbf{x} \in \mathbf{F}_{\mathbf{U}}$. Let $\mathbf{V} \in \mathfrak{F}^\bullet$ and suppose that $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$. If $\mathbf{V} \in \mathfrak{F}^\bullet_+$, then by definition $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{x})$. Otherwise, $\mathcal{T}^\bullet\mathbf{V}$ is a single point, and $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{x})$ automatically. Hence, by Definition 4.9, $(\mathbf{X}, \mathfrak{F}^\bullet)$ has nonempty products. □

Geometrically, the above $\mathbb{R}$–cubing structure is identical to the one from Theorem 5.1, except formally we have taken its product with a point. This was just because it will be more convenient later to work with all of $\mathfrak{F}^\infty$ than to discard the elements of $\mathfrak{F}^\infty - \mathfrak{F}^\infty_+$ — which depend a priori on the rescaling — everywhere.

Remark 29.8 ($\pi_{\mathbf{U}}$ surjectivity on the ultralimits $\mathbf{P}_{\mathbf{U}}$ and $\mathbf{F}_{\mathbf{U}}$)**.** Let $\mathbf{U} \in \mathfrak{F}^\infty_+$. Represent $\mathbf{U}$ by a legal sequence (U_n) and let $\mathbf{P}_{\mathbf{U}} = \lim_\omega P_{U_n}$, so that $\mathbf{P}_{\mathbf{U}} = \mathbf{F}_{\mathbf{U}} \times \mathbf{E}_{\mathbf{U}}$, where $\mathbf{F}_{\mathbf{U}} = \lim_\omega F_{U_n}$. Recall that $\mathcal{W}(\mathbf{F}_{\mathbf{U}}) = \mathcal{W}_{\mathbf{U}}$, essentially by the definition of Col. Hence, as we saw before, $\mathbf{F}_{\mathbf{U}}$ (the ultralimit) is a representative of the parallelism class of closed convex subspaces associated to the filter $\sigma_{\mathbf{U}}$ from the proof of Theorem 5.1 (these spaces were also called $\mathbf{F}_{\mathbf{U}}$ in that proof, and there is no ambiguity because any $\mathbf{F}_{\mathbf{U}}$ in the sense of the proof of Theorem 5.1 is parallel to a subspace $\mathbf{F}_{\mathbf{U}} \subset \mathbf{P}_{\mathbf{U}}$ arising as above as an ultralimit). By construction, $\pi_{\mathbf{U}} : \mathrm{Cone}^\omega(\mathcal{X}) \to \mathcal{T}^\bullet\mathbf{U}$ is surjective and factors through the gate map to (anything parallel to) $\mathbf{F}_{\mathbf{U}}$. Hence the restriction of $\pi_{\mathbf{U}}$ to $\mathbf{F}_{\mathbf{U}}$ and to $\mathbf{P}_{\mathbf{U}}$ is surjective. For $\mathbf{U} \in \mathfrak{F}^\infty - \mathfrak{F}^\infty_+$, the same conclusion holds just because $\mathcal{T}^\bullet\mathbf{U}$ is a point. □

30. Homogeneity of the real cubing in the HHG case

Let $(G, \mathfrak{F})$ be a hierarchically hyperbolic group. Let $\mathrm{Cone}^\omega(G)$ be an asymptotic cone with scaling sequence $(j_n)_n$, ultrafilter ω, and, without loss of generality, observation point $(1)_n = \mathbf{1}$.

Let $(\mathrm{Cone}^{\omega}(G), \mathfrak{F}^{\infty})$ be the ℝ–cubing provided by Theorem 26.3, so that the initial metric $\mathsf{d}_{\mathrm{Cone}^{\omega}(G)}$ and the ℝ–cubing metric $\mathbf{D}$ on $\mathrm{Cone}^{\omega}(G)$ are bilipschitz-equivalent.

Recall that $G^{*} = \lim_{\omega} G$, the group of sequences $(g_n)_n$ in G up to the ultrafilter ω, with multiplication and inversion defined pointwise. Note that G^{*} is independent of the rescaling, but depends on ω. We also recall the admissible subgroup $G^{*}_{ad} \leqslant G^{*}$.

Lemma 30.1. *The action of G^{*}_{ad} on $(\mathrm{Cone}^{\omega}(\mathcal{X}), \mathbf{D}, \boldsymbol{\mu})$ is an action by median-preserving isometries.*

Proof. Let $\mathbf{g}, \mathbf{h}, \mathbf{k} \in \mathrm{Cone}^{\omega}(G)$ and let $a \in G^{*}_{ad}$. Let $(g_n)_n, (h_n)_n, (k_n)_n$ be admissible sequences representing $\mathbf{g}, \mathbf{h}, \mathbf{k}$ respectively, and let $(a_n)_n$ be a sequence representing a. Let $\mathbf{U} \in \mathfrak{F}^{\infty}$ be represented by the legal sequence $(U_n)_n$.

Fix n. Then by Definition 10.11, the coarse median of

$$\pi_{a_n U_n}(a_n g_n), \pi_{a_n U_n}(a_n h_n), \pi_{a_n U_n}(a_n k_n)$$

is uniformly close to the image under $a_n : \mathcal{C}U_n \to \mathcal{C}U_n$ of the coarse median of $\pi_{U_n}(g_n), \pi_{U_n}(h_n), \pi_{U_n}(k_n)$, so by the uniqueness axiom and the definition of the coarse median on G, left-multiplication by a_n coarsely preserves the coarse median. Hence $\boldsymbol{\mu}(a\mathbf{g}, a\mathbf{h}, a\mathbf{k}) = a\boldsymbol{\mu}(\mathbf{g}, \mathbf{h}, \mathbf{k})$, so a is a median homomorphism.

Since a preserves the median and is an isometry of the original metric on the asymptotic cone, it is also a $\mathbf{D}$–isometry by Proposition 23.3. □

Recall that we have an action of G on $\mathfrak{F}$ preserving $\sqsubseteq, \perp, \pitchfork$. Moreover, if $U \in \mathfrak{F}$ and $g \in G$, then $gP_U = P_{gU}$, by Remark 15.11. There is a global constant C such that for all $x, g \in G$ and $U \in \mathfrak{F}$, we have, by e.g. Lemma 4.15 of [RST18], $\mathsf{d}_G(g\mathfrak{g}_{P_U}(x), \mathfrak{g}_{P_{gU}}(gx)) \leqslant C$. So, if $(U_n)_n$ is legal in $(\mathrm{Cone}^{\omega}(G), \mathbf{1})$, then $(a_n U_n)_n$ is legal in $(\mathrm{Cone}^{\omega}(G), (a_n))_n$. So $a \in G^{*}_{ad}$ provides a bijection $a : \mathfrak{F}^{\infty} \to \mathfrak{F}^{\infty}$ preserving $\sqsubseteq, \perp, \pitchfork$.

Moreover, for all $\mathbf{U} \in \mathfrak{F}^{\infty}$, and $\mathbf{x} \in \mathrm{Cone}^{\omega}(G)$, and $a \in G^{*}_{ad}$, we have $a\mathbf{P}_{\mathbf{U}} = \mathbf{P}_{a\mathbf{U}}$ and $a\mathfrak{g}_{\mathbf{U}}(\mathbf{x}) = \mathfrak{g}_{a\mathbf{U}}(a\mathbf{x})$. The same is true with $\mathbf{P}_{\mathbf{U}}$ replaced by any parallel copy $\mathbf{F}_{\mathbf{U}} \subset \mathbf{P}_{\mathbf{U}}$, in the sense that $a\mathbf{F}_{\mathbf{U}} \subset \mathbf{P}_{a\mathbf{U}}$ is a parallel copy $\mathbf{F}_{a\mathbf{U}}$ and $a\mathfrak{h}_{\mathbf{U}}(\mathbf{x}) = \mathfrak{h}_{a\mathbf{U}}(a\mathbf{x})$, where the latter gate is taken in the parallel copy $a\mathbf{F}_{\mathbf{U}}$.

Hence, since median isometries preserve the set $\mathcal{W}$ of walls, and $\hat{w}$ crosses $\mathbf{F}_{\mathbf{U}}$ only if $a\hat{w}$ crosses $a\mathbf{F}_{\mathbf{U}}$, which is parallel to $\mathbf{F}_{a\mathbf{U}}$, we have:

Lemma 30.2. *The poset-colouring $Col : \mathcal{W} \to \mathfrak{F}^{\infty}$ is G^{*}_{ad}–equivariant.*

So, by Proposition 6.16, the action of G^{*}_{ad} on $\mathrm{Cone}^{\omega}(\mathcal{X})$ is an action by ℝ–cubing automorphisms.

Specifically, for each $a \in G^{*}_{ad}$, the ℝ–cubing automorphism data is:

- the isometry $a : \mathrm{Cone}^{\omega}(\mathcal{X}) \to \mathrm{Cone}^{\omega}(\mathcal{X})$;
- the bijection $a : \mathfrak{F}^{\infty} \to \mathfrak{F}^{\infty}$ discussed above, which preserves the three relations;
- for each $\mathbf{U} \in \mathfrak{F}^{\infty}$, an isometry $a_{\mathbf{U}} : \mathcal{T}^{\bullet}\mathbf{U} \to \mathcal{T}^{\bullet}a\mathbf{U}$ which is either the unique isometry between one-point spaces (when $\mathbf{U} \in \mathfrak{F}^{\infty} - \mathfrak{F}^{\infty}_{+}$), or given by $a_{\mathbf{U}}(\pi_{\mathbf{U}}(\mathbf{x})) = \pi_{a\mathbf{U}}(a\mathbf{x})$ when $\mathbf{U} \in \mathfrak{F}^{\infty}_{+}$.

By construction, $(ab)_{\mathbf{U}} = a_{b\mathbf{U}} \circ b_{\mathbf{U}}$ and $a(\rho^{\mathbf{U}}_{\mathbf{V}}) = \rho^{a\mathbf{U}}_{a\mathbf{V}}$ whenever $\mathbf{U} \subsetneq \mathbf{V}$ or $\mathbf{U} \pitchfork \mathbf{V}$.

Remark 30.3. In summary, if $a \in G^{*}_{ad}$, then the triple

$$(a : \mathrm{Cone}^{\omega}(G) \to \mathrm{Cone}^{\omega}(G), a : \mathfrak{F}^{\infty} \to \mathfrak{F}^{\infty}, \{a_{\mathbf{U}} : \mathbf{U} \in \mathfrak{F}^{\infty}\})$$

satisfies all the conditions from the definition of a 1–morphism (Definition 4.30). □

31. Local structure of $\mathrm{Cone}^\omega(G)$

We now discuss the local real cubing structure of $\mathrm{Cone}^\omega(G)$.

Let $(G,\mathfrak{F})$ be an HHG. Fix $\mathbf{x}\in\mathrm{Cone}^\omega(G)$. Recall the definition of the *local* $\mathbb{R}$*–cubing* $(\mathrm{Cone}^\omega(G)_{\mathbf{x}},\mathfrak{F}^\infty_{\mathbf{x}})$ from Section 4.9:

(1) $\mathfrak{F}^\infty_{\mathbf{x}}$ is the set of $\mathbf{U}\in\mathfrak{F}^\infty$ such that for all $\mathbf{V}\in\mathfrak{F}^\infty$ with $\mathbf{V}\subsetneq\mathbf{U}$ or $\mathbf{V}\pitchfork\mathbf{U}$, we have $\rho^{\mathbf{U}}_{\mathbf{V}}=\pi_{\mathbf{V}}(\mathbf{x})$.

(2) $\mathrm{Cone}^\omega(G)_{\mathbf{x}}$ is the set of $\mathbf{y}\in\mathrm{Cone}^\omega(G)$ such that $\pi_{\mathbf{U}}(\mathbf{x})\neq\pi_{\mathbf{U}}(\mathbf{y})$ only if $\mathbf{U}\in\mathfrak{F}^\infty_{\mathbf{x}}$.

Choose $a\in G^*$ such that $a\mathbf{1}=\mathbf{x}$. Recall that a induces a 1–morphism of $\mathbb{R}$–cubings from $(\mathrm{Cone}^\omega(G),\mathfrak{F}^\infty)$ to $(\mathrm{Cone}^\omega(G),\mathfrak{F}^\infty)$, denoted $(a,I_a,\{a_{\mathbf{U}}\})$, where $a:\mathrm{Cone}^\omega(G)\to\mathrm{Cone}^\omega(G)$ is an isometry, $I_a:\mathfrak{F}^\infty\to\mathfrak{F}^\infty$ is a bijection preserving $\sqsubseteq,\perp,\pitchfork$, and each $a_{\mathbf{U}}:\mathcal{T}^\bullet\mathbf{U}\to\mathcal{T}^\bullet I_a(\mathbf{U})$ is an isometry. Also, $(a^{-1},I_a^{-1},\{a_{\mathbf{U}}^{-1}\})$ is an inverse for the above 1–morphism, and is again a 1–morphism.

By Remark 4.36, the bijection I_a restricts to a bijection $I:\mathfrak{F}^\infty_{\mathbf{1}}\to\mathfrak{F}^\infty_{\mathbf{x}}$ preserving the nesting, orthogonality, and transversality relations.

31.1. **Refining the local structure.** Let $(G,\mathfrak{F})$ be an HHG with $\mathfrak{F}$ having the wedge property and clean containers. Under the following extra assumption, we will refine the local real cubing structure, by modifying the underlying real trees so that they are *universal*. This will be useful in subsequent parts of the paper.

Remark 31.1 (HHG relative to subgroups)**.** We assume in this section that for each $U\in\mathfrak{F}$, the subgroup $\mathrm{Stab}_G(U)$ acts on P_U coboundedly. We will revisit this definition in more detail in Section 35. □

We freely use:

Lemma 31.2. *For each U, the group $\mathrm{Stab}_G(U)$ acts uniformly coboundedly on $\mathcal{C}U$.*

Proof. As explained in [BHS19, Section 1], we can and shall assume that $\pi_U:G\to\mathcal{C}U$ is E–coarsely surjective. So, by, for instance, considering gates, $\pi_U:P_U\to\mathcal{C}U$ is uniformly coarsely surjective. Since P_U uniformly coarse coincides with the $\mathrm{Stab}_G(U)$–orbit of $\mathfrak{g}_{P_U}(1)$, the action of the latter group on $\mathcal{C}U$ is cobounded. □

Recall from the previous section that we have shown that each local $\mathbb{R}$–cubing $(\mathrm{Cone}^\omega(G)_{\mathbf{x}},\mathfrak{F}^\infty_{\mathbf{x}})$ is isomorphic to the local $\mathbb{R}$–cubing $(\mathrm{Cone}^\omega(G)_{\mathbf{1}},\mathfrak{F}^\infty_{\mathbf{1}})$.

For later purposes, we now show that there is a more convenient $\mathbb{R}$–cubing structure on $\mathrm{Cone}^\omega(G)_{\mathbf{1}}$. Specifically, for each $\mathbf{U}\in\mathfrak{F}^\infty$, represented by a legal sequence $(g_nU),g_n\in G,U\in\mathfrak{F}$, it will be convenient to know that the $\mathbb{R}$–tree $\mathcal{T}^\bullet\mathbf{U}$ has isometry type independent of everything except the cardinality of the Gromov boundary of the hyperbolic space $\mathcal{C}U$. One way to do this is to show that $\mathcal{T}^\bullet\mathbf{U}$ is a *universal* $\mathbb{R}$*–tree* of the appropriate valence. However, as constructed, it is not true in general that $\mathcal{T}^\bullet\mathbf{U}$ is complete.

This is because of the following phenomenon: consider a (D,D)–hierarchy ray γ in G that starts at 1, projects close to an unbounded geodesic ray in $\mathcal{C}S$ (recall that $S\in\mathfrak{F}$ is the $\sqsubseteq$–maximal element), and spends arbitrarily large amounts of time in standard product regions $P_U,U\subsetneq S$. Let $\boldsymbol{\gamma}$ be the geodesic in $\mathrm{Cone}^\omega(G)$ arising as the ultralimit of γ. It is possible to choose γ so that $\boldsymbol{\gamma}$ projects to a copy of $[0,1)$ in $\mathcal{T}^\bullet\mathbf{S}$ that cannot be extended to a geodesic $[0,1]$.

On the other hand, such a $\boldsymbol{\gamma}$ cannot lie in $\mathrm{Cone}^\omega(G)_{\mathbf{1}}$, because it necessarily has nontrivial intersection with product regions $\mathbf{P_U}$ that do not contain $\mathbf{1}$. In other words, $\pi_{\mathbf{S}}$ is not surjective on the local real cubing. So, we should be using a better $\mathbb{R}$–cubing structure on $\mathrm{Cone}^\omega(G)_{\mathbf{1}}$, in which each $\mathcal{T}^\bullet\mathbf{U}$ is replaced by the $\mathbb{R}$–tree $\mathrm{im}\,\pi_{\mathbf{U}}|_{\mathrm{Cone}^\omega(G)_{\mathbf{1}}}$.

This motivates the following construction.

Definition 31.3 (Smaller $\mathbb{R}$–trees for the local structure)**.** Let $\mathbf{x} \in \mathrm{Cone}^\omega(G)$. Let $\mathfrak{F}^\infty_{\mathbf{x}}$ be defined as before, i.e. it is the set of $\mathbf{U} \in \mathfrak{F}^\infty$ such that $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{x})$ whenever $\mathbf{V} \in \mathfrak{F}^\infty$ satisfies $\mathbf{U} \subsetneq \mathbf{V}$ or $\mathbf{V} \pitchfork \mathbf{U}$.

Again, let $\mathrm{Cone}^\omega(G)_{\mathbf{x}}$ be the set of $\mathbf{y} \in \mathrm{Cone}^\omega(G)$ such that $\pi_{\mathbf{U}}(\mathbf{x}) \neq \pi_{\mathbf{U}}(\mathbf{y})$ implies $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{x}}$. For each $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{x}}$, let $\mathcal{L}\mathbf{U} = \pi_{\mathbf{U}}(\mathrm{Cone}^\omega(G)_{\mathbf{x}})$.

Let $\pi_{\mathbf{U}} : \mathrm{Cone}^\omega(G)_{\mathbf{x}} \to \mathcal{L}\mathbf{U}$ be the restriction of $\pi_{\mathbf{U}}$ to $\mathrm{Cone}^\omega(G)_{\mathbf{x}}$.

Given $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty_{\mathbf{x}}$ with $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \subsetneq \mathbf{V}$, let $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{x})$, which coincides with the original definition. □

Lemma 31.4. *$\mathrm{Cone}^\omega(G)_{\mathbf{x}}$ is closed and convex in $\mathrm{Cone}^\omega(G)$ in the sense of Definition 4.17. In particular, $\mathcal{L}\mathbf{U}$ is a closed subtree of $\mathcal{T}^\bullet\mathbf{U}$ for all $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{x}}$, and whenever $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty_{\mathbf{x}}$ satisfy $\mathbf{U} \subsetneq \mathbf{V}$, the map $\rho^{\mathbf{V}}_{\mathbf{U}} : \mathcal{T}^\bullet\mathbf{V} \to \mathcal{T}^\bullet\mathbf{U}$ restricts to a map $\mathcal{L}\mathbf{V} \to \mathcal{L}\mathbf{U}$, after possibly redefining it in an arbitrary way on $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$.*

Proof. We first check convexity. Let $\mathbf{y}, \mathbf{z} \in \mathrm{Cone}^\omega(G)_{\mathbf{x}}$. Let $\mathbf{a} \in \mathrm{Cone}^\omega(G)$ be arbitrary. Let $\mathbf{n} = \boldsymbol{\mu}(\mathbf{y}, \mathbf{z}, \mathbf{a})$.

If $\mathbf{V} \in \mathfrak{F}^\infty - \mathfrak{F}^\infty_{\mathbf{x}}$, then $\pi_{\mathbf{V}}(\mathbf{y}) = \pi_{\mathbf{V}}(\mathbf{z}) = \pi_{\mathbf{V}}(\mathbf{x})$. Since $\pi_{\mathbf{V}}$ takes geodesics to geodesics, and $\mathbf{n}$ lies on a geodesic from $\mathbf{y}$ to $\mathbf{z}$, we have $\pi_{\mathbf{V}}(\mathbf{n}) = \pi_{\mathbf{V}}(\mathbf{x})$, so $\mathbf{n} \in \mathrm{Cone}^\omega(G)_{\mathbf{x}}$. This proves convexity, and hence connectedness of $\mathcal{L}\mathbf{U}$, since $\pi_{\mathbf{U}}$ is a median homomorphism.

Restricting $\rho^{\mathbf{V}}_{\mathbf{U}}$: Suppose that $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty_{\mathbf{x}}$ satisfy $\mathbf{U} \subsetneq \mathbf{V}$. We define $\rho^{\mathbf{V}}_{\mathbf{U}}(\pi_{\mathbf{V}}(\mathbf{x}))$ to be an arbitrary point in $\mathcal{L}\mathbf{U}$.

For any other $\pi_{\mathbf{V}}(\mathbf{y}) \in \mathcal{L}\mathbf{V}$ — with $\mathbf{y}$ lying, without loss of generality, in $\mathrm{Cone}^\omega(G)_{\mathbf{x}}$ — we have by definition that $\rho^{\mathbf{V}}_{\mathbf{U}}(\pi_{\mathbf{V}}(\mathbf{y})) = \pi_{\mathbf{U}}(\mathbf{y})$, which lies in $\mathcal{L}\mathbf{U}$ by our choice of $\mathbf{y}$ (which was justified by the definition of $\mathcal{L}\mathbf{V}$ and the bounded geodesic image property).

$\mathcal{L}\mathbf{U}$ is closed in $\mathcal{C}\mathbf{U}$: Lemma 31.5 below will show that $\mathcal{L}\mathbf{U}$ is complete, which immediately implies that it is closed in $\mathcal{T}^\bullet\mathbf{U}$. This completes the proof. □

Lemma 31.5. *For each $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{x}}$, the $\mathbb{R}$–tree $\mathcal{L}\mathbf{U}$ is complete.*

Proof. Suppose that $\mathcal{L}\mathbf{U}$ is not complete, and let $\overline{\mathcal{L}\mathbf{U}}$ be its completion. By [MNO92, Theorem 1.11] (see also [AB87]), $\overline{\mathcal{L}\mathbf{U}} - \mathcal{L}\mathbf{U}$ is a set of valence–1 points in $\overline{\mathcal{L}\mathbf{U}}$, which we aim to show is empty. Toward a contradiction, let $\mathbf{p} \in \overline{\mathcal{L}\mathbf{U}} - \mathcal{L}\mathbf{U}$ be a valence–1 point. Let α be a geodesic in $\overline{\mathcal{L}\mathbf{U}}$ based at $\mathbf{p}$ and representing the unique direction at $\mathbf{p}$. Then α contains a sequence in $\mathcal{L}\mathbf{U}$ converging to $\mathbf{p}$, so since $\mathcal{L}\mathbf{U}$ is connected, there is an isometric embedding $\boldsymbol{\gamma} : [0, \epsilon) \to \mathcal{L}\mathbf{U}$ such that $\boldsymbol{\gamma}(t) \to \mathbf{p}$ as $t \to \epsilon$.

For each $n \in \mathbb{N}$, choose $\mathbf{p}_n \in \mathrm{im}\,\boldsymbol{\gamma}$ such that $\mathbf{D}_{\mathbf{U}}(\mathbf{p}_n, \mathbf{p}) < 1/n$. Choose $\mathbf{x}_n \in \mathrm{Cone}^\omega(G)_{\mathbf{x}}$ such that $\pi_{\mathbf{U}}(\mathbf{x}_n) = \mathbf{p}_n$. By replacing each $\mathbf{x}_n$ with $\mathfrak{h}_{\mathbf{U}}(\mathbf{x}_n)$, we can assume that $\mathbf{x}, \mathbf{x}_n$ lie in a common parallel copy $\mathbf{F}_{\mathbf{U}}$.

This does not change the fact that $\mathbf{x}_n \in \mathrm{Cone}^\omega(G)_{\mathbf{x}}$.

Let $m, n \in \mathbb{N}$ and suppose that $\mathbf{V} \in \mathfrak{F}^\infty$ satisfies $\mathbf{D}_{\mathbf{V}}(\mathbf{x}_n, \mathbf{x}_m) > 0$. If $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$, then $\pi_{\mathbf{V}}(\mathbf{x}_n) = \pi_{\mathbf{V}}(\mathbf{x}_1)$, so $\mathbf{V} \sqsubseteq \mathbf{U}$ or $\mathbf{V} \perp \mathbf{U}$. But our choice of $\mathbf{x}_n, \mathbf{x}_m \in \mathbf{F}_{\mathbf{U}}$ rules out the latter possibility.

If $\mathbf{V} \subsetneq \mathbf{U}$, then since $\mathbf{x}_n, \mathbf{x}_m \in \mathrm{Cone}^\omega(G)_{\mathbf{x}}$, we have $\pi_{\mathbf{V}}(\mathbf{x}_n) = \pi_{\mathbf{V}}(\mathbf{x}_m) = \pi_{\mathbf{V}}(\mathbf{x})$ unless $\mathbf{V} \in \mathfrak{F}^\infty_{\mathbf{x}}$. All such $\mathbf{V}$ have the property that $\pi_{\mathbf{U}}(\mathbf{x}) = \rho^{\mathbf{V}}_{\mathbf{U}}$.

Now, by bounded geodesic image, $\rho^{\mathbf{V}}_{\mathbf{U}}$ must lie on $\boldsymbol{\gamma}$ between $\mathbf{p}_n, \mathbf{p}_m$. For sufficiently large m, n, this means that there are no such $\mathbf{V}$. Hence $\mathbf{D}(\mathbf{x}_n, \mathbf{x}_m) = \mathbf{D}_{\mathbf{U}}(\mathbf{x}_n, \mathbf{x}_m) \to 0$ as $m, n \to \infty$. Since $\mathrm{Cone}^\omega(G)$ is complete, there exists $\mathbf{y} \in \mathrm{Cone}^\omega(G)$ such that $\mathbf{x}_n \to \mathbf{y}$ as $n \to \infty$.

By continuity of $\pi_{\mathbf{U}}$, we have $\pi_{\mathbf{U}}(\mathbf{x}_n) \to \pi_{\mathbf{U}}(\mathbf{y}) \in \mathcal{T}^\bullet\mathbf{U}$.

Now, since $\mathbf{F}_{\mathbf{U}}$ is closed, $\mathbf{y} \in \mathbf{F}_{\mathbf{U}}$. So, by bounded geodesic image, $\mathbf{y} \in \mathrm{Cone}^\omega(G)_{\mathbf{x}}$, because the same was true for each $\mathbf{x}_n$. Hence $\pi_{\mathbf{U}}(\mathbf{y}) \in \mathcal{L}\mathbf{U}$. Thus $(\mathbf{p}_n)_n$ has a limit in $\mathcal{L}\mathbf{U}$, contradicting our choice of $\mathbf{p}$. Hence $\mathcal{L}\mathbf{U}$ is complete. □

Corollary 31.6. *The pair* $(\mathrm{Cone}^\omega(G)_{\mathbf{1}}, \mathfrak{F}^\infty_{\mathbf{1}})$*, with the* $\mathbb{R}$*–trees* $\mathcal{L}\mathbf{U}, \mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{1}}$*, is a local* $\mathbb{R}$*–cubing. Moreover, choosing* $a \in G^*$ *such that* $\mathbf{x} = a\mathbf{1}$*, the* 1*–morphism* $(a, I_a, \{a_{\mathbf{U}}\})$ *restricts to an invertible* 1*–morphism* $(\mathrm{Cone}^\omega(G)_{\mathbf{1}}, \mathfrak{F}^\infty_{\mathbf{1}}) \to (\mathrm{Cone}^\omega(G)_{\mathbf{x}}, \mathfrak{F}^\infty_{\mathbf{x}})$ *whose inverse is a* 1*–morphism.*

By construction, each $\pi_{\mathbf{U}} : \mathrm{Cone}^\omega(G)_{\mathbf{x}} \to \mathcal{L}\mathbf{U}, \mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{x}}$ *is surjective, so* $\mathrm{Cone}^\omega(G)_{\mathbf{x}}$ *is exactly the set of points in* $\prod_{\mathbf{U}\in\mathfrak{F}^\infty_{\mathbf{x}}} \mathcal{L}\mathbf{U}$ *that lie at finite* ℓ_1*–distance from* $\mathbf{x}$ *and satisfy the consistency equations from Definition 4.2.(5).*

Proof. Lemma 31.4 and Lemma 4.23 prove that $(\mathrm{Cone}^\omega(G)_{\mathbf{x}}, \mathfrak{F}^\infty_{\mathbf{x}})$, with the $\mathbb{R}$–trees $\mathcal{L}\mathbf{U}, \mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{x}}$ and projections from Definition 31.3, is an $\mathbb{R}$–cubing. The last assertion is immediate since each $\pi_{\mathbf{U}} : \mathrm{Cone}^\omega(G)_{\mathbf{x}} \to \mathcal{L}\mathbf{U}$ is surjective by definition.

So, it remains to check the assertion about morphisms. We just need to check that if $a \in G^*_{ad}$ takes $\mathbf{1}$ to $\mathbf{x}$, then for each $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{1}}$, the isometry $a_{\mathbf{U}} : \mathcal{T}^\bullet U \to \mathcal{T}^\bullet I_a(\mathbf{U})$ takes $\pi_{\mathbf{U}}(\mathrm{Cone}^\omega(G))_{\mathbf{1}})$ to $\pi_{I_a(\mathbf{U})}(\mathrm{Cone}^\omega(G)_{\mathbf{x}})$. But since $\pi_{I_a(\mathbf{U})} \circ a = a_{\mathbf{U}} \circ \pi_{\mathbf{U}}$, it suffices to observe from the definitions that a takes $\mathrm{Cone}^\omega(G)_{\mathbf{1}}$ to $\mathrm{Cone}^\omega(G)_{\mathbf{x}}$. ☐

Now we come to the reason for preferring $\mathcal{L}\mathbf{U}$ over $\mathcal{T}^\bullet\mathbf{U}$.

Definition 31.7 (Universal $\mathbb{R}$–tree)**.** Let μ be a cardinal. A *universal* μ*–tree* is an $\mathbb{R}$–tree $\mathcal{T}^\bullet$ such that any $\mathbb{R}$–tree of valence bounded by μ isometrically embeds in $\mathcal{T}^\bullet$, and the valency of $\mathcal{T}^\bullet$ at every point is μ. (The *valency* of $p \in \mathcal{C}$ is the cardinality of the set of connected components of $\mathcal{C} - \{p\}$, or, equivalently, the cardinality of the space of directions in $\mathcal{C}$ at p; see e.g. Definition 11.15 in [DK18] for the definition of the space of directions). ☐

Dyubina-Polterovich explicitly constructed a universal μ–tree $\mathcal{T}(\mu)$, showed that its isometry group acts transitively, and any complete $\mathbb{R}$–tree with valence μ at every point is isometric to $\mathcal{T}(\mu)$ (Theorem 1.1 of [DP01]).

Proposition 31.8 (Universality of $\mathcal{L}\mathbf{U}$)**.** *Let* $(G, \mathfrak{F})$ *be an HHG relative to subgroups, with* $\mathfrak{F}$ *having the wedge property and clean containers. Let* $\mathrm{Cone}^\omega(G)$ *be an asymptotic cone of* G *and let* $(\mathrm{Cone}^\omega(G), \mathfrak{F}^\infty)$ *be the* $\mathbb{R}$*–cubing structure from Theorem 26.3. Let* $\mathbf{x} \in \mathrm{Cone}^\omega(G)$ *and let* $(\mathrm{Cone}^\omega(G)_{\mathbf{x}}, \mathfrak{F}^\infty_{\mathbf{x}})$ *be the local* $\mathbb{R}$*–cubing from Corollary 31.6.*

Let $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{x}}$ *and let* $(g_n)_n$ *be an admissible sequence such that* $(g_nU)_n$ *is a legal sequence representing* $\mathbf{U}$*, where* $U \in \mathfrak{F}$*. Then one of the following holds:*

- $\mathcal{C}U$ *is bounded, and* $\mathcal{L}\mathbf{U}$ *is a single point, i.e. a universal* 0*–tree.*
- $\mathcal{C}U$ *is* 2*–ended, and* $\mathcal{L}\mathbf{U}$ *is isometric to* $\mathbb{R}$*, i.e. a universal* 2*–tree.*
- $|\partial\mathcal{C}U| = 2^{\aleph_0}$*, and* $\mathcal{L}\mathbf{U}$ *is isometric to the complete homogeneous* $\mathbb{R}$*–tree with valence* $2^{\aleph_0}$ *at each point.*

Proof. By Theorem 1.1 in [DP01], it suffices to check that $\mathcal{L}\mathbf{U}$ is complete and has the valence at each point demanded by the statement. Completeness was proven in Lemma 31.5. So, we just need to compute the valencies of points in $\mathcal{L}\mathbf{U}$.

Setup: Let $(g_nU)_n$ be as in the statement. By Corollary 31.6, we can assume for simplicity that $\mathbf{x} = 1$ and $g_n = 1$ for ω–a.e. n.

Three sub-cases: Since $(G, \mathfrak{F})$ is an HHG relative to subgroups, we have that $\mathrm{Stab}_G(U)$ acts coboundedly on P_U and hence acts coboundedly on $\mathcal{C}U$, by Lemma 31.2. By Gromov's classification of actions on hyperbolic spaces [Gro87] and coboundedness, one of the following three possibilities holds:

- $\mathcal{C}U$ is bounded. There are two ways to proceed; we give both. In either case, we can assume that $\mathrm{Stab}_G(U)$ is infinite, for otherwise P_U is bounded, whence $\mathbf{P}_{\mathbf{U}}$, and hence $\mathcal{L}\mathbf{U}$, is a single point (see Remark 29.8), as required.

  First method: consider the action of $\mathrm{Stab}_G(U)$ on the HHS $(F'_U, \mathfrak{F}_U)$, where $\mathfrak{F}_U$ is the set of $V \sqsubseteq U$, and F'_U is an HHS quasi-isometric to F_U (the modification is to ensure that $\mathrm{Stab}_G(U)$ actually acts on F'_U; see Section 2.2 of [DHS20]). We would like to apply Proposition 9.2 of [DHS17] to this action. As noted in [DHS20], the properness hypothesis in [DHS17, Proposition 9.2] is only used to know that $\partial F'_U$ is compact, but this follows from the fact that F_U is proper (as a subspace of G) and the map $(F'_U, \mathfrak{F}) \to (F_U, \mathfrak{F})$, which is an isometry on each $\mathcal{C}V$, induces a homeomorphism of boundaries. Proposition 9.2 of [DHS17] now provides $V \sqsubsetneq U$ such that $\mathrm{Stab}_G(U)$ virtually stabilises V, and F_U coarsely coincides with P_V. So any wall separating points in $\mathbf{F_U}$ has colour nested in $\mathbf{V}$ or $\mathbf{V}^\perp$. Hence $\mathcal{T}^\bullet\mathbf{U}$ is a single point, i.e. a universal 0–tree.

Second method: this is almost the same, except we use results of Petyt-Spriano, which depend on elementary arguments, instead of the above result of Durham-Hagen-Sisto using stationary measures on HHS boundaries. Since $\mathrm{Stab}_G(U)$ acts properly and coboundedly on the hierarchically quasiconvex (hence quasigeodesic) space P_U, it is finitely generated. Theorem 5.1 of [PS20] provides a nonempty set of pairwise orthogonal $V_i, \ldots, V_j$, stabilised setwise by $\mathrm{Stab}_G(U)$, such that $\pi_{V_i}(\mathrm{Stab}_G(U))$ is unbounded for each i and every V such that $\pi_V(\mathrm{Stab}_G(U))$ is unbounded is nested in some V_i. Now, for all i, we cannot have $V_i \pitchfork U$ or $U \sqsubsetneq V_i$, since $\mathrm{Stab}_G(U)$–orbits in $\mathcal{C}V_i$ would then coarsely concide with the bounded set $\rho^U_{V_i}$. So, $V_i \sqsubseteq U$ or $V_i \perp U$ for all i. Hence P_U coarsely coincides with the image of the quasimedian quasi-isometric embedding $\prod_i F_{V_i} \to G$. As above, it follows that $\mathcal{T}^\bullet\mathbf{U}$, and hence $\mathcal{L}\mathbf{U}$, is a single point.

- $\mathcal{C}U$ is 2–ended, and the action of $\mathrm{Stab}_G(U)$ on $\mathcal{C}U$ is *lineal*, i.e. any orbit coarsely coincides with the orbit of $\langle h \rangle$ for some $h \in \mathrm{Stab}_G(U)$ acting loxodromically. Now, if $V \sqsubsetneq U$, then by Proposition 19.1, $\pi_V(P_U)$ has diameter bounded independently of V. Hence F_U is two-ended, and therefore $\mathbf{F_U}$ is isometric to $\mathbb{R}$, and if $\mathbf{V} \sqsubsetneq \mathbf{U}$, then $\mathbf{F_V}$ is a single point. Hence $\pi_\mathbf{U} : \mathbf{F_U} \to \mathcal{C}\mathbf{U}$ is an isometry, and its image is $\mathcal{L}\mathbf{U}$. Thus $\mathcal{L}\mathbf{U} \cong \mathbb{R}$ in this case.
- The action of $\mathrm{Stab}_G(U)$ on $\mathcal{C}U$ is of general type: every element is elliptic or loxodromic, and there exists a pair h, h' of loxodromics such that the limit set of $\langle h \rangle$ is disjoint from that of $\langle h' \rangle$. We analyse this case below.

Lower bound on valence in the general-type case: Since $\mathrm{Stab}_G(U)$ has a general-type action on $\mathcal{C}U$, there is a free subgroup $Q \leqslant \mathrm{Stab}_G(U)$ of rank 2 such that the Q–orbit of $\pi_U(1)$ is quasiconvex in $\mathcal{C}U$ and, when Q is equipped with the word-metric coming from, say, a free basis, the map $Q \to \mathcal{C}U$ given by $q \mapsto \pi_U(q)$ is a (L, L)–quasi-isometric embedding for some L.

Note that for $V \in \mathfrak{F}$ such that $U \sqsubsetneq V$ or $U \pitchfork V$, we have that $\pi_V(Q)$ uniformly coarsely concides with ρ^V_U since $Q \subset \mathrm{Stab}_G(U)$. We next produce a constant C such that, if $V \sqsubsetneq U$, then $\pi_V(Q)$ has diameter at most C. Indeed, if not then for all N there exist $x, y \in Q$ such that $\mathsf{d}_V(x, y) > N$ for some $V \sqsubsetneq U$. Bounded geodesic image (Definition 10.1.(7)) implies that there exist $x', y' \in Q$ such that $\mathsf{d}_V(x, x') \leqslant E$ and $\mathsf{d}_V(y, y') \leqslant E$, so $\mathsf{d}_V(x', y') > N - 2E$, and $\mathsf{d}_U(x', \rho^V_U), \mathsf{d}_U(y', \rho^V_U) \leqslant E$.

So, if no such C exists, then for all N there exists $V \sqsubsetneq U$ and $x, y \in Q$ such that

- $\mathsf{d}_V(x, y) > N$, and
- $\mathsf{d}_U(x, y) \leqslant 4E$.

Then $\mathsf{d}_Q(x, y) \leqslant L(4E + L)$, since $\pi_U : Q \to \mathcal{C}U$ is an (L, L)–quasi-isometric embedding. Since the inclusion $Q \to G$ is (L_1, L_1)–coarsely lipschitz, where L_1 just depends on the word metrics on Q and G, we have that $\mathsf{d}_G(x, y) \leqslant L_1\mathsf{d}_Q(x, y) \leqslant L_1L(4E + L) + L_1$, which is independent of x and y. Finally, since π_V is (E, E)–coarsely lipschitz by Definition 10.1, we get $\mathsf{d}_V(x, y) \leqslant EL_1L(4E + L) + E(L_1 + 1)$, which is a contradiction if N is sufficiently large.

Hence there exists C such that $\mathsf{diam}(\pi_V(Q)) \leqslant C$ whenever $V \subsetneq U$. (This argument is essentially due to Abbott-Behrstock-Durham [ABD21, Corollary 6.2].)

Fix a parallel copy F_U and let $\bar{Q}$ be the image of Q under the gate map to F_U. Since Q is coarsely contained in P_U, the gate map changes the coordinate of each $q \in Q$ by more than a uniformly bounded amount only in various $\mathcal{C}V$ with $V \perp U$. In particular, $\pi_U(Q)$ and $\pi_U(\bar{Q})$ coarsely coincide.

So, the map $f : Q \to G$ given by including Q into G and then taking the gate to F_U is a quasi-isometric embedding (by the distance formula). Since $Q \to G \to \mathcal{C}U$ is a quasi-isometric embedding in a hyperbolic space, it is quasimedian, and hence f is quasimedian. We now have that $\pi_V(\bar{Q})$ has uniformly bounded diameter for $V \neq U$.

Hence we have a bilipschitz median homomorphism $q : \mathrm{Cone}^\omega(Q) \to \mathrm{Cone}^\omega(G)$ such that for all $\mathbf{V} \neq \mathbf{U}$, the subspace $\mathfrak{h}_{\mathbf{V}}(\mathrm{im}\, q)$ of $\mathbf{F_V}$ is a single point, and $\mathbf{1} \in \mathrm{im}\, q$. Hence, for all $\mathbf{y} \in \mathrm{im}(q)$ and all $\mathbf{V} \neq \mathbf{U}$, no wall separating $\mathbf{1}, \mathbf{y}$ has colour $\mathbf{V}$. Hence $\pi_{\mathbf{U}} : \mathrm{im}\, q \to \mathcal{T}^\bullet\mathbf{U}$ is an isometric embedding, and a median homomorphism, and its image lies in $\mathcal{L}\mathbf{U}$. Since $\mathrm{Cone}^\omega(Q)$ is a universal $2^{\aleph_0}$–tree [DP01], $\pi_{\mathbf{U}}(\mathrm{im}\, q)$ has valence $2^{\aleph_0}$ at every point, and in particular the valence of $\pi_{\mathbf{U}}(\mathbf{1})$ in $\mathcal{L}\mathbf{U}$ is at least $2^{\aleph_0}$.

Now let $\mathbf{x} \in \mathrm{Cone}^\omega(G)_{\mathbf{1}}$. Choose $a \in G^*$ such that $a\mathbf{U} = \mathbf{U}$ and $a\mathbf{1} = \mathfrak{g}_{\mathbf{U}}(\mathbf{x})$. This is possible since $\mathrm{Stab}_G(U)$ acts on P_U coboundedly.

Then $\pi_{\mathbf{U}}(a(\mathrm{im}\, q)) = a(\pi_{\mathbf{U}}(\mathrm{im}\, q))$ is an isometrically embedded universal $2^{\aleph_0}$–tree in $\mathcal{C}\mathbf{U}$ containing $\pi_{\mathbf{U}}(\mathbf{x})$.

Moreover, if $\mathbf{y} \in a(\mathrm{im}\, q)$, and $\mathbf{V} \in \mathfrak{F}^\infty$ satisfies $\pi_{\mathbf{V}}(\mathbf{x}) \neq \pi_{\mathbf{V}}(\mathbf{y})$, then $\mathbf{V} = \mathbf{U}$. Hence, if $\mathbf{V}$ satisfies $\pi_{\mathbf{V}}(\mathbf{y}) \neq \pi_{\mathbf{V}}(\mathbf{1})$, then $\pi_{\mathbf{V}}(\mathbf{1}) \neq \pi_{\mathbf{V}}(\mathbf{x})$, so $\mathbf{V} = \mathbf{U}$. Hence $\pi_{\mathbf{U}}(\mathbf{y}) \in \mathcal{L}\mathbf{U}$. Thus $\pi_{\mathbf{U}}(a\, \mathrm{im}\, q) \subset \mathcal{L}\mathbf{U}$, which therefore has valence at least $2^{\aleph_0}$ at $\pi_{\mathbf{U}}(\mathbf{x})$.

Upper bound on valence in the general-type case: Since G is finitely generated, it is countable, so $\mathrm{Cone}^\omega(G)$ has cardinality $2^{\aleph_0}$. Since $\pi_{\mathbf{U}} : \mathrm{Cone}^\omega(G) \to \mathcal{T}^\bullet\mathbf{U}$ is surjective, $|\mathcal{T}^\bullet\mathbf{U}| \leqslant 2^{\aleph_0}$. Hence every point in $\mathcal{T}^\bullet\mathbf{U}$ has valence at most $2^{\aleph_0}$. □

Remark 31.9. In fact, the same argument shows that $\mathcal{T}^\bullet\mathbf{U}$ is a homogeneous ℝ–tree with the right valence at every point, for all $\mathbf{U} \in \mathfrak{F}^\infty$. So we prefer $\mathcal{L}\mathbf{U}$ over $\mathcal{T}^\bullet\mathbf{U}$ for exactly two reasons: first $\pi_{\mathbf{U}} : \mathrm{Cone}^\omega(G)_{\mathbf{1}} \to \mathcal{L}\mathbf{U}$ is surjective, and second, $\mathcal{L}\mathbf{U}$ is complete. □

32. Negligible sequences and the local index set

Let $(G, \mathfrak{F})$ be an HHG such that $\mathfrak{F}$ has the **wedge property and clean containers**, and, as in Remark 31.1, $\mathrm{Stab}_G(U)$ **acts coboundedly** on P_U for each $U \in \mathfrak{F}$. Fix a non-principal ultrafilter ω on ℕ and a rescaling sequence $(j_n)_n$. Let $\mathrm{Cone}^\omega(G)$ be the associated asymptotic cone of G.

Equip $\mathrm{Cone}^\omega(G)$ with the median $\boldsymbol{\mu}$, the median metric $\mathbf{D}$, and the ℝ–cubing structure $(\mathrm{Cone}^\omega(G), \mathfrak{F}^\infty)$ from Section 26. We now analyse the local ℝ–cubing $(\mathrm{Cone}^\omega(G)_{\mathbf{1}}, \mathfrak{F}^\infty_{\mathbf{1}})$ from Section 31.

Choose a finite set $\overline{\mathfrak{F}} \subset \mathfrak{F}$ with the property that $G \cdot \overline{\mathfrak{F}} = \mathfrak{F}$. Such a set exists by the definition of an HHG. In later sections, we will put additional constraints on $\overline{\mathfrak{F}}$, but this is all we need at the moment.

Note that the inclusion $\overline{\mathfrak{F}} \hookrightarrow \mathfrak{F}$ induces an injection $\overline{\mathfrak{F}} \to \mathfrak{F}^\infty$ sending each $U \in \overline{\mathfrak{F}}$ to the constant sequence whose terms are all U.

Lemma 32.1. *Letting $\mathfrak{F}^\infty$ denote the index set for the ℝ–cubing structure on $\mathrm{Cone}^\omega(G)$ constructed in Section 27, we have $\mathfrak{F}^\infty = G^*_{ad} \cdot \overline{\mathfrak{F}}$.*

Proof. The inclusion $G^*_{ad} \cdot \overline{\mathfrak{F}} \subseteq \mathfrak{F}^\infty$ is clear. Let $\mathbf{U} \in \mathfrak{F}^\infty$, and choose a legal sequence (U_n) representing $\mathbf{U}$. For each n, choose $\bar{U}_n \in \overline{\mathfrak{F}}$ and $g_n \in G$ such that $g_n\bar{U}_n = U_n$. Since $\overline{\mathfrak{F}}$ is finite, there exists $U \in \overline{\mathfrak{F}}$ such that $\bar{U}_n = U$ for ω–a.e. n. We need to choose an admissible sequence (h_n) such that $h_n U = g_n U$ for ω–a.e. n.

Choose $h \in P_U$. Then $g_n h \in g_n P_U = P_{g_n U}$. Let $z_n = \mathfrak{g}_{g_n U}(1)$, so that by legality, (z_n) is an admissible sequence.

By coboundedness of the $\mathrm{Stab}_G(U)^{g_n}$–action on $P_{g_n U}$, we have $a_n \in \mathrm{Stab}_G(U)^{g_n}$ such that $\mathsf{d}_G(a_n g_n h, z_n)$ is bounded independently of n, say by B. By construction, $a_n(g_n U) = g_n U$.

On the other hand,

$$\mathsf{d}_G(1, a_n g_n) \leqslant \mathsf{d}_G(a_n g_n h, a_n g_n) + \mathsf{d}_G(a_n g_n h, z_n) + \mathsf{d}_G(z_n, 1) \leqslant |h| + B + \mathsf{d}_G(1, z_n),$$

which is ω–a.e. bounded, after rescaling by j_n, by legality. Hence $(a_n g_n)$ is admissible and $a_n g_n U = U_n$ for ω–a.e. n. Thus $\mathbf{U} \in G^*_{ad} \cdot \overline{\mathfrak{F}}$. □

Let $G^*_{neg} \leqslant G^*_{ad}$ be the negligible subgroup.

Proposition 32.2. *Let $\mathbf{U} = (g_n)_n \bar{U} \in \mathfrak{F}^\infty$. Then $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{1}}$ if and only if there exists $(h_n)_n \in G^*_{neg}$ such that $h_n \bar{U} = g_n \bar{U}$ for ω–a.e. n.*

First we need a lemma. Recall from Section 26 that if $(U_n)_n$ is a legal sequence representing $\mathbf{U} \in \mathfrak{F}^\infty$, then $\mathbf{P_U}$ is by definition $\lim_\omega P_{U_n} \subset \mathrm{Cone}^\omega(G)$. Moreover, with respect to μ and $\mathbf{D}$, we have $\mathbf{P_U} \cong \mathbf{F_U} \times \mathbf{F_{U^\perp}}$, where $\mathbf{F_U}$ and $\mathbf{F_{U^\perp}}$ are ultralimits of $(F_{U_n})_n$ and $(F_{U_n^\perp})_n$ respectively. There is also an intrinsic notion of a product region in an $\mathbb{R}$–cubing, from Section 4.9, and the next lemma will show that these two notions coincide in $\mathrm{Cone}^\omega(G)$.

Lemma 32.3. *Let $\mathbf{U} \in \mathfrak{F}^\infty$ and let $\mathbf{x} \in \mathrm{Cone}^\omega(G)$. Then $\mathbf{x} \in \mathbf{P_U}$ if and only if $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ whenever $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \subsetneq \mathbf{V}$.*

Proof. Suppose that $\mathbf{x} \in \mathbf{P_U}$. Note that $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{P_U}) = \pi_{\mathbf{V}}(\mathbf{x})$ whenever $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \subsetneq \mathbf{V}$. This is because of how $\rho^{\mathbf{U}}_{\mathbf{V}}$ was defined: if $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty_+$, then $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{F_U})$ for any parallel copy $\mathbf{F_U} \subset \mathbf{P_U}$ (and these points all coincide), so $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{P_U})$ since $\mathbf{P_U}$ is the union of such parallel copies.

If $\mathbf{V} \in \mathfrak{F}^\infty - \mathfrak{F}^\infty_+$, then $\mathcal{T}^\bullet\mathbf{V}$ is a point and $\pi_{\mathbf{V}}(\mathbf{P_U}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ irrespective of whether $\mathbf{U} \in \mathfrak{F}^\infty_+$. If $\mathbf{U} \in \mathfrak{F}^\infty - \mathfrak{F}^\infty_+$ and $\mathbf{V} \in \mathfrak{F}^\infty_+$, then we have $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{F_U})$, again by definition, as required.

This shows that $\mathbf{P_U} = \lim_\omega P_{U_n}$ is contained in the standard product region of $\mathbf{U}$ (in the real cubing sense of Section 4.10).

Conversely, suppose that $\mathbf{x} \in \mathrm{Cone}^\omega(G)$ satisfies $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}}$ whenever $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \subsetneq \mathbf{V}$.

Suppose that $\mathbf{x} \notin \mathbf{P_U}$. Let $\mathbf{y} = \mathfrak{g}_{\mathbf{P_U}}(\mathbf{x})$, so that our assumption ensures that $\mathbf{D}(\mathbf{x}, \mathbf{y}) > 0$. Hence there exists $\mathbf{V} \in \mathfrak{F}^\infty$ such that $\mathbf{D_V}(\mathbf{x}, \mathbf{y}) > 0$. Now, if $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \subsetneq \mathbf{V}$, then our other assumption ensures that $\mathbf{D_V}(\mathbf{x}, \mathbf{y}) = 0$.

Now, any wall $\hat{w}$ in $\mathrm{Cone}^\omega(G)$ separating $\mathbf{x}$ from $\mathbf{y} = \mathfrak{g}_{\mathbf{P_U}}(\mathbf{x})$ separates $\mathbf{x}$ from $\mathbf{P_U}(\mathbf{x})$, and consequently $Col(\hat{w}) \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq Col(\hat{w})$. Indeed, if $\mathbf{V} \sqsubseteq \mathbf{U}$, then any wall coloured $\mathbf{V}$ intersects $\mathbf{P_U}$ because $\mathbf{P_U}$ contains a parallel copy of $\mathbf{F_V}$. Similarly, if $\mathbf{V} \perp \mathbf{U}$, then $\mathbf{P_U}$ contains a parallel copy of $\mathbf{F_V}$ since each P_{U_n} coarsely contains a parallel copy of F_{V_n}. Thus $\hat{w}$ induces a nontrivial partition of $\mathbf{P_U}$ and hence cannot separate $\mathbf{x}, \mathbf{y}$.

Hence $\mathbf{D_V}(\mathbf{x}, \mathbf{y}) > 0$ only if $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$. But for such $\mathbf{V}$, we have $\pi_{\mathbf{V}}(\mathbf{x}) = \rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{y})$. Hence $\mathbf{D}(\mathbf{x}, \mathbf{y}) = 0$, so $\mathbf{x} \in \mathbf{P_U}$, as required. □

Proof of Proposition 32.2. Let $\mathbf{U} \in \mathfrak{F}^\infty$. Then we can represent $\mathbf{U}$ by a legal sequence $(g_n)_n \bar{U}$, where $\bar{U} \in \overline{\mathfrak{F}}$ and $(g_n)_n \in G^*_{ad}$. Now, by the definition of $\mathfrak{F}^\infty_{\mathbf{1}}$ and Lemma 32.3, $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{1}}$ if and only if $\mathbf{1} \in \mathbf{P_U}$. If $(g_n)_n \in G^*_{neg}$, then $\mathbf{1} \in \mathbf{P_U}$ because $g_n \in P_{g_n \bar{U}}$ and $\mathsf{d}_G(1, g_n)/j_n \to 0$.

Conversely, if $\mathbf{1} \in \mathbf{P_U}$, then (since $\mathsf{d}_{\mathrm{Cone}^\omega(G)}$ and $\mathbf{D}$ are bilipschitz–equivalent metrics), we have $\mathsf{d}_G(1, g_n P_{\bar{U}})/j_n \to 0$. Let $h_n = \mathfrak{g}_{P_{g_n \bar{U}}}(1)$. Fix $k \in \mathrm{Stab}_G(\bar{U})$, so that $g_n k \in g_n P_{\bar{U}} = P_{g_n \bar{U}}$. Note that $g_n^{-1} h_n \in g_n^{-1} P_{g_n \bar{U}} = P_{\bar{U}}$. Also note that k is uniformly close to $P_{\bar{U}}$ (more precisely, the distance can be bounded in terms of the finite set $\overline{\mathfrak{F}}$ and independently of n, using that $\mathrm{Stab}_G(\bar{U})$ acts on $P_{\bar{U}}$ coboundedly).

Using coboundedness, choose $a_n \in \mathrm{Stab}_G(\bar{U})$ so that $\mathsf{d}_G(a_n k, g_n^{-1} h_n) \leqslant B$, where B is independent of n. So $\mathsf{d}_G(g_n a_n k, h_n) \leqslant B$.

Note that $g_n a_n \bar{U} = g_n \bar{U}$, so $(g_n a_n)\bar{U} = (g_n)\bar{U} = \mathbf{U}$. Hence, to conclude, we have to show that $(g_n a_n) \in G^* neg$. By the triangle inequality,

$$\mathsf{d}_G(g_n a_n, 1) \leqslant \mathsf{d}_G(g_n a_n k, h_n) + \mathsf{d}_G(h_n, 1) + \mathsf{d}_G(k, 1).$$

The third term is independent of n, and the first term is bounded by B. So $(g_n a_n) \in G^*_{neg}$ provided $(h_n) \in G^*_{neg}$. Now, since $h_n = \mathfrak{g}_{P_{g_n \bar{U}}}(1)$, by [BHS17c, Corollary 1.28], the distance formula, and the definition of the gate, we have $\mathsf{d}_G(1, g_n P_{\bar{U}})/j_n \to 0$ implies $\mathsf{d}_G(1, h_n)/j_n \to 0$, i.e. $(h_n)_n \in G^*_{neg}$. □

So, we can think of elements of the local grove in $\mathrm{Cone}^\omega(G)$ as translates of elements of $\overline{\mathfrak{F}}$ by negligible sequences. We need two more useful facts about $\mathfrak{F}^\infty_{\mathbf{1}}$.

Proposition 32.4. *Let $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{1}}$. Then $\mathbf{U}^\perp \in \mathfrak{F}^\infty_{\mathbf{1}}$.*

Proof. Let $(U_n)_n$ be a legal sequence representing $\mathbf{U}$. By Lemma 27.3, $(U_n^\perp)_n$ is a legal sequence representing $\mathbf{U}^\perp$.

We recall that P_{U_n} is uniformly coarsely contained in $P_{U_n^\perp}$. Indeed, let $V \in \mathfrak{F}$ and suppose that $V \pitchfork U_n^\perp$ or $U_n^\perp \subsetneq V$. Then V cannot be nested in U_n (for otherwise it would be orthogonal to $U_n^\perp$) and V cannot be orthogonal to U_n, for otherwise it would be nested in $U_n^\perp$. Hence $V \pitchfork U_n$ or $U_n \subsetneq V$. Thus $\rho^{U_n}_V$ and $\rho^{U_n^\perp}_V$ are both well-defined bounded sets in $\mathcal{C}V$. By Lemma 15.10, $\rho^{U_n}_V$ and $\rho^{U_n^\perp}_V$ coarsely coincide for such V. Hence each point in P_{U_n} is uniformly close to $P_{U_n^\perp}$, as required.

Since P_{U_n} is uniformly coarsely contained in $P_{U_n^\perp}$, we have $\mathbf{P_U} \subseteq \mathbf{P_{U^\perp}}$. Now, by Lemma 32.3, we have $\mathbf{U} \in \mathfrak{F}^\infty_{\mathbf{1}}$ if and only if $\mathbf{1} \in \mathbf{P_U}$. But the latter condition implies that $\mathbf{1} \in \mathbf{P_{U^\perp}}$, so $\mathbf{U}^\perp \in \mathfrak{F}^\infty_{\mathbf{1}}$. □

Proposition 32.5. *Let $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty_{\mathbf{1}}$. Then there exists a unique $\sqsubseteq$–minimal $\mathbf{U} \vee \mathbf{V}$ into which $\mathbf{U}, \mathbf{V}$ are nested, and $\mathbf{U} \vee \mathbf{V} \in \mathfrak{F}^\infty_{\mathbf{1}}$.*

Proof. Let $(U_n)_n$ and $(V_n)_n$ be legal sequences representing $\mathbf{U}, \mathbf{V}$ respectively. For each n, let W_n be the unique $\sqsubseteq$–minimal element of $\mathfrak{F}$ into which U_n and V_n are both nested. That such an element exists is shown by Berlai and Robbio in [BR20a, Section 3], where it is shown that there exist $T^n_1, \ldots, T^n_{c_n}$, where c_n is bounded in terms of the complexity, such that $W_n = \bigwedge_{i=1}^{c_n} T^n_i$ has the required property, where $\bigwedge$ is defined by applying the wedge property iteratively.

Hence we can apply Lemma 27.3 boundedly many times to see that $(W_n)_n$ is legal, and hence represents an element $\mathbf{W} \in \mathfrak{F}^\infty$. We let $\mathbf{U} \vee \mathbf{V} = \mathbf{W}$. By construction, $\mathbf{U}, \mathbf{V} \sqsubseteq \mathbf{W}$. If there exists $\mathbf{W'}$ such that $\mathbf{U}, \mathbf{V} \sqsubseteq \mathbf{W'} \subsetneq \mathbf{W}$, then we could choose a legal sequence $(W'_n)_n$ such that $U_n, V_n \sqsubseteq W'_n \subsetneq W_n$ for ω–a.e. n, contradicting the definition of W_n. Hence $\mathbf{W}$ is $\sqsubseteq$–minimal with the property that $\mathbf{U}, \mathbf{V} \sqsubseteq \mathbf{W}$, and uniqueness of $\mathbf{W}$ follows from Lemma 27.3.

It remains to prove that $\mathbf{W} \in \mathfrak{F}^\infty_{\mathbf{1}}$, provided $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^\infty_{\mathbf{1}}$. In view of Lemma 32.3, it suffices to show that $\mathbf{1} \in \mathbf{P_W}$. By the same lemma, we can assume that $\mathbf{1} \in \mathbf{P_U} \cap \mathbf{P_V}$.

By construction, $\mathbf{P_U} \cong \mathbf{F_U} \times \mathbf{F_{U^\perp}}$, and a similar equality holds for $\mathbf{V}$, so we can choose parallel copies $\mathbf{F_U}, \mathbf{F_V}$ that both contain $\mathbf{1}$.

Choose a parallel copy $\mathbf{F_W}$ containing $\mathfrak{g}_{\mathbf{P_W}}(\mathbf{1})$. Note that if $\mathfrak{g}_{\mathbf{P_W}}(\mathbf{1}) = \mathbf{1}$ (equivalently, $\mathbf{1} \in \mathbf{P_W}$), then $\mathbf{W} \in \mathfrak{F}^\infty_{\mathbf{1}}$, by Lemma 32.3, and we are done.

Choose parallel copies $\mathbf{F}'_{\mathbf{U}}, \mathbf{F}'_{\mathbf{V}}$ of $\mathbf{F}_{\mathbf{U}}, \mathbf{F}_{\mathbf{V}}$ such that $\mathbf{F}'_{\mathbf{U}}, \mathbf{F}'_{\mathbf{V}} \subset \mathbf{F}_{\mathbf{W}}$. Let $\mathbf{a} = \mathfrak{g}_{\mathbf{F}'_{\mathbf{U}}}(\mathbf{1})$ and let $\mathbf{b} = \mathfrak{g}_{\mathbf{F}'_{\mathbf{V}}}(\mathbf{1})$. By convexity of $\mathbf{F}_{\mathbf{W}}$, we have $\mu(\mathbf{1}, \mathbf{a}, \mathbf{b}) = \mu \in \mathbf{F}_{\mathbf{W}}$.

If $\mathbf{T}$ satisfies $\pi_{\mathbf{T}}(\mathbf{1}) \neq \pi_{\mathbf{T}}(\mu)$, then $\mathbf{T}$ separates $\mathbf{a}$ and $\mathbf{b}$ from $\mathbf{1}$, and hence $\mathbf{T} \perp \mathbf{U}, \mathbf{V}$. Since $(\mathbf{U} \vee \mathbf{V})^{\perp} = \mathbf{U}^{\perp} \wedge \mathbf{V}^{\perp}$, we have $\mathbf{T} \perp \mathbf{W}$. Hence, since $\mathbf{a} \in \mathbf{P}_{\mathbf{W}}$, so is $\mathbf{1}$. Hence, by Lemma 32.3, $\mathbf{W} \in \mathfrak{F}^{\infty}_{\mathbf{1}}$. □

Remark 32.6. Although we do not require it later, a similar argument shows that $\mathbf{U} \wedge \mathbf{V} \in \mathfrak{F}^{\infty}_{\mathbf{1}}$ whenever that wedge is defined and $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\infty}_{\mathbf{1}}$. □

33. Summary of results

We summarise the conclusions as follows.

Let $(G, \mathfrak{F})$ be an HHG relative to subgroups, as in Remark 31.1. Let $\mathrm{Cone}^{\omega}(G)$ be an asymptotic cone of G.

By Theorem 26.3, $(\mathrm{Cone}^{\omega}(G), \mathfrak{F}^{\infty})$ is an ℝ–cubing, where $\mathrm{Cone}^{\omega}(G)$ is equipped with a median metric $\mathbf{D}$ that is bilipschitz equivalent to $\mathsf{d}_{\mathrm{Cone}^{\omega}(G)}$.

For each $\mathbf{U} \in \mathfrak{F}^{\infty}$, we denote by $\mathcal{C}\mathbf{U}$ the associated ℝ–tree and by $\pi_{\mathbf{U}} : \mathrm{Cone}^{\omega}(G) \to \mathcal{C}\mathbf{U}$ the (surjective) projection.

Each $a \in G^*$ induces a 1–morphism $(a, I_a, \{a_{\mathbf{U}}\}) : \mathrm{Cone}^{\omega}(G) \to \mathrm{Cone}^{\omega}(G)$, where

- $a : \mathrm{Cone}^{\omega}(G) \to \mathrm{Cone}^{\omega}(G)$ is an isometry (for the metric $\mathbf{D}$);
- $I_a : \mathfrak{F}^{\infty} \to \mathfrak{F}^{\infty}$ is a bijection preserving $\sqsubseteq, \pitchfork, \perp$;
- for each $\mathbf{U} \in \mathfrak{F}^{\infty}$, the map $a_{\mathbf{U}} : \mathcal{C}\mathbf{U} \to \mathcal{C}I_a(\mathbf{U})$ is an isometry such that $\pi_{I_a(\mathbf{U})} \circ a = a_{\mathbf{U}} \circ \pi_{\mathbf{U}}$ and $\rho^{I_a(\mathbf{U})}_{I_a(\mathbf{V})} = a_{\mathbf{V}}(\rho^{\mathbf{U}}_{\mathbf{V}})$ whenever $\mathbf{U} \pitchfork \mathbf{V}$ or $\mathbf{U} \sqsubsetneq \mathbf{V}$.

Given $\mathbf{x} \in \mathrm{Cone}^{\omega}(G)$, we denote by $\mathfrak{F}^{\infty}_{\mathbf{x}}$ the set of $\mathbf{V} \in \mathfrak{F}^{\infty}$ such that $\rho^{\mathbf{V}}_{\mathbf{U}} = \pi_{\mathbf{U}}(\mathbf{x})$ whenever $\mathbf{V} \sqsubsetneq \mathbf{U}$ or $\mathbf{V} \pitchfork \mathbf{U}$. We denote by $\mathrm{Cone}^{\omega}(G)_{\mathbf{x}}$ the set of $\mathbf{y} \in \mathrm{Cone}^{\omega}(G)$ such that the $\mathbf{V}$–coordinates of $\mathbf{x}, \mathbf{y}$ differ only when $\mathbf{V} \in \mathfrak{F}^{\infty}_{\mathbf{x}}$.

Corollary 31.6 shows that $(\mathrm{Cone}^{\omega}(G)_{\mathbf{x}}, \mathfrak{F}^{\infty}_{\mathbf{x}})$ is a local ℝ–cubing, and is isomorphic to $(\mathrm{Cone}^{\omega}(G)_{\mathbf{1}}, \mathfrak{F}^{\infty}_{\mathbf{1}})$.

Given $\mathbf{U} \in \mathfrak{F}^{\infty}_{\mathbf{1}}$, we have a subtree $\mathcal{L}\mathbf{U} = \pi_{\mathbf{U}}(\mathrm{Cone}^{\omega}(G)_{\mathbf{1}})$. The ℝ–trees in the above local ℝ–cubing structure are the trees $\mathcal{L}\mathbf{U}$.

By Proposition 31.8, we have the following. Let $\mathbf{U} \in \mathfrak{F}^{\infty}_{\mathbf{1}}$ and let $(g_n U)_n$ be a legal sequence representing it, where $U \in \mathfrak{F}$ and (g_n) is an admissible sequence.

Let $\alpha \in \{0, 2, 2^{\aleph_0}\}$ be the cardinality of the Gromov boundary of the hyperbolic space $\mathcal{C}U$. Then $\mathcal{L}\mathbf{U}$ is a complete ℝ–tree with valence α at every point, and is therefore the universal α–tree.

Finally, under the additional hypothesis that $(G, \mathfrak{F})$ has wedges and clean containers, we get the following. For any finite $\overline{\mathfrak{F}} \subset \mathfrak{F}$ containing at least one element of each G–orbit, $\mathfrak{F}^{\infty}_{\mathbf{1}}$ consists exactly of $G^*_{neg} \cdot \overline{\mathfrak{F}}$, by Proposition 32.2. Also, by Proposition 32.4 and Proposition 32.5, the set $\mathfrak{F}^{\infty}_{\mathbf{1}}$ is closed under joins and orthogonal complements.

We will see below that all of the above conclusions hold when G is one of our main examples, i.e. the fundamental group of a compact special cube complex, or the mapping class group of a hyperbolic surface with finite genus and finitely many punctures.

34. ☺ Questions and remarks

We conclude with some miscellaneous questions and remarks.

34.1. ☺ **What happened to large links when we passed to the cone, and the difference between $\mathcal{T}^{\bullet}\mathbf{U}$ and $\lim_{\omega} \mathcal{C}U_n$.** Let $(\mathcal{X}, \mathfrak{F})$ be a hierarchically hyperbolic space. Let $(\mathrm{Cone}^{\omega}\mathcal{X}, \mathfrak{F}^{\infty})$ be the resulting real cubing with $\mathrm{Cone}^{\omega}(\mathcal{X})$ a (non-principal) asymptotic cone of $\mathcal{X}$ (with the median metric).

In $(\mathcal{X}, \mathfrak{F})$, we have the large link axiom: given $x, y \in \mathcal{X}$, and a sufficiently large constant C, there is a bounded-cardinality list of N elements $U_i \subsetneq S$ such that if $\mathsf{d}_U(x,y) > C$, then either $U = S$ or $U \sqsubseteq U_i$ for some i, and N is bounded by a linear function of $\mathsf{d}_S(x,y)$.

We have already seen that the most naive version of the large links axiom does not apply to real cubings. Specifically, a typical situation is that, for $\mathbf{x}, \mathbf{y} \in \mathrm{Cone}^\omega(\mathcal{X})$, there are countably many $\mathbf{U} \in \mathfrak{F}^\infty$ such that $\mathbf{x}, \mathbf{y}$ have distinct projections onto $\mathcal{T}^\bullet\mathbf{U}$.

We now discuss some concrete examples, to illustrate the proof of Theorem 26.3 and show why one should not expect any analogue of large links for the asymptotic cone.

Let $F_2 = \langle a, b \mid \rangle$ be equipped with the obvious choice of word-metric, and let $\mathfrak{F}$ consist of the set of left cosets of $\langle a \rangle$. So, $\mathcal{C}F_2$ is the Cayley graph, with each coset in $\mathfrak{F}$ coned off, and $\mathcal{C}g\langle a \rangle$ is a copy of $\mathbb{R}$.

Let us consider some points in $\mathrm{Cone}^\omega(F_2)$, where we use the rescaling sequence $j_n = n$ and observation point the constant sequence 1.

First, let

$$x_n = b(a^n)b(a^{n/2})b \cdots b(a^{n/2^{k_n}}),$$

where (k_n) is some nondecreasing sequence of natural numbers. For $0 \leqslant i \leqslant k$, let

$$A_i(n) = b(a^n)b(a^{n/2})b \cdots b(a^{n/2^{i-1}})b\langle a \rangle \in \mathfrak{F}.$$

Then for $U \in \mathfrak{F}$, we have

- $\mathsf{d}_U(1, x_n) = 0$ if $U \neq F_2$ and $U \neq A_i$ for $i \leqslant k_n$,
- $\mathsf{d}_{Cay(F_2)}(1, x_n) = (k_n + 1) + n\sum_{i=0}^{k_n} 2^{-i}$, and
- $\mathsf{d}_{A_i(n)}(1, x_n) = n \cdot 2^{-i}$.

So, a priori, if $\mathbf{U} \in \mathfrak{F}^\infty$ has $\mathbf{D_U}(\mathbf{1}, \mathbf{x}) > 0$, then $\mathbf{U}$ is represented by a sequence $(A_{f(n)})_n$ for some $f : \mathbb{N} \to \mathbb{N}$, where $f(n) \leqslant k_n + 1$ for each n, or $\mathbf{U} = \mathbf{S}$ is represented by the constant sequence F_2. Specifically, for $(A_{f(n)})_n = \mathbf{U}$ as above, we have

$$\mathbf{D_U}(\mathbf{1}, \mathbf{x}) = \lim_\omega 2^{-f(n)},$$

so we are interested in those $f(n)$ for which this is positive. But this would imply that $f(n)$ is bounded, so since it takes integer values, up to ω, we have that $f(n)$ is constant.

So $\sum_{\mathbf{U} \subsetneq \mathbf{S}} \mathbf{D_U}(\mathbf{1}, \mathbf{x}) = \sum_{i=0}^\infty 2^{-i} = 2$. Now let's vary k_n.

First, consider the case where $k_n = \lfloor \sqrt{n} \rfloor$. In this case, $\mathsf{d}_{\mathrm{Cone}^\omega(F_2)}(\mathbf{1}, \mathbf{x}) = 2$. Indeed, for each $i \geqslant 0$, we have a non-zero term coming from $(A_i(n))_n$ contributing 2^{-i}.

So $\mathbf{D_S}(\mathbf{1}, \mathbf{x}) = 0$. In particular, the points $\rho^{\mathbf{U}}_{\mathbf{S}}$ with $\mathbf{D_U}(\mathbf{1}, \mathbf{x}) > 0$ all coincide.

Next, consider the case where $k_n = n$. Then $\mathsf{d}_{\mathrm{Cone}^\omega(F_2)}(\mathbf{1}, \mathbf{x}) = 3$, but again the total contribution from properly nested elements of $\mathfrak{F}^\infty$ is 2, so $\mathbf{D_S}(\mathbf{1}, \mathbf{x}) = 3 - 2 = 1$. Again, the relevant points $\rho^{\mathbf{U}}_{\mathbf{S}}$ are all at $\pi_{\mathbf{S}}(\mathbf{1})$. On the other hand, there exist $\mathbf{U}$ such that $\mathbf{D_U}(\mathbf{1}, \mathbf{x}) = 0$ and $\rho^{\mathbf{U}}_{\mathbf{S}} \neq \pi_{\mathbf{S}}(\mathbf{1})$.

This indicates that in $(\mathrm{Cone}^\omega(F_2), \mathfrak{F}^\infty)$, one can't expect some sort of "infinitary large link axiom". For example, in each of the two choices of (x_n), for each $\epsilon > 0$, we have exactly the same number of $\mathbf{U} \subsetneq \mathbf{S}$ for which $\mathbf{D_U}(\mathbf{1}, \mathbf{x}) > \epsilon$, and those lengths are all independent of $\mathbf{x}$. But for the two choices of $\mathbf{x}$, the length in $\mathcal{T}^\bullet\mathbf{S}$ is different (and can obviously be chosen arbitrarily with small modifications to $(x_n)_n$).

Already in this example there are points $\mathbf{y} \in \mathrm{Cone}^\omega(F_2)$ where the relevant $\rho^{\mathbf{U}}_{\mathbf{S}}$ points are distributed more interestingly in $[\pi_{\mathbf{S}}(\mathbf{1}), \pi_{\mathbf{S}}(\mathbf{y})]$. For example, we leave as an exercise for the reader to construct $\mathbf{y}$ so that $\mathsf{d}_{\mathrm{Cone}^\omega(F_2)}(\mathbf{1}, \mathbf{y}) = 1$ and the set of $\rho^{\mathbf{U}}_{\mathbf{S}}$ with $\mathbf{U} \subsetneq \mathbf{S}$ and $\mathbf{D_U}(\mathbf{1}, \mathbf{y}) > 0$ correspond to the components of the complement in $[0, 1]$ of a fat Cantor set.

Relatively simple examples like this also illustrate another point, the difference between $\mathcal{T}^\bullet\mathbf{S}$ and the corresponding asymptotic cone of $\mathcal{C}F_2$. For example, for integral $i \geqslant 0$, let

$$z_n^i = b^n a^n b^{n/2} a^n \cdots b^{n/2^i} a^n.$$

Then $(z_n^i)_n$ is admissible for each i, and satisfies

$$\mathsf{d}_{\mathrm{Cone}^\omega(F_2)}(\mathbf{1}, \mathbf{z}^i) = i + 3 - 2^{1-i},$$

while $\sum_{\mathbf{U}} \mathbf{D}_{\mathbf{U}}(\mathbf{1}, \mathbf{z}^i) = i + 1$. So $\mathbf{D}_{\mathbf{S}}(\mathbf{1}, \mathbf{z}^i) = 2 - 2^{1-i}$, and since z_n^i is a subword of z_n^{i+1} for each i, n, one can check by a similar computation that $(\pi_{\mathbf{S}}(\mathbf{z}^i))_i$ is Cauchy. But one can deduce from the fact that $(\mathbf{z}^i)_i$ is unbounded in $\mathrm{Cone}^\omega(F_2)$ that $(\pi_{\mathbf{S}}(\mathbf{z}^i))_i$ cannot converge in $\mathcal{T}^\bullet\mathbf{S}$. So $\mathcal{T}^\bullet\mathbf{S}$ is not complete, but asymptotic cones of $\mathcal{C}F_2$ are.

Intuitively, this reflects the fact that traveling distance j_n in certain directions in $\mathcal{C}U_n$, ending at some $\pi_{U_n}(x_n)$, entails traveling through points $\rho_{U_n}^V$ for which $\mathsf{d}_V(1, x_n)$ may be large compared to j_n, so the point $(\pi_{U_n}(x_n)) \in \lim_\omega(\mathcal{C}U_n, \mathsf{d}_{U_n}/j_n)$ is not visible in $\mathcal{T}^\bullet\mathbf{U}$. So in this sense, $\mathcal{T}^\bullet\mathbf{U}$ contains "less" than $\lim_\omega(\mathcal{C}U_n, \mathsf{d}_{U_n}/j_n)$.

However, we saw already that if $\mathbf{U}$ (represented by (U_n)) satisfies $\mathbf{D}_{\mathbf{U}}(\mathbf{1}, \mathbf{x}) > 0$, then $(U_n) \in \mathrm{Rel}((1), (x_n))$, i.e.

$$\lim_\omega \mathsf{d}_{U_n}(1, x_n) = \infty.$$

It does *not* follow that $\lim_\omega \mathsf{d}_{U_n}(1, x_n)/j_n > 0$. Indeed, in our same F_2 example, we can take

$$x_n = \left(b(a^{\sqrt{n}})\right)^{\sqrt{n}}.$$

Then $\mathsf{d}_{\mathrm{Cone}^\omega(F_2)}(\mathbf{1}, \mathbf{x}) = 1$. For any properly nested $\mathbf{U}$, we have $\mathbf{D}_{\mathbf{U}}(\mathbf{1}, \mathbf{x}) = 0$, since $\sqrt{n}/n \to 0$. Hence $\mathbf{D}_{\mathbf{S}}(\mathbf{1}, \mathbf{x}) = 1$. On the other hand, $\mathbf{S}$ is represented by a constant sequence for which the associated hyperbolic spaces are all $\mathcal{C}F_2$, in which the distance is $3\sqrt{n}$, which is unbounded (reflecting that the constant sequence $\mathcal{C}F_2$ is in $\mathrm{Rel}((1), (x_n))$) but sublinear. So the real tree $\mathcal{T}^\bullet(F_2)_n$ contains "more" than $\mathrm{Cone}^\omega(\mathcal{C}F_2)$.

So there is no direct relation between $\mathrm{Cone}^\omega(\mathcal{C}F_2)$ and the real tree $\mathcal{T}^\bullet(F_2)_n$, and this reflects the general situation that there is no direct relationship between $\lim_\omega(\mathcal{C}U_n, \mathsf{d}_{U_n}/j_n)$ and $\mathcal{T}^\bullet\mathbf{U}$, where (U_n) is a legal sequence representing $\mathbf{U} \in \mathfrak{F}^\infty$.

Finally, although the large link axiom for HHSes is lost when passing to the asymptotic cone, there is a vestige from this axiom that holds for asymptotic cones of HHSes, namely, for any $\mathbf{x}, \mathbf{y} \in \mathrm{Cone}^\omega(x)$ and any admissible sequences $(x_n)_n, (y_n)_n$ representing the points correspondingly, the set of maximal relevant elements is finite, see Definition 41.3 and Lemma 41.4. But the latter is a property of asymptotic cones of HHSes, not a property that can even be sensibly stated for general real cubings.

34.2. ☺ **$\mathbf{P}_{\mathbf{U}}$ versus the parallel set of $\mathbf{F}_{\mathbf{U}}$.** Fix an HHS $(\mathcal{X}, \mathfrak{F})$ and let $(\mathrm{Cone}^\omega(\mathcal{X}), \mathfrak{F}^\infty)$ be the resulting real cubing structure on an asymptotic cone.

For each $\mathbf{U} \in \mathfrak{F}^\infty$, represented by a legal sequence $(U_n)_n$, let $\mathbf{P}_{\mathbf{U}}$ be the standard product region in the real cubing sense, i.e. the set of $\mathbf{x}$ such that $\pi_{\mathbf{V}} = \rho_{\mathbf{V}}^{\mathbf{U}}$ whenever the $\rho_{\mathbf{V}}^{\mathbf{U}}$ is defined and a point.

Let $\mathbf{P}_{\mathbf{U}}^\omega = \lim_\omega P_{U_n}$. Although we phrased it for HHGs with clean containers, Lemma 32.3 says that $\mathbf{P}_{\mathbf{U}} = \mathbf{P}_{\mathbf{U}}^\omega$ (the proof only used the HHS structure and the resulting real cubing structure on the asymptotic cone).

Now, consider $\mathbf{F}_{\mathbf{U}} = \lim_\omega F_{U_n}$. We have that $\mathbf{F}_{\mathbf{U}}$ is also the subspace determined by the colour $\mathbf{U}$ and the associated filter $\sigma_{\mathbf{U}}$, so by the discussion right after Lemma 5.21, $\mathbf{F}_{\mathbf{U}}$ is also the subspace from Proposition 4.37, and $\mathbf{P}_{\mathbf{U}} = \mathbf{F}_{\mathbf{U}} \times \mathbf{E}_{\mathbf{U}}$.

In particular, letting $Para(\mathbf{U})$ be the union of all closed convex subspaces parallel to $\mathbf{F_U}$, we have

$$\mathbf{P_U} = \mathbf{P}^{\omega}_{\mathbf{U}} \subseteq Para(\mathbf{U}).$$

The subtle difference between $\mathbf{P_U} = \mathbf{P}^{\omega}_{\mathbf{U}}$ and $Para(\mathbf{U})$ initially caused us some cognitive dissonance, which is resolved by the following proposition.

Proposition 34.1. *Let $\mathbf{U} \in \mathfrak{F}^{\infty}_{+}$. Then $\mathbf{P_U} = Para(\mathbf{U})$.*

Moreover, if $\mathbf{U} \in \mathfrak{F}^{\infty}$ satisfies $\mathbf{P_U} \subsetneq Para(\mathbf{U})$, then there exist $\mathbf{V} \in \mathfrak{F}^{\infty}_{+}$ and $\{\mathbf{W}_i\}_i \subset \mathfrak{F}^{\infty}_{+}$ such that

- *$\mathbf{V} \perp \mathbf{W}_i$ for all i;*
- *$\mathbf{W}_i \subsetneq \mathbf{U}$ for all i;*
- *if $\mathbf{x}, \mathbf{y} \in \mathbf{F_U}$, then $\mathsf{d}_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) > 0$ implies $\mathbf{W} = \mathbf{W}_i$ for some i.*

Proof. First assume $\mathbf{U} \in \mathfrak{F}^{\infty}_{+}$ and let $\mathbf{F}$ be a closed convex set parallel to $\mathbf{F_U}$. Let $\mathbf{F}'$ be the image of $\mathbf{F}$ under the gate map to $\mathbf{P_U}$. Then $\mathbf{F}'$ is a parallel copy of $\mathbf{F_U}$ lying in $\mathbf{P_U}$, so we are justified in writing $\mathbf{F_U} = \mathbf{F}'$.

Let $\mathbf{x}, \mathbf{x}' \in \mathbf{F_U}$ be distinct. Let $\mathbf{y}, \mathbf{y}' \in \mathbf{F}$ respectively denote the gates in $\mathbf{F}$ of $\mathbf{x}, \mathbf{x}'$. If $\mathbf{x} = \mathbf{y}$ we are done, so suppose not.

Then $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{x}'$ is a median rectangle, so every wall in $\mathcal{W}(\mathbf{x}, \mathbf{x}')$ crosses every wall in $\mathcal{W}(\mathbf{x}, \mathbf{y})$, and both associated sets of halfspaces have positive measure.

Let $\mathbf{V}$ be such that $\mathbf{D_V}(\mathbf{x}, \mathbf{y}) > 0$. Then $\mathcal{H}_{\mathbf{V}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ is an inseparable set of positive measure.

Apply Definition 3.1.(IV) to find $\{\mathbf{U}_i\}$ and $\{\mathbf{V}_j\}$ such that

- $\mathbf{U}_i \sqsubseteq \mathbf{U}$ for all i, and
- $\mathbf{V}_j \sqsubseteq \mathbf{V}$ for all j, and
- $\mathcal{H}_{\mathbf{U}_i} \cap \mathcal{H}(\mathbf{x}, \mathbf{x}')$ has positive measure for each i, and
- $\mathcal{H}_{\mathbf{V}_j} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$ has positive measure for each j, and
- the sets $\cup_i \mathcal{H}_{\mathbf{U}_i}$ and $\cup_j \mathcal{H}_{\mathcal{V}_j}$ respectively cover $\mathcal{H}(\mathbf{x}, \mathbf{x}')$ and $\mathcal{H}_{\mathbf{V}} \cap \mathcal{H}(\mathbf{x}, \mathbf{y})$, up to null sets, and
- $\mathbf{U}_i \perp \mathbf{V}_j$ for all i, j.

Now, we cannot have $\mathbf{V}_j \subsetneq \mathbf{V}$ for all j, because this would imply $\pi_{\mathbf{V}}(\mathbf{x}) = \pi_{\mathbf{V}}(\mathbf{y})$. So $\mathbf{V} \perp \mathbf{U}_i$ for all i. On the other hand, any $\mathbf{W}$ with $\pi_{\mathbf{W}}(\mathbf{x}) \neq \pi_{\mathbf{W}}(\mathbf{x}')$ must be nested in some $\mathbf{U}_i$, and hence orthogonal to $\mathbf{V}$. Thus the gate map to $\mathbf{P_V}$ is an embedding on $I(\mathbf{x}, \mathbf{x}')$. Since $\mathbf{x}, \mathbf{x}' \in \mathbf{F_U}$ were arbitrary, we see that $\mathbf{F_U}$ is parallel to some $\mathbf{F}'$ such that there is a product $\mathbf{F}' \times \mathbf{F_V}$.

Now, $\mathbf{U} \in \mathfrak{F}^{\infty}_{+}$ by hypothesis, and $\mathbf{V} \in \mathfrak{F}^{\infty}_{+}$ since $\mathcal{T}^{\bullet}\mathbf{V}$ is nontrivial. So, by Lemma 29.4, the existence of this product implies $\mathbf{U} \perp \mathbf{V}$.

Since this holds for every $\mathbf{V}$ on which $\mathbf{x}, \mathbf{y}$ have different projections, we see that $\mathbf{F} \subset \mathbf{P_U}$, as required.

But by the definition of the gate map to $\mathbf{P_U}$, we have either $\mathbf{V} \pitchfork \mathbf{U}$ or $\mathbf{U} \subsetneq \mathbf{V}$, a contradiction. Hence $\mathbf{V}$ cannot exist, i.e. $\mathbf{F}' \subset \mathbf{P_U}$. This proves the first assertion.

The above argument also verified the "moreover" part of the statement. $\square$

We hope this proposition will also make the reader feel better when combined with the reminder that we only showed that orthogonality in the poset-colouring sense coincides with orthogonality in the ultralimit sense on $\mathfrak{F}^{\infty}_{+}$. Given $\mathbf{U}, \mathbf{V} \in \mathfrak{F}^{\infty}$, it is possible that e.g. U_n and V_n are transverse for ω–a.e. n, but F_{U_n}, F_{V_n} nonetheless form a coarse product and so the cone contains $\mathbf{F_U} \times \mathbf{F_V}$. In this case — which one can easily visualise in an HHS — any unbounded sequences of distance formula terms for points in F_{U_n} have to be properly nested in U_n and orthogonal to any unbounded sequence of distance formula terms for points in

F_{V_n}, or vice versa. Lemma 29.2 then says that no wall has colour, say, $\mathbf{U}$, and so either $\mathbf{U}$ or $\mathbf{V}$ is not in $\mathfrak{F}^{\infty}_{+}$.

A related caveat: having $\mathbf{U} \in \mathfrak{F}^{\infty} - \mathfrak{F}^{\infty}_{+}$ implies $\mathcal{T}^{\bullet}\mathbf{U}$ is trivial, but the converse need not hold. It is possible to make examples where, say, the set of walls crossing $\mathbf{F_U}$ but not crossing $\mathbf{F_V}$ for any properly nested $\mathbf{V}$ is nonempty but has measure 0. So, in characterising orthogonality between such elements (Lemma 29.4 and similar arguments above), we really used extra information only available in HHS structures of asymptotic cones, namely Lemma 29.2 (which says that $\mathbf{U}$ appears in $\max \mathrm{Rel}((x_n),(y_n))$ for some sequences) and the fact that orthogonality can be detected in the space $(\mathcal{X},\mathfrak{F})$, as in the argument for Claim 18.

34.3. ☺ **Simplification in the presence of wedges and clean containers.** Theorem 26.3 and Corollary 29.7 are considerably simplified for an HHS $(\mathcal{X},\mathfrak{F})$ with wedges and clean containers. In particular, considerations about max-relevant sets can be removed. Roughly:

- The colour of a wall is well-defined just because, if it crosses $\mathbf{F_U}$ and $\mathbf{F_V}$, then it crosses $\mathbf{F_{U\wedge V}}$, in view of Lemma 27.3. So we can define Col as before.
- Verifying Definition 3.1.(III) is simplified since, given $\mathbf{U},\mathbf{V}$ and some positive-measure set crossing both $\mathbf{F_U}, F_{\mathbf{V}}$, all such walls cross $F_{\mathbf{U}\wedge\mathbf{V}}$, and we can take $\{\mathbf{U}\wedge\mathbf{V}\}$ as the set required by the definition.
- The key point is that $Col(\hat{h})\perp Col(\hat{v})$ if and only if $\hat{h},\hat{v}$ cross. This involves an argument like the one for Claim 18, except simplified by the fact that we have joins and clean containers in the HHS index set. Here is the only place where one needs to argue using sequences in $\mathfrak{F}$. As above, we end up in a situation where $\mathrm{Rel}_M(a_n,b_n)$ is nonempty and has elements orthogonal to those in $\mathrm{Rel}_M(a_n,d_n)$, but this implies that the join of the first set is orthogonal to the join of the second set, which removes the need for considering "max-relevant" sets.
- Verifying Definition 3.1.(IV) is an application of the previous application of the previous point, along with the existence of joins and clean containers.
- As usual, we have to verify that, for elements in the image of Col, the two notions of orthogonality agree, and this is again enabled by the equivalence of crossing of walls with orthogonality (in the ultralimit sense) of their colours.

So the argument is morally similar, but simplified at every stage.

34.4. ☺ **Structure of the real trees associated to elements of the index set of an asymptotic cone.** As we indicated in the previous section, see also Section 31.1, in general, the real tree associated to $\mathbf{U} \in \mathfrak{F}^{\infty}$ is not a universal tree as, in particular, it is not complete. However, it does have some type of universality and completeness with the following adjustments.

The $2^{\aleph_0}$–universal real tree can be characterised by the germ of directions at any point. More precisely, it is a complete, contractible, geodesic metric space such that for each point in the space, the space of directions is a sheaf of $2^{\aleph_0}$ real lines, i.e. $2^{\aleph_0}$ copies of the real line identified all of them at 0.

One can generalise the germ of directions, from considering real lines, to considering open intervals containing 0. More precisely, let $I_\alpha = (-\alpha,\alpha)$, $\alpha \in (0,\infty]$ and let L be a sheaf of $2^{\aleph_0}$ intervals I_α for all $\alpha \in (0,\infty]$ identified at 0.

The real line is homeomorphic to each interval I_α. Indeed, it is routine to check that the maps $f : I_\alpha \to \mathbb{R}$ defined as $f_\alpha(x) = x/(\alpha - |x|)$ and $f'_\alpha : \mathbb{R} \to I_\alpha$ defined as $f'_\alpha(x) = x/(\alpha + |x|)$ are homeomorphisms and inverse to each other. We can then induce the metric on the real line to I_α, i.e. given $x, y \in I_\alpha$, we define $d_\alpha(x,y)$ to be the distance $d(f(x), f(y))$ between $f(x)$ and $f(y)$ in the real line. We thus have two distance maps on I_α, the distance d induced as a subspace of $\mathbb{R}$ and the distance d_α induced via the homeomorphism with the real line.

Notice that sequences converging to α are Cauchy with respect to d but not with respect to d_α and so I_α is not complete with respect to the distance d but it is complete with respect to the distance d_α.

The *universal punctured real tree* is defined as a complete (with respect to the metric d_α), contractible, geodesic metric space (the metric is induced by d) such that for each point in the space, the germ of directions at each point is L. More precisely, the metric on the space is induced by the metric of the intervals as subspaces of $\mathbb{R}$ but the notion of Cauchy sequence is relative to the metric d_α (the interval I_α as subspace of the universal punctured real tree is complete).

One can then show that the real trees associated to elements of the index set in the asymptotic cone of an HHG relative to subgroups are either universal or universal punctured real trees. Furthermore, since L contains a sheaf of $2^{\aleph_0}$ lines, one can show that the universal real tree is a subspace of the universal punctured one and by vice-versa, since any real tree is a subspace of the universal one.

34.5. ☺ **Asymptotically ℝ–cubing and asymptotically CAT(0) groups.** Hyperbolic groups are characterised as the class of finitely generated groups whose (non-principal) asymptotic cones are all real trees.

Following this approach, one can consider the class of finitely generated groups G such that every non-principal asymptotic cone $\mathrm{Cone}^\omega(G)$ is bilipschitz equivalent to a real cubing. We call this class of groups *asymptotically real cubical*.

In [Kar11], Kar introduced and studied the class of asymptotically CAT(0) groups, namely those groups acting geometrically on spaces, all of whose non-principal asymptotic cones are CAT(0) spaces. In [Bow16b], Bowditch examines groups, all of whose asymptotic cones are bilipschitz equivalent to CAT(0) spaces. Bowditch shows that spaces whose asymptotic cones are bilipschitz equivalent to finite-rank median metric spaces have this property. The class considered by Kar is contained in the class considered by Bowditch.

Bowditch's results combine with the fact that any HHS is a finite-rank coarse median space (Lemma 23.1) to show that if $(\mathcal{X}, \mathfrak{F})$ is an HHS, then any asymptotic cone $\mathrm{Cone}^\omega(\mathcal{X})$ is bilipschitz equivalent to a CAT(0) space. In fact, as shown by Theorem 26.3, the asymptotic cone of an HHS is bilipschitz equivalent to a real cubing, that is, HHS are asymptotically real cubical.

Now, if G is a group whose asymptotic cones are all bilipschitz equivalent to CAT(0) spaces, then G has various nice properties, some of which have been established for HHGs by other means. The fact that asymptotic cones of hierarchically hyperbolic spaces are contractible, along with some other homological propertes, are also discussed in [BHS19].

- G is of type F_∞. Indeed, our assumptions imply that all asymptotic cones of G are contractible, and we apply [Ril02, 2.6.D].
- G has solvable word problem, by a result of Dru tu [Dru02]. When G is hierarchically hyperbolic, or, more generally, coarse median, the stronger property of satisfying a quadratic isoperimetric inequality was proved in [Bow13, BHS19].

If G is in the class considered by Kar, then moreover G has finitely many conjugacy classes of finite subgroups, and every quasi-isometrically embedded nilpotent group is virtually abelian [Kar11]. When G is an HHG, the former statement is proved by other means in [HHP20], and the latter statement can be strengthened: any finitely generated polcyclic subgroup is quasi-isometrically embedded and virtually abelian [HHP20].

A priori, being asymptotically real cubical is a stronger property than having all asymptotic cones bilipschitz to CAT(0) spaces, which motivates a vague question:

Question 34.2. Find useful group-theoretic consequences of being asymptotically real-cubical that do not follow from having all asymptotic cones bilipschitz equivalent to CAT(0) spaces.

There should be asymptotically CAT(0) groups that are not asymptotically real cubical. Indeed, as a candidate, consider the $\mathbb{R}$-completion of the infinitely generated RAAG defined in the example 21.4, i.e. $G = \mathbb{G}^{\mathbb{R}}$.

On the other hand, it follows from our results that hierarchically hyperbolic groups are asymptotically real cubical. This brings to the question:

Question 34.3. Are there finitely generated groups all of whose asymptotic cones are real cubings and their Cayley graphs are not HHS?

Note that there are asymptotically real cubical groups that are not HHGs, which is why we phrased the question how we did. Indeed, the $(3, 3, 3)$–triangle group is quasi-isometric to $\mathbb{Z}^2$, and hence every asymptotic cone is bilipschitz equivalent to an asymptotic cone of $\mathbb{Z}^2$, and $\mathbb{Z}^2$ is an HHS and hence asymptotically real cubical. But the $(3, 3, 3)$–triangle group is not an HHG [PS20].

Finally, it would be interesting to know if hierarchically hyperbolic groups are actually asymptotically CAT(0) in Kar's stronger sense:

Question 34.4. Let $(G, \mathfrak{F})$ be a hierarchically hyperbolic group. When does G act properly and coboundedly on a space X such that every asymptotic cone of X is CAT(0)?[9]

[9] In personal communication from Alessandro Sisto, we have learned that it is likely that such an X exists provided $\mathfrak{F}$ is *colourable*, using methods similar to those in [DMS20].

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List of HHS constants

$B_1 = B_1(M_0)$**:** Realisation constant, page 122.
$B_2 = B_2(M)$**:** Threshold on non-relevance, see Proposition 16.4, page 123
$B_0^{aut} = B_0^{aut}(x, U, g)$**:** Bigset constant, see Lemma 19.2, page 131
$B_1^{aut} = B_1^{aut}(x, U, g)$**:** Bigset constant, see Lemma 19.2, page 131
C**:** Quasi-isomtetry constant of the cubical approximation, see Proposition 16.1, page 119
C_0**:** constant (depending only on E) so that for all $W \in \mathfrak{F}$ and $x, y, z, \in G$, the point $\pi_W(\mu(x, y, z))$ lies C_0–close to any geodesic joining $\pi_W(a), \pi_W(b)$ whenever $a, b \in \{x, y, z\}$ are distinct, page 112
χ**:** complexity of HHS, page 105
E**:** large links HHS constant, page 105
K**:** Constant so that π_W is (K, K)–coarsely lipschitz, where $W \in \mathfrak{F}$, page 104;
κ_0**:** HHS constant, see Definition 10.1, page 104
$\kappa^{\times}$**:** a function, depending only on the HHS constants, such that each P_U, E_U, F_U is $\kappa^{\times}$–hierarchically quasiconvex, page 115
λ**:** large links HHS constant, see Definition 10.1, page 105
M_0, M_1**:** Distance threshold constants of the cubical approximations, see Proposition 16.1, page 119
θ_0**:** constant depending only on the HHS, such that for all $\theta \geqslant \theta_0$, there is a function κ_0 such that $H_\theta(A)$ is κ_0–hierarchically quasiconvex for any A, page 113
ξ**:** HHS constant, see Definition 10.1, page 104

Glossary of Notation

$Big(g)$**:** the set of $U \in \mathfrak{F}$ such that $\pi_U(\langle g \rangle \cdot x)$ is unbounded in $\mathcal{C}U$ (for some, and hence any, $x \in \mathcal{X}$). 130

$Col : \mathcal{W} \to (\mathfrak{F}^{\bullet}, \sqsubseteq)$**:** poset-colouring from the set of walls to a partially ordered set. 33
$\mathrm{Cone}^{\omega}(G)_{\mathbf{x}}$**:** the set of $\mathbf{y} \in \mathrm{Cone}^{\omega}(G)$ such that $\pi_{\mathbf{U}}(\mathbf{x}) \neq \pi_{\mathbf{U}}(\mathbf{y})$ implies $\mathbf{U} \in \mathfrak{F}^{\infty}_{\mathbf{x}}$. 166
χ**:** complexity of HHS. 105
$\chi^{\bullet}$**:** the bound on lengths of $\sqsubseteq$–chains and cardinalities of pairwise–$\perp$ sets in $\mathfrak{F}^{\bullet}$. 47

$\mathbf{D}$**:** metric making $(\mathrm{Cone}^{\omega}(\mathcal{X}), \mathbf{D}, \boldsymbol{\mu})$ a finite-rank complete connected geodesic median metric space. 143
$\hat{\mathbf{d}}_{\infty}$**:** metric on $\mathrm{Cone}^{\omega}(\mathcal{X})$. 141
$\mathbf{d}_{\mathbf{X}}$**:** the metric on the $\mathbb{R}$–cubing $\mathbf{X}$. 47

E_U**:** the set of tuples $(p_V)_{V \in \mathfrak{F}_U^{\perp}} \in \prod_{V \in \mathfrak{F}_U^{\perp}} \mathcal{C}V$ that are E–consistent. 114, 115
$E^p_{\mathbf{U}}$**:** factor of the product region in the real cubing, $F^p_{\mathbf{U}} \times E^p_{\mathbf{U}}$ is median-preserving isometric to $\mathbf{P}_{\mathbf{U}}$. 60

F_U**:** the set of tuples $(p_V)_{V \in \mathfrak{F}_U} \in \prod_{V \in \mathfrak{F}_U} \mathcal{C}V$ that are E–consistent. 114
$F^p_{\mathbf{U}}$**:** factor of the product region in the real cubing, $F^p_{\mathbf{U}} \times E^p_{\mathbf{U}}$ is median-preserving isometric to $\mathbf{P}_{\mathbf{U}}$. 60
$\mathfrak{F}^{\infty}_{+}$**:** the image of the poset-colouring map. 160
$\mathfrak{F}^{\infty}_{\mathbf{x}}$**:** the set of $\mathbf{U} \in \mathfrak{F}^{\infty}$ such that for all $\mathbf{V} \in \mathfrak{F}^{\infty}$ with $\mathbf{V} \sqsubsetneq \mathbf{U}$ or $\mathbf{V} \pitchfork \mathbf{U}$, we have $\rho^{\mathbf{U}}_{\mathbf{V}} = \pi_{\mathbf{V}}(\mathbf{x})$. 165
$\mathfrak{F}^{\infty}$**:** the set of equivalence classes of legal sequences in $\mathfrak{F}$. 145
$\mathfrak{F}$**:** index set of a hierarchically hyperbolic space. 104
$\overline{\mathfrak{F}}$**:** a finite set $\overline{\mathfrak{F}} \subset \mathfrak{F}$ with the property that $G \cdot \overline{\mathfrak{F}} = \mathfrak{F}$. 169
$\mathfrak{F}^{\bullet}$**:** index set of the $\mathbb{R}$–cubing structure. 47

G^*_{ad}**:** subgroup of G^* consisting of admissible sequences. 141
G^*_{neg}**:** subgroup of G^* and G^*_{ad} consisting of negligible sequences. 142
G^***:** ultrapower of G. 141
$\mathfrak{g} : \mathbf{X} \to \mathbf{Y}$**:** gate map. 26, 54

$\mathcal{H}(\mathbf{Y})$**:** set of halfspaces associated to walls crossing $\mathbf{Y}$. 26
$\hat{h} = \{h, h^*\}$**:** wall; halfspace and its dual. 26
h**:** halfspace. 26
$\mathcal{H}$**:** set of all halfspaces. 26
$\mathrm{Hull}^{\bullet}(A)$**:** convex hull of A. 54
$\mathrm{Hyp}(\mathbf{Y})$**:** the set of hyperplanes in $\mathbf{Y}$. 122

$I(a, b)$**:** median interval between two points. 24

$\kappa^{\times}$**:** a function, depending only on the HHS constants, such that each P_U, E_U, F_U is $\kappa^{\times}$–hierarchically quasiconvex. 115

$Lab(h(i, U))$**:** label of the hyperplane. 121
$\mathrm{Level}(U)$**:** level of an element $U \in \mathfrak{F}$. 110
$\mathcal{L}\mathbf{U}$**:** the set $\pi_{\mathbf{U}}(\mathrm{Cone}^{\omega}(G)_{\mathbf{x}})$. 166

$\boldsymbol{\mu}$**:** the median operator on $\mathrm{Cone}^{\omega}(\mathcal{X})$. 142

$\mu_{\mathbf{W}}$**:** the median operator on $\mathcal{T}^\bullet\mathbf{W}$ coming from the $\mathbb{R}$–tree metric (the map $\pi_{\mathbf{W}}$ takes μ to $\mu_{\mathbf{W}}$). 49

μ**:** the median operator on $\mathbf{X}$, the coarse median operator on an HHG. 24, 49, 112

$\sqsubseteq$**:** the relation of nesting on $\mathfrak{F}^\bullet$ or $\mathfrak{F}$. 47, 104

ω**:** non-principal ultrafilter on $\mathbb{N}$. 141

$\perp$**:** the relation of orthogonality on $\mathfrak{F}^\bullet$ or $\mathfrak{F}$. 47, 104

P_U**:** product region in the HHS. 114

$\mathbf{P_U}$**:** product region in the real cubing. 50

$\pi_{\mathbf{W}}$**:** the natural projection $\prod_{\mathbf{U}\in\mathfrak{F}^\bullet}\mathcal{T}^\bullet\mathbf{U}\to\mathcal{T}^\bullet\mathbf{W}$, generally restricted to a map $\mathbf{X}\to\mathcal{T}^\bullet\mathbf{W}$. 47

$\prec$**:** partial order on relevant elements. 110

$\mathrm{Rel}_C(x,y)$**:** the set of $V\in\mathfrak{F}$ with $\mathsf{d}_V(x,y)\geqslant C$. 109

$\rho^{\mathbf{W}}_{\mathbf{V}}:\mathcal{T}^\bullet\mathbf{W}\to\mathcal{T}^\bullet\mathbf{V}$**:** the map that is defined whenever $\mathbf{V}\subsetneq\mathbf{W}$ (the bounded geodesic image condition ensures that $\rho^{\mathbf{W}}_{\mathbf{V}}$ is constant on each component of $\pi_{\mathbf{W}}(\mathbf{X})-\{\rho^{\mathbf{V}}_{\mathbf{W}}\}$). 47

$\rho^{\mathbf{V}}_{\mathbf{W}}$**:** the point in $\mathcal{T}^\bullet\mathbf{W}$ associated to $\mathbf{V}$, that exists whenever $\mathbf{V}\pitchfork\mathbf{W}$ or $\mathbf{V}\subsetneq\mathbf{W}$ (the position of $\rho^{\mathbf{V}}_{\mathbf{W}}$ is constrained by the consistency conditions). 47

ρ^V_W**:** ρ-sets and maps in HHS. 104

$s_{\mathbf{U}}(\mathbf{x},\mathbf{y})$**:** s–distance in a median space with poset-colouring. 65

σ_A**:** filter of halfspaces that contain A. 28

$\sigma_{\mathbf{U}}$**:** filter associated to a colour. 34

$\mathcal{C}W$**:** the δ–hyperbolic geodesic metric spaces associated to $W\in\mathfrak{F}$. 104

$\mathcal{T}^\bullet\mathbf{W}$**:** the $\mathbb{R}$–tree associated to $\mathbf{W}\in\mathfrak{F}^\bullet$. 47

$t_{\mathbf{U}}(\mathbf{x},\mathbf{y})$**:** t–distance in a median space with poset-colouring. 64

$\pitchfork$**:** the relation of transversality on $\mathfrak{F}^\bullet$ or $\mathfrak{F}$. 47, 104

$\mathcal{V}$**:** the set of $V\in\mathrm{Rel}_C(x,y)$ such that $U\subsetneq V$. 109

$\mathcal{W}_{\mathbf{U}}$**:** set of walls with colour nested in $\mathbf{U}$. 33

$\mathcal{W}$**:** set of all walls. 26

$\mathbf{X}(\mathbf{U},\mathbf{V})$**:** semialgebraic set defined by a cubical system relative to $I(\mathbf{U}),I(\mathbf{V})$. 91

$\mathbf{X}$**:** the underlying space of an $\mathbb{R}$–cubing. 47

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