

DEHN FILLING DEHN TWISTS

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ABSTRACT. Let $\Sigma_{g,p}$ be the genus- g oriented surface with p punctures, with either $g > 0$ or $p > 3$. We show that $MCG(\Sigma_{g,p})/DT$ is acylindrically hyperbolic where DT is the normal subgroup of the mapping class group $MCG(\Sigma_{g,p})$ generated by K^{th} powers of Dehn twists about curves in $\Sigma_{g,p}$ for suitable K .

Moreover, we show that in low complexity $MCG(\Sigma_{g,p})/DT$ is in fact hyperbolic. In particular, for $3g - 3 + p \leq 2$, we show that the mapping class group $MCG(\Sigma_{g,p})$ is fully residually non-elementary hyperbolic and admits an affine isometric action with unbounded orbits on some L^q space. Moreover, if every hyperbolic group is residually finite, then every convex-cocompact subgroup of $MCG(\Sigma_{g,p})$ is separable.

The aforementioned results follow from general theorems about composite rotating families, in the sense of [Dah18], that come from a collection of subgroups of vertex stabilisers for the action of a group G on a hyperbolic graph X . We give conditions ensuring that the graph X/N is again hyperbolic and various properties of the action of G on X persist for the action of G/N on X/N .

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INTRODUCTION

Thurston’s Dehn filling theorem has an algebraic counterpart in the context of relatively hyperbolic groups [Osi07, GM08], which has numerous important applications such as in the proof of the Virtual Haken conjecture [Ago13] and in the solution of the isomorphism problem for certain relatively hyperbolic groups [DG18, DT19]. A Dehn filling of a relatively hyperbolic group G is the quotient of G by the normal closure of normal subgroups $N_i \triangleleft H_i$ of its peripheral subgroups H_i , and the Dehn filling theorem for relatively hyperbolic groups says that such quotients are still relatively hyperbolic provided that the N_i are sufficiently “sparse”. Mapping class groups are not non-trivially relatively hyperbolic except in very low complexity [AAS07, BDM09], but it is still natural to think of their subgroups generated by Dehn twists around curves in a pants decomposition as peripheral subgroups. Hence, we think of the following theorem (Theorem 5.2 below), as a Dehn filling theorem for mapping class groups:

Theorem 1. *Suppose $g > 0$ or $p > 3$, and consider $\Sigma_{g,p}$, an oriented surface of genus g with p punctures. There exists a positive integer K_0 so that for all non-zero multiples K of K_0 , if DT_K denotes the normal subgroup of the mapping class group $MCG(\Sigma_{g,p})$ generated by all K^{th} powers of Dehn twists, then the group $MCG(\Sigma_{g,p})/DT_K$ is acylindrically hyperbolic.*

Residual properties of mapping class groups in low complexity.

Recall that the complexity of $\Sigma_{g,p}$ is defined as $\xi(g,p) = 3g + p - 3$. In low complexity, we can press a bit further and study residual properties of mapping class groups. First, we produce many hyperbolic quotients. Recall that, given a class of groups \mathcal{P} , a group G is fully residually \mathcal{P} if for every finite set $F \subseteq G - \{1\}$ there exists a group H in \mathcal{P} and a surjective homomorphism $\phi : G \rightarrow H$ with $\phi(F) \subseteq H - \{1\}$.

Theorem 2. *Suppose that $(g,p) \in \{(0,5), (1,2)\}$. Then $MCG(\Sigma_{g,p})$ is fully residually non-elementary hyperbolic.*

It was not previously known whether either of $MCG(\Sigma_{0,5})$ or $MCG(\Sigma_{1,2})$ admits an infinite hyperbolic quotient.

We deduce Theorem 2 from a different statement, Theorem 5.8, which is interesting even in complexity 1. Specifically, for $(g,p) \in \{(0,4), (1,0), (1,1)\}$,

this says that $MCG(\Sigma_{g,p})/DT_K$ is hyperbolic for all suitably large multiples K of the constant K_0 from Theorem 5.8, and when $(g, p) \in \{(0, 5), (1, 2)\}$, the quotient $MCG(\Sigma_{g,p})/DT_K$ is hyperbolic relative to subgroups isomorphic to $MCG(\Sigma_{g',p'})/DT_K$ with $(g', p') \in \{(0, 4), (1, 1)\}$ and is therefore again hyperbolic.

From Theorem 2 and results of Yu and Nica (see also Alvarez and Laforgue), we obtain:

Corollary 3. *Suppose that $(g, p) \in \{(0, 5), (1, 2)\}$. Then $MCG(\Sigma_{g,p})$ admits an affine isometric action with unbounded orbits on some L^q space.*

Specifically, Theorem 2 yields a non-elementary hyperbolic quotient Q , which in turn admits a proper affine isometric action on $L^q(\partial Q \times \partial Q)$ and $\ell^q(Q \times Q)$, by [Nic13, Yu05, AL17], for sufficiently large q .

This is related to the question of which mapping class groups have property (T) because, if Q can be chosen as above so that one could take $q = 2$, we would get an affine isometric action of $MCG(\Sigma_{g,p})$, with unbounded orbits, on a Hilbert space.

Recall from [FM02] that a subgroup $H \leq MCG(\Sigma_{g,p})$ is *convex-cocompact* if some (hence any) H -orbit in the Teichmüller space of $\Sigma_{g,p}$ is quasiconvex. There are several equivalent characterisations of convex-cocompactness, see [KL08, Ham05, DT15], and one reason this notion is interesting is its connection with hyperbolicity of fundamental groups of surface bundles over surfaces [FM02, Ham05].

In [Rei06], Reid posed the question of whether convex-cocompact subgroups of $MCG(\Sigma_{g,p})$ are separable. Recall that a subgroup $H < G$ is separable if for every $x \in G - H$ there exists a finite group F a surjective homomorphism $\phi : G \rightarrow F$ with $\phi(x) \notin \phi(H)$.

Note that in general $MCG(\Sigma_{g,p})$ contains non-separable subgroups, and in fact this is already the case for $MCG(\Sigma_{0,5})$ [LM07]. (Nonetheless, various geometrically natural subgroups, e.g. curve-stabilisers, are known to be separable in $MCG(\Sigma_{g,p})$ [LM07].)

The techniques of the present paper engage with this question in a somewhat mysterious way:

Theorem 4. *Assume that all hyperbolic groups are residually finite. Suppose $(g, p) \in \{(0, 5), (1, 2)\}$. Then any convex-cocompact subgroup $Q < MCG(\Sigma_{g,p})$ is separable.*

The proof of Theorem 4 relies on the hyperbolic quotients of $MCG(\Sigma_{g,p})$ arising in the proof of Theorem 2. The extra work, done in Proposition 5.12, is to show that in all but finitely many such quotients, the subgroup Q survives as a quasiconvex subgroup, and arbitrary $g \in \Sigma_{g,p} \setminus Q$ can be separated from Q in such quotients. At this point, the assumption about residual finiteness is invoked: using a result of [AGM09] (namely, if all hyperbolic groups are residually finite then all hyperbolic groups are QCERF), we then separate the image of g from this quasiconvex subgroup in a finite quotient.

More general context and proof strategy. In order to prove our results we actually work in a more general context (that we plan on using in future work, see the next subsection). Roughly speaking, we consider a group G acting on a hyperbolic graph X so that the vertex set of X fits the framework of composite projection systems introduced in [Dah18], and consider quotients G/N of G by normal subgroups N generated by subgroups of vertex stabilisers consisting of “big rotations”. The main technical innovation we introduce in this paper is our method for proving that the graph X/N is hyperbolic. The strategy is as follows (see Proposition 4.3). Consider a geodesic triangle in X/N . We can lift the 3 sides of the triangle to a concatenation of 3 geodesics, which need not close up. If it doesn’t close up, there is a non-trivial element $n \in N$ that maps the initial vertex x of the concatenation to the terminal vertex nx . We then want to change the lifts so that the new concatenation is either a triangle, or at least n is “simpler”. What allows us to do this is Corollary 3.6, which we think of as an analogue of the Greendlinger lemma from [DGO17]. What the corollary says is that we have a large rotation γ_v around some vertex v so that $\gamma_v n$ is “simpler”, and v needs to lie within distance 1 of one of the lifts. Applying γ_v to part of the concatenation yields new lifts with the required property. The measure of complexity of elements of N is actually rather complicated, but the same proof strategy works in other contexts as long as there is a version of the Greendlinger lemma so that some notion of complexity gets reduced when applying it. In particular, it can replace the use of the Cartan-Hadamard Theorem (for hyperbolic space) in the context of very rotating families from [DGO17] (thereby making for a simpler proof). On the other hand, Cartan-Hadamard cannot be applied to X/N in our context, and in fact we could not see any way to apply it to any space quasi-isometric to it.

Future work in the hierarchically hyperbolic setting. A natural strategy for extending the above applications beyond complexity 2 involves combining the techniques in the present paper with the theory of *hierarchically hyperbolic groups* [BHS17b, BHS15] (of which mapping class groups are one of the “type species”).

We believe that the quotients $MCG(\Sigma_{g,p})/DT_K$ are in fact hierarchically hyperbolic. One could then apply again our results on composite rotating graphs, and take further quotients. The complexity (in the hierarchically hyperbolic sense) decreases at each step, and hierarchically hyperbolic groups of minimal complexity are known to be hyperbolic [BHS17a]. In particular, under the assumption that every hyperbolic group is residually finite, one should be able to prove that for arbitrary g, p , the group $MCG(\Sigma_{g,p})$ admits a non-elementary hyperbolic quotient, and that every convex-cocompact subgroup is separable.

Outline of the paper. In Section 1, we recall the notions of *composite projection systems* and *composite rotating families* from [Dah18] and establish

some useful facts, relying on the *transfer lemma* from [Dah18]. The notion of a composite projection system relies on the projection axioms from [BBF15].

In Section 2, we introduce hyperbolic graphs into the picture, and define the notion of a composite projection graph and a composite rotating family on it. Here, we state our main technical result, Theorem 2.1, and give a proof which relies on statements proved in subsequent sections.

The reader mainly interested in the proof of Theorem 2.1 is advised to focus on the main result of Section 3, which is Corollary 3.6; this is the statement, mentioned above, that allows one to lower the “complexity” of elements $n \in N$ by applying large rotations. This is used in Section 4 to construct the lifts mentioned above. From the lifting procedure, one obtains the facts used in the proof of Theorem 2.1. In Section 4, we also prove Proposition 4.8, which describes how stabilisers of vertices in X intersect N . This is not used in the proof of Theorem 2.1, but does play a role in Section 5.

Finally, in Section 5, we consider the case of $MCG(\Sigma_{g,p})$ acting on the curve graph $\mathcal{C}(\Sigma_{g,p})$, which is a composite projection graph by Proposition 5.1 (see also [Dah18]). Large powers of Dehn twists generate a collection of rotation subgroups forming a composite rotating family, and we can invoke Theorem 2.1 to obtain Theorem 1. The rest of Section 5 is devoted to the proofs of Theorem 2 and Theorem 4.

1. COMPOSITE PROJECTION SYSTEMS AND ROTATING FAMILIES

We now recall the notion of a composite projection system from [Dah18], and establish some basic facts. The reader familiar with mapping class groups might want to keep in mind that in that context \mathbb{Y}_* is the collection of (isotopy classes of simple closed) curves, that two curves are active if they intersect, and that d_y^π is defined using subsurface projection.

1.1. Composite projection systems. Given y in a partitioned set $\mathbb{Y}_* = \sqcup_{i=1}^m \mathbb{Y}_i$, denote by $i(y)$ the index such that $y \in \mathbb{Y}_{i(y)}$.

Definition 1.1. [Dah18, Definition 1.2] Let \mathbb{Y}_* be the disjoint union of finitely many countable sets $\mathbb{Y}_1, \dots, \mathbb{Y}_m$. A *composite projection system* on $(\mathbb{Y}_i)_{i=1..m}$ (or on \mathbb{Y}_*), for the constant θ , consists of

- a family of subsets $\text{Act}(y) \subset \mathbb{Y}_*$ for $y \in \mathbb{Y}_*$ (the *active set* for y) such that $\mathbb{Y}_{i(y)} \subset \text{Act}(y)$, and such that $x \in \text{Act}(y)$ if and only if $y \in \text{Act}(x)$ (*symmetry in action*),
- and a family of functions $d_y^\pi : (\text{Act}(y) \setminus \{y\})^2 \rightarrow \mathbb{R}_+$, satisfying:
 - **Symmetry:** $d_y^\pi(x, z) = d_y^\pi(z, x)$ for $x, z \in \text{Act}(y) \setminus \{y\}$;
 - **Triangle inequality:** $d_y^\pi(w, x) \leq d_y^\pi(w, z) + d_y^\pi(z, x)$ for all $w, x, z \in \text{Act}(y) \setminus \{y\}$;
 - **Behrstock inequality:** $\min\{d_y^\pi(x, z), d_z^\pi(x, y)\} \leq \theta$ whenever both quantities are defined;
 - **Properness:** $|\{y \in \mathbb{Y}_i, d_y^\pi(x, z) \geq \theta\}| < \infty$ for all $x, z \in \mathbb{Y}_i$;

- **Separation:** $d_y^\pi(z, z) < \theta$ for $z \in \text{Act}(y) \setminus \{y\}$;
- **Closeness in inaction:** if $x \notin \text{Act}(z)$ then, for all $y \in \text{Act}(x) \cap \text{Act}(z)$, we have $d_y^\pi(x, z) \leq \theta$;
- **Finite filling:** for all $\mathcal{Z} \subset \mathbb{Y}_*$, there is a finite collection $x_j \in \mathcal{Z}$ such that $\cup_j \text{Act}(x_j)$ covers $\cup_{x \in \mathcal{Z}} \text{Act}(x)$.

By [BBF15, Theorem 3.3], for each $i \leq m$, and $y \in \mathbb{Y}_i$, and for a suitable choice of θ , there exists a modified function $d_y : \mathbb{Y}_i \times \mathbb{Y}_i \rightarrow \mathbb{R}_+$, satisfying the *monotonicity* property of [BBF15, Theorem 3.3] (see also [BBFS17, Axiom (SP3)] for a strengthened property).

This function is unfortunately not defined on $\text{Act}(y) \setminus \mathbb{Y}_i$. However, d_y^π is defined on all of $\text{Act}(y)$. Therefore, we define $d_y^\triangleleft : (\text{Act}(y) \setminus \{y\})^2 \rightarrow \mathbb{R}_+$ as follows. Let $x, z \in \text{Act}(y) \setminus \{y\}$. If $x, z \in \mathbb{Y}_i$, we let $d_y^\triangleleft(x, z) = d_y(x, z)$, and otherwise, we let $d_y^\triangleleft(x, z) = d_y^\pi(x, z)$.

Let $\mathbb{Y}_M^j(x, z) = \{y \in \mathbb{Y}_j \cap \text{Act}(x) \cap \text{Act}(z), d_y^\triangleleft(x, z) \geq M\}$ (the set of M -large projections between x and z in the j -coordinate). The elements x, y, z need not be in the same coordinate.

We now introduce the first of various constraints on the constants that will appear. Fix a composite projection system with constant θ . Let $\Theta = \Theta(\theta)$ be the constant provided by applying Theorem 3.3 of [BBF15] to each \mathbb{Y}_i to obtain the maps d_y as above. In particular, d_y now has the monotonicity property: if $d_y(x, z) \geq \Theta$, then $d_w(x, y), d_w(y, z) \leq d_w(x, z)$ where $w, x, y, z \in \mathbb{Y}_i$. Within \mathbb{Y}_i , the maps d_y continue to be symmetric and satisfy the properness property, with Θ replacing θ . The same theorem also provides a constant κ so that, for all pairwise distinct $x, y, w, z \in \mathbb{Y}_i$, we have

- $d_y^\pi - \kappa \leq d_y \leq d_y^\pi$;
- $d_y(x, z) - \kappa \leq d_y(x, w) + d_y(z, w)$;
- $\min\{d_y(x, z), d_x(y, z)\} \leq \kappa$.

We emphasise that the constants κ, Θ have been chosen so that the above properties hold within each \mathbb{Y}_i .

Remark 1.2. From the proof of [BBF15, Theorem 3.3], we see that we can take $\Theta = 4\theta + 1$. Indeed, any choice of $\Theta > 4\theta$ guarantees all of the properties of d_y that we will need. We can also take any $\kappa \geq 3\theta$, by [BBF15, Proposition 3.2].

1.2. Composite rotating families. We now recall the notion of a composite rotating family. The main idea to keep in mind is that we want Γ_x to consist of “large rotations” around x , where d_x^\triangleleft is thought of as the angle at x .

Definition 1.3. (Composite rotating family) Consider a composite projection system \mathbb{Y}_* endowed with an action of a group G by isomorphisms, i.e. G acts on \mathbb{Y}_* , preserving the partition $\mathbb{Y}_* = \bigsqcup_{i=1}^m \mathbb{Y}_i$, and satisfying $\text{Act}(gy) = g \text{Act}(y)$ for all $g \in G$ and $y \in \mathbb{Y}_*$. Moreover, suppose that if $x, y, z \in \mathbb{Y}_*$ are such that $d_y^\triangleleft(x, z)$ is defined, then $d_{gy}^\triangleleft(gx, gz) = d_y^\triangleleft(x, z)$ for all $g \in G$.

A *composite rotating family* on (\mathbb{Y}_*, G) , with rotating control $\Theta_{rot} > 0$ is a family of subgroups $\Gamma_v, v \in \mathbb{Y}_*$ such that

- for all $x \in \mathbb{Y}_*$, $\Gamma_x < \text{Stab}_G(x)$, is an infinite group;
- Γ_x acts by **rotations** around x (i.e. whenever $y = x$ or $y \notin \text{Act}(x)$, the subgroup Γ_x fixes y and d_y^{\triangleleft}), with
- **proper isotropy** (i.e. for all $R > 0, y \in \text{Act}(x)$, the set $\{\gamma \in \Gamma_x, d_x^{\triangleleft}(y, \gamma y) < R\}$ is finite);
- for all $g \in G$, and all $x \in \mathbb{Y}_*$, we have $\Gamma_{gx} = g\Gamma_x g^{-1}$;
- if $x \notin \text{Act}(z)$ then Γ_x and Γ_z commute;
- for all $i \leq m$ and for all $x, y, z \in \mathbb{Y}_i$, if $d_y(x, z) \leq \Theta$, then

$$d_y(x, gz) \geq \Theta_{rot}$$

for all $g \in \Gamma_y \setminus \{1\}$.

Standing assumptions 1.4. From now and until the end of the subsection, we fix a composite rotating family, and we use the notation from Definition 1.3. Moreover, we assume that the constants are chosen as in Remark 1.2. Finally, we set $\Theta_0 = 2\Theta + 3\kappa$.

Recall the useful transfer lemma, which allows one to reduce to “transfer” various configurations to a single coordinate.

Lemma 1.5. (*Transfer Lemma*, [Dah18, Lemma 1.4, Prop. 1.6]) *Let $x \in \mathbb{Y}_*$. For all i , there exists $x^t \in \mathbb{Y}_i$, such that*

- for all $\gamma_x \in \Gamma_x$, and all $z \in \mathbb{Y}_i$, one has $d_z^{\triangleleft}(\gamma_x x^t, x^t) \leq \Theta_0$, and
- for all $y \in \mathbb{Y}_i$ that is x -active, for all but finitely many elements γ_x of Γ_x , one has $d_y^{\triangleleft}(x, \gamma_x x^t) \leq \kappa$,
- there is $\gamma_x \in \Gamma_x$ so that for all $y \in \mathbb{Y}_i$ that is x -active, we have either $d_y^{\triangleleft}(x, \gamma_x x^t) \leq \kappa$, or $d_y^{\triangleleft}(x, x^t) \leq \kappa$.

Moreover, if Γ_x has a fixed point in \mathbb{Y}_i , we can choose x^t to be such a fixed point.

Remark 1.6. The reader familiar with the construction of projection complexes from [BBF15] will notice that the first bullet says that Γ_x has an orbit of diameter at most 1 in the projection complex of \mathbb{Y}_i . This is in fact the defining property of x^t in [Dah18], and the reason why the “moreover” part holds.

Remark 1.7. Notice that by the first and third bullets, we have $d_y^{\triangleleft}(x, x^t) \leq \Theta_0 + \kappa$ (regardless of which case from the third bullet applies).

Corollary 1.8. *Let $x, y \in \mathbb{Y}_*$ so that x is y -active, and let $\gamma_y \in \Gamma_y - \{1\}$. Then $d_y^{\triangleleft}(x, \gamma_y x) \geq \Theta_{rot} - 2(\Theta_0 + 2\kappa)$.*

Proof. Let x^t be as in Lemma 1.5. Recall that we have $d_y^{\triangleleft}(x, x^t) \leq \Theta_0 + \kappa$ (Remark 1.7). By equivariance, $d_y^{\triangleleft}(\gamma_y x, \gamma_y x^t) \leq \Theta_0 + \kappa$, and the conclusion follows from the (approximate) triangular inequality for d_y^{\triangleleft} . \square

Also, the transfer lemma allows one to transfer properness, from Definition 1.1, which will be useful.

Lemma 1.9. *For all i and all $x, z \in \mathbb{Y}_i$ the set $I = \{y \in \mathbb{Y}_*, d_y^\pi(x, z) \geq \theta + 2\kappa\}$ is finite.*

Proof. Assume it is infinite, and extract an infinite family of elements of the same coordinate j . We use Lemma 1.5 to transfer x and z in \mathbb{Y}_j : for x^t and z^t as in the transfer lemma, we have that for all $y \in \mathbb{Y}_j \cap I$, and for $\phi_x(y)$ either the specific element $\gamma_x \in \Gamma_x$ from the lemma (third point), or the identity, (and ϕ_z similarly), one has $d_y^\pi(\phi_x(y)x^t, \phi_z(y)z^t) > \theta$. One may extract an infinite family of elements y for which the $\phi_x(y)$ are all equal, and the $\phi_z(y)$ are all equal. This provides two elements $\phi_x x^t$ and $\phi_z z^t$ of \mathbb{Y}_j such that $\{y \in \mathbb{Y}_j, d_y^\pi(\phi_x x^t, \phi_z z^t) \geq \theta\}$ is infinite, which is a contradiction with properness. \square

Notation 1.10. Given a composite rotating family, let $N \leq G$ be the subgroup generated by all the subgroups $\Gamma_x, x \in \mathbb{Y}_*$.

2. COMPOSITE PROJECTION GRAPHS

We say that (X, \mathbb{Y}_*) is a *G-composite projection graph* if

- (1) X hyperbolic graph and G acts on X by simplicial automorphisms.
- (2) **Composite projection system:** $\mathbb{Y}_* = X^{(0)}$ has the structure of a composite projection system on which G acts by isomorphisms. We let θ be the constant from Definition 1.1 and let Θ, κ be the constants, depending on θ , from the discussion following that definition.
- (3) **Bounded geodesic image (BGIT):** There exists C so that the following holds. For each $x, y, s \in \mathbb{Y}_*$ so that $d_s(x, y)$ is defined and larger than C , on any geodesic $[x, y]$ there exists a vertex w with $d_X(w, s) = 1$.

Moreover, $(\{\Gamma_s\}, \mathbb{Y}_*^\tau)$ is a *composite rotating family* with constant Θ_{rot} on the *G-composite projection graph* (X, \mathbb{Y}_*) if:

- (1) $\{\Gamma_s\}$ is a composite rotating family on a composite projection system $\mathbb{Y}_*^\tau \subseteq \mathbb{Y}_*$, with constant Θ_{rot} .
- (2) Γ_s fixes any w with $d_X(w, s) = 1$.

Our main technical statement is about properties of the action of G on X that persist for the action of G/N on X/N when the rotations are sufficiently large.

Recall that, given a group G acting on a metric space X , the element $g \in G$ is *WPD* (weakly proper discontinuous) if for every $x_0 \in X$ and $r \geq 0$ there exists n_0 so that for all $n \geq n_0$ the set

$$\{h \in G : d_X(x_0, hx_0), d_X(g^n x_0, hg^n x_0) \leq r\}$$

is finite.

Our main goal in this section is to prove Theorem 2.1. The proof refers to various statements which are postponed to subsequent sections, so that the high-level strategy is made clear before the technicalities are introduced.

Theorem 2.1. *Let (X, \mathbb{Y}_*) be a G -composite projection graph. If Θ_{rot} is sufficiently large, in terms of $X, G, \mathbb{Y}_*, \mathbb{Y}_*^T, C$, then for $N = \langle \{\Gamma_s\} \rangle$, we have:*

- X/N is hyperbolic.
- If the action of G on X has a loxodromic then so does the action of G/N on X/N . If the action of G on X has a WPD element, then so does the action of G/N on X/N . If the action of G on X is non-elementary, then so is the action of G/N on X/N .

Proof. We will refer to three facts established in the next section, Corollary 4.5, Proposition 4.6, and Lemma 4.7. Let θ, Θ, κ be the constants from above, which depend on the composite projection system, and recall that C is the BGIT constant associated to the composite rotating family. Suppose that Θ_{rot} satisfies $\frac{3}{10}\Theta_{rot} > \frac{5}{2}\Theta + 3\kappa + 2C$.

Corollary 4.5 implies that X/N is hyperbolic.

Suppose that $\gamma \in G$ acts loxodromically on X . Suppose, moreover, that there exists $x_0 \in X$ such that $d_s(x_0, \gamma^n x_0) < \Theta_{rot}/10$ for all $n \in \mathbb{Z}$ and all s for which the preceding quantity is defined. Then Proposition 4.6 ensures that the image $\bar{\gamma} \in G/N$ of γ is loxodromic on X/N , and, moreover, if γ acts on X as a WPD isometry, then $\bar{\gamma}$ acts on X/N as a WPD isometry. We need to show that for a suitably large Θ_{rot} , there exists a loxodromic (resp. loxodromic WPD) element $\gamma \in G$ with the desired small-projection property from Proposition 4.6.

Fix a base vertex $x_0 \in X$ and let $\gamma \in G$ be loxodromic on X (we choose it loxodromic WPD if there is such an element in G). We now show that $d_s(x_0, \gamma^n x_0)$ is uniformly bounded whenever it is defined. This is ultimately a consequence of BGIT, which implies that the “tails” of the orbit of γ do not affect projection distances very much.

Let $L_\gamma = L_\gamma(x_0)$ be the supremum over all $n \in \mathbb{Z}$, and over all s for which the quantity is defined, of $d_s(x_0, \gamma^n x_0)$. We claim that $L_\gamma < \infty$.

Indeed, let δ be the hyperbolicity constant for X . Then there exists $\mu = \mu(\gamma, \delta)$ such that each geodesic $[x_0, \gamma^n x_0]$ lies at Hausdorff distance at most μ from the (quasigeodesic) sequence $x_0, \gamma x_0, \dots, \gamma^n x_0$.

Fix s . By BGIT, either $d_s(x_0, \gamma^n x_0) \leq C$, or s is adjacent to some vertex of $[x_0, \gamma^n x_0]$. Hence there exist integers p, q with $1 \leq p \leq q \leq n$ and $|p - q|$ depending only on γ , so that s is not adjacent to a vertex of $[x_0, \gamma^i x_0]$ or $[\gamma^i x_0, \gamma^n x_0]$ unless $p \leq i \leq q$. Thus $d_s(x_0, \gamma^n x_0) \leq d_s(\gamma^p x_0, \gamma^q x_0) + 2C$, by BGIT and the triangle inequality. It follows that $d_s(x_0, \gamma^n x_0) \leq 2C + d_{\gamma^{-p}s}(x_0, \gamma^{q-p} x_0)$. Now, either $d_{\gamma^{-p}s}(x_0, \gamma^{q-p} x_0) \leq \theta + 2\kappa$, in which case $d_s(x_0, \gamma^n x_0) \leq \theta + 2\kappa + 2C$, or $\gamma^{-p}s$ is one of finitely many elements for which $d_{\gamma^{-p}s}(x_0, \gamma^{q-p} x_0) > \theta + 2\kappa$, by Lemma 1.9. Letting L'_γ be the maximum of $d_{\gamma^{-p}s}(x_0, \gamma^{q-p} x_0)$ over these finitely many elements gives $d_s(x_0, \gamma^n x_0) \leq L'_\gamma + 2C$. Hence $L_\gamma < \infty$ for all γ .

Now let $L = \inf L_\gamma$, where the infimum is taken over the set of $\gamma \in G$ that are loxodromic on X . Note that L depends only on the composite projection system, its associated constants, the BGIT constant C , and the G -action, but not on the choice of rotation subgroups. Suppose that $\Theta_{rot} > 10(L + 1)$. Then, any loxodromic γ with $L_\gamma \leq L + 1$ satisfies the hypothesis of Proposition 4.6 and thus has image $\bar{\gamma}$ which is loxodromic on X/N , and WPD if γ is itself WPD.

Finally, suppose that γ is as above and that γ' is a loxodromic element that is independent of γ and has the property that $\bar{\gamma}'$ is also loxodromic on X/N . (If G contains independent loxodromics γ, γ_1 , then we can choose γ' to be a conjugate of γ by a sufficiently high power of γ_1 , and see from the above argument that $\bar{\gamma}'$ is again loxodromic on X/N .)

Given such a pair γ, γ' , let $L_{\gamma, \gamma'}$ be the supremum, over all $m, n \in \mathbb{Z}$ and all s where the following quantity is defined, of $d_s(\gamma^m x_0, (\gamma')^n x_0)$. We claim that $L_{\gamma, \gamma'} < \infty$. Indeed, since γ, γ' are independent loxodromics, there exists μ such that we have the following. For all $m, n \in \mathbb{Z}$, any geodesic $[\gamma^m x_0, (\gamma')^n x_0]$ lies at Hausdorff distance $\leq \mu$ from

$$\{\gamma^m x_0, \gamma^{m-1} x_0, \dots, x_0\} \cup \{x_0, \gamma' x_0, \dots, (\gamma')^n x_0\}.$$

We can now argue as above, using BGIT and replacing $\{x_0, \gamma x_0, \dots, \gamma^n x_0\}$ by $\{\gamma^m x_0, \gamma^{m-1} x_0, \dots, x_0\} \cup \{x_0, \gamma' x_0, \dots, (\gamma')^n x_0\}$.

Now, letting γ, γ' vary over all pairs of independent loxodromic elements of G such that $\bar{\gamma}, \bar{\gamma}' \in G/N$ are loxodromic on X/N , take $M = \inf_{\gamma, \gamma'} L_{\gamma, \gamma'}$. Suppose that $\Theta_{rot} > 10(M + 1)$. Then any pair $\gamma, \gamma' \in G$ of independent loxodromics such that $\bar{\gamma}, \bar{\gamma}'$ are loxodromic and $L_{\gamma, \gamma'} \leq M + 1$ has the property that $\bar{\gamma}, \bar{\gamma}'$ are independent loxodromics, by Lemma 4.7. Hence, if $\Theta_{rot} > 10(M + 1)$, the action of G/N on X/N is non-elementary. \square

3. SHORTENINGS AND THEIR APPLICATIONS

We work in the setting of Section 2, keeping all notation.

The results of this section support the lifting procedure developed in Section 4, which is vital for proving the statements (Proposition 4.6, Corollary 4.5, Lemma 4.7) used in the proof of Theorem 2.1.

The main statements are Corollary 3.6 and Proposition 3.5, on which the corollary depends. In fact, the reader interested in the proof of Theorem 2.1 is advised to read the statement of Corollary 3.6 and then proceed to Section 4. In order to understand the statement of Corollary 3.6, one needs to know the following. For each $\gamma \in N \setminus \{1\}$, there is an associated complexity $(\alpha(\gamma), n(\gamma))$, where $\alpha(\gamma)$ is a countable ordinal and $n(\gamma) \in \mathbb{N}$, and Θ_{short} is a constant depending on Θ_{rot} , Θ_0 , and κ . (This value is one of the sources of the ‘‘sufficiently large’’ constraint on Θ_{rot} in Theorem 2.1.)

3.1. Structure of the kernel N and complexity of elements. The aim of [Dah18] was to investigate the structure of N . We may extract the

following statement, combining the construction from [Dah18, §2.4.2] with [Dah18, Lem 2.16, Prop. 2.13, Lem 2.17]:

Theorem 3.1. *For any countable ordinal α there exists a subset $\mathbb{Y}_{*\alpha}$ of \mathbb{Y}_* such that, denoting by N_α the subgroup of N generated by $\Gamma_v, v \in \mathbb{Y}_{*\alpha}$, we have:*

- (1) $N_0 = \{1\}$ and $\mathbb{Y}_{*0} = \emptyset$
- (2) *if α is not a limit ordinal, there exists $i(\alpha)$, and a subset $R_\alpha \subset \mathbb{Y}_{i(\alpha)}$, such that N_α is an amalgamated free product of $N_{\alpha-1}$ with the groups*

$$\Gamma_v \times \langle \Gamma_w, w \in (\mathbb{Y}_{*\alpha-1} \setminus \text{Act}(v)) \rangle$$

for $v \in R_\alpha$. ([Dah18, Lem. 2.16])

Notice that each element in the N_α -orbit of R_α naturally corresponds to a vertex in the Bass-Serre tree of the previous decomposition; from now on we implicitly identify any such element with the corresponding vertex.

- (3) *Suppose that α is not a limit ordinal, and let T_α be the Bass-Serre tree of the previous decomposition. Also, let v_1, v_2, v_3 be three vertices of T_α in the N_α -orbit of the vertices of R_α , with $v_2 \in [v_1, v_3]$ in T_α . Then, when seen as elements in $\mathbb{Y}_{i(\alpha)}$, one has $d_{v_2}^<(v_1, v_3) \geq \Theta_{rot} - \Theta_0$, and there exists $\gamma_{v_2} \in \Gamma_{v_2}$ such that $d_{v_2}^<(v_1, \gamma_{v_2}v_3) \leq \Theta_0$. ([Dah18, Lem. 2.16 with Prop 2.13])*
- (4) *if α is a limit ordinal, N_α is the direct union of $N_\beta, \beta < \alpha$. ([Dah18, Lem. 2.17])*
- (5) $\bigcup_\alpha \mathbb{Y}_{*\alpha} = \mathbb{Y}_*$. ([Dah18, Lem. 2.19])

We can now define the complexity $(\alpha(\gamma), n(\gamma))$ of $\gamma \in N \setminus 1$.

Definition 3.2. Given $\gamma \in N$, let $\alpha(\gamma)$ be the smallest ordinal α for which γ is in a conjugate of N_α in N . Observe that $\alpha(\gamma)$ is never a limit ordinal, by the third point of Theorem 3.1, and if $\alpha(\gamma) = 0$ then $\gamma = 1$.

Definition 3.3. Given $\gamma \in N \setminus \{1\}$, consider the amalgamated free product decomposition of a conjugate of $N_{\alpha(\gamma)}$ containing γ , given by the second point of Theorem 3.1.

Consider the cyclic normal form of the conjugacy class $[\gamma]$, which is either an element of Γ_v for some $v \in R_{\alpha(\gamma)}$, or a cyclic word $w = \ell_1 \ell_2 \dots \ell_{2r}$, where, for all i , we have $\ell_{2i} \in N_{\alpha(\gamma)-1} \setminus \{1\}$ and $\ell_{2i+1} \in \Gamma_v \setminus \{1\}$ for some $v \in R_{\alpha(\gamma)}$. Let $n(\gamma)$ be the length of this cyclic normal form, namely $n(\gamma) = 1$ if γ is conjugate into some Γ_v for some $v \in R_{\alpha(\gamma)}$, and it is $n(\gamma) = 2r$ for the above r otherwise.

Remark 3.4. Note that if $\gamma \in N_{\alpha(\gamma)}$ is seen as an element of $N_{\alpha(\gamma)+1}$, the length of its cyclic normal form is 1, but we do not set this in the notation $n(\gamma)$ since this notation is reserved to the amalgam decomposition of $N_{\alpha(\gamma)}$. This way, no $n(\gamma)$ has been multiply defined. We adopt the convention that $n(1) = 0$.

3.2. Angles and shortenings. The main point of the following proposition is to relate the normal form of γ with vertices at which one sees a large projection between γ -translates. The move that will allow us to shorten normal forms can be pictorially described as follows: Consider the axis of γ , a vertex v on the axis, and an element γ_v that stabilises v and rotates an edge on the axis containing v to the other such edge. Then $\gamma_v\gamma$ has shorter normal form than γ .

Proposition 3.5 (Angles and shortenings). *Let γ be an element of $N_{\alpha(\gamma)}$ with $n(\gamma) > 1$. Then all of the following hold:*

- *The element γ is hyperbolic in the tree $T_{\alpha(\gamma)}$, and its axis in this tree contains some vertex v_0 in the $N_{\alpha(\gamma)}$ -orbit of $R_{\alpha(\gamma)}$.*
- *For all v in the axis of γ and in the $N_{\alpha(\gamma)}$ -orbit of $R_{\alpha(\gamma)}$, there exists $\gamma_v \in \Gamma_v \setminus \{1\}$ such that the cyclic normal form of $[\gamma_v\gamma]$ is strictly shorter than that of $[\gamma]$.*
- *If w is a vertex in the tree $T_{\alpha(\gamma)}$, then there exists a vertex v in the $N_{\alpha(\gamma)}$ -orbit of $R_{\alpha(\gamma)}$ such that v lies in the intersection of the interior of the segment $[w, \gamma w]$ of $T_{\alpha(\gamma)}$, and of the axis of γ . Moreover, for all such v , one has $d_v^{\triangleleft}(w, \gamma w) > \Theta_{rot} - \Theta_0$.*
- *For all $\nu \in \mathbb{Y}_{i(\gamma)}$, there is a vertex v in the $N_{\alpha(\gamma)}$ -orbit of $R_{\alpha(\gamma)}$ on the axis of γ such that $d_v^{\triangleleft}(\nu, \gamma\nu) > (\Theta_{rot} - \Theta_0)/2 - \kappa$.*
- *Suppose that $j \neq i(\gamma)$ and let $\nu' \in \mathbb{Y}_j$. Then either there exists v in the $N_{\alpha(\gamma)}$ -orbit of $R_{\alpha(\gamma)}$ that is ν' -inactive, and $\gamma_v \in \Gamma_v$ so that $\gamma_v\gamma$ has shorter cyclic normal form than γ , or there exists v that is active for ν' and $\gamma\nu'$, with the property that $d_v^{\triangleleft}(\nu', \gamma\nu') > (\Theta_{rot} - \Theta_0)/2 - 2\Theta_0 - 3\kappa$.*

Proof. The first point is a general fact for elements in amalgamated free products. This is also true of the second point, in the specific setting where there is only one orbit of edges around v under the action of its stabilizer, as is the case in our situation. The first part of the third point is also general. The distance estimate of the third point follows from Theorem 3.1 (3).

For the fourth point, we can assume that ν is not a vertex of $T_{\alpha(\gamma)}$, for otherwise we can just use the third bullet. Consider the bi-infinite sequence of points $(\gamma^n v)_{n \in \mathbb{Z}}$ for some vertex v in the $N_{\alpha(\gamma)}$ -orbit of $R_{\alpha(\gamma)}$ on the axis of γ . This sequence can be thought of as ranging over points of the axis of γ in $T_{\alpha(\gamma)}$ or over points of $\mathbb{Y}_{i(\gamma)}$. Notice that $d_{\gamma^n v}^{\triangleleft}(\gamma^{n-1}v, \gamma^{n+1}v) \geq \Theta_{rot} - \Theta_0$ for every n , by the third point of Theorem 3.1. One can then see, using the Behrstock inequality and induction, that the set $\{n \in \mathbb{Z} : d_{\gamma^n v}^{\triangleleft}(\gamma^{n-1}v, \nu) \leq \kappa\}$ is an open interval $(n_0, +\infty)$, for $n_0 \in \mathbb{Z} \cup \{-\infty\}$ (notice that $d_{\gamma^n v}^{\triangleleft}(\gamma^{n-1}v, \nu)$ is defined for each n). Moreover, n_0 cannot be equal to $-\infty$, since in that case, for all negative n , $\gamma^n v$ would be in $\mathbb{Y}_{\Theta_{rot} - \Theta_0 - \kappa}^i(\nu, \gamma\nu)$, contradicting the properness of the projection system. Thus, $d_{\gamma^{n_0} v}^{\triangleleft}(\gamma^{n_0-1}v, \nu) > \kappa$, while $d_{\gamma^{n_0+1} v}^{\triangleleft}(\gamma^{n_0}v, \nu) \leq \kappa$. The Behrstock

inequality again ensures that $d_{\gamma^{n_0-1}v}^{\leftarrow}(\gamma^{n_0}v, \nu) \leq \kappa$. After translation by γ , we have $d_{\gamma^{n_0}v}^{\leftarrow}(\gamma^{n_0+1}v, \gamma\nu) \leq \kappa$.

There are two cases to consider. First, suppose that $d_{\gamma^{n_0}v}^{\leftarrow}(\gamma^{n_0-1}v, \nu) \leq (\Theta_{rot} - \Theta_0)/2$. In that case, the triangle inequality gives $d_{\gamma^{n_0}v}^{\leftarrow}(\nu, \gamma\nu) \geq (\Theta_{rot} - \Theta_0)/2 - \kappa$ (we used $d_{\gamma^{n_0}v}^{\leftarrow}(\gamma^{n_0-1}v, \gamma^{n_0+1}v) \geq \Theta_{rot} - \Theta_0$). The second case is when $d_{\gamma^{n_0}v}^{\leftarrow}(\gamma^{n_0-1}v, \nu) > (\Theta_{rot} - \Theta_0)/2$. Then $d_{\gamma^{n_0}v}^{\leftarrow}(\nu, \gamma^{-1}\nu) \geq (\Theta_{rot} - \Theta_0)/2 - \kappa$ (we used $d_{\gamma^{n_0}v}^{\leftarrow}(\gamma^{n_0-1}v, \gamma^{-1}\nu) = d_{\gamma^{n_0+1}v}^{\leftarrow}(\gamma^{n_0}v, \nu) \leq \kappa$). In both cases, we obtained the desired conclusion.

Let us prove the fifth point. The first case is when $\Gamma_{\nu'}$ has no fixed point in $\mathbb{Y}_{i(\gamma)}$. In that case, the conjugate $\gamma\Gamma_{\nu'}\gamma^{-1}$ has no fixed point either, since a fixed point for one would give a fixed point for the other by translation by γ^{-1} . Then for all $v \in \mathbb{Y}_{i(\gamma)}$ (so in particular in the axis of γ in the tree $T_{\alpha(\gamma)}$), ν' and $\gamma\nu'$ are v -active. We now consider ν'^t as in the transfer lemma, and we apply the previous point for ν'^t . There is v on the axis of γ such that $d_v^{\leftarrow}(\nu'^t, \gamma\nu'^t) > (\Theta_{rot} - \Theta_0)/2 - \kappa$. By Remark 1.7, we have $d_v^{\leftarrow}(\nu', \gamma\nu'^t) > (\Theta_{rot} - \Theta_0)/2 - 2\kappa - \Theta_0$.

Notice that $d_v^{\leftarrow}(\gamma\nu', \gamma\nu'^t) = d_{\gamma^{-1}v}^{\leftarrow}(\nu', \nu'^t)$, and the latter quantity is again bounded by $\Theta_0 + \kappa$ by Remark 1.7. Hence, we get

$$d_v^{\leftarrow}(\nu', \gamma\nu') > (\Theta_{rot} - \Theta_0)/2 - 3\kappa - 2\Theta_0.$$

Let us now treat the case where $\Gamma_{\nu'}$ has a fixed point in $\mathbb{Y}_{i(\gamma)}$. Pick one fixed point ν'^t and consider first the case where $\gamma\nu'$ is not active for one of the vertices of the axis. Then, by the second bullet, one could shorten the length of $[\gamma]$ using the element γ_v associated to this vertex in the normal form of $[\gamma]$, so we are done. We may thus assume that $\gamma\nu'$ is v active for all v in the axis. It follows that ν' is active for all v in the axis as well, and the argument of the previous case can be applied (in view of the ‘‘moreover’’ part of the transfer lemma). \square

3.3. Rotations to reduce complexity. For convenience, let $\Theta_{short} = (\Theta_{rot} - \Theta_0)/2 - 2\Theta_0 - 3\kappa$.

Corollary 3.6 (Rotating to reduce the complexity (α, n)). *For all $\gamma \in N \setminus \{1\}$, and all $x \in \mathbb{Y}_*$, there is (s, γ_s) (here $s \in \mathbb{Y}_*$ and $\gamma_s \in \Gamma_s$) so that $(\alpha(\gamma_s\gamma), n(\gamma_s\gamma)) < (\alpha(\gamma), n(\gamma))$ in lexicographic order and either*

- (1) x is s -inactive, or
- (2) x and γx are s -active and $d_s^{\leftarrow}(x, \gamma x) > \Theta_{short}$.

Proof. After conjugation by a suitable element h and replacing x with $h^{-1}x$, one can assume that $\gamma \in N_{\alpha(\gamma)}$. If $x \in \mathbb{Y}_{i(\gamma)}$ then we consider v as in the fourth bullet from Proposition 3.5 and set $s = v$. The cyclic normal form of $\gamma_s\gamma$ is shorter than that of γ by the second bullet from Proposition 3.5. Otherwise, if $x \in \mathbb{Y}_j$ with $j \neq i(\gamma)$, then by the fifth bullet from Proposition 3.5 we either proceed as above, or we find some x -inactive s so that $\gamma_s\gamma$ has shorter cyclic normal form.

In either case, the cyclic length of the conjugacy class is reduced. Either the result is still greater than 2, and in that case $\alpha(\gamma_s\gamma) = \alpha(\gamma)$, or it is reduced to 1 (or 0) and $\gamma_s\gamma$ is actually conjugate into $N_{\alpha-1}$. In the latter case, one has $\alpha(\gamma_s\gamma) < \alpha(\gamma)$. Thus $(\alpha(\gamma_s\gamma), n(\gamma_s\gamma)) < (\alpha(\gamma), n(\gamma))$. \square

Note that in the last case of the proof (in which $\alpha(\gamma_s\gamma) < \alpha(\gamma)$), the value of $n(\gamma_s\gamma)$ can be arbitrary.

4. LIFTING AND PROJECTING

In this section, we describe how to lift quadrilaterals and triangles in X/N to X . This will allow us to prove the various statements referenced in the proof of Theorem 2.1.

Standing assumptions 4.1. We fix the notation of Theorem 2.1, and fix constants as in Standing assumption 1.4. Let $p : X \rightarrow X/N$ be the quotient map. We assume that $3\Theta_{rot}/10 > 2C + 3\kappa + 5\Theta_0/2$.

4.1. **Lifting.** First, paths (and, more particularly, geodesics) lift:

Lemma 4.2. *For each combinatorial path $\bar{\gamma}$ in X/N starting at \bar{x} , and any point x in the preimage of \bar{x} (henceforth: a lift of x), there exists a combinatorial path in X so that $p \circ \gamma = \bar{\gamma}$, which we call a lift of $\bar{\gamma}$. Moreover, if $\bar{\gamma}$ is a geodesic, then so is γ .*

Proof. In order to lift combinatorial paths it suffices to lift edges, given a lift of the starting point of the edge in the quotient. This can be done since the action of N on X is simplicial. Lifting a geodesic yields a geodesic because the quotient map is 1-Lipschitz. \square

The following proposition is the key to our approach to study X/N and G/N : it allows us to lift geodesic triangles and quadrilaterals in X/N to triangles/quadrilaterals in G/N , thereby allowing us to translate properties of X (e.g. hyperbolicity) to properties of X/N . By requiring the constant Θ_{rot} to be even larger, we could ensure that we can lift n -gons for any given n , but we will only need the cases $n = 3$ and $n = 4$. The “moreover” part will only be needed for the WPD property.

Proposition 4.3. *For each geodesic quadrangle \bar{Q} in X/N there exists a geodesic quadrangle Q in X so that $p(Q) = \bar{Q}$. We call Q a lift.*

Moreover, if the geodesics $[\bar{v}_1, \bar{w}_1]$, $[\bar{v}_2, \bar{w}_2]$ of Q have lifts $[v_i, w_i]$ so that $d_s(v_i, w_i) \leq \Theta_{rot}/10$ whenever the quantity is defined, then the lift $[v'_i, w'_i]$ of $[\bar{v}_i, \bar{w}_i]$ contained in Q is an N -translate of $[v_i, w_i]$.

Proof. Let $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ be the vertices of a quadrangle in X/N . Lift \bar{x} as $x \in X$, and lift all four geodesic segments to get geodesics $[x, y]$, $[y, z]$, $[z, t]$, $[t, x']$. In the setting of the “moreover” part, choose an N -translate of $[v_i, w_i]$ as the lift of $[\bar{v}_i, \bar{w}_i]$. There is an element $\gamma \in N$ such that $\gamma x = x'$. We argue, by transfinite induction on the pair $(\alpha(\gamma), n(\gamma))$ (for lexicographic order).

If $(\alpha(\gamma), n(\gamma)) = (0, n)$, then $\gamma = 1$ and $x' = x$, so we are done. We thus assume that $(\alpha(\gamma), n(\gamma)) = (\alpha, n)$, with $\alpha > 0$.

Let us consider (s, γ_s) as in Corollary 3.6. If x is s -inactive, we have $\gamma_s x = x$. We can then apply γ_s to all lifts, and conclude by induction hypothesis.

Otherwise, we have $d_s(x, \gamma_s x) > \Theta_{short}$. Viewing s as a vertex of X , and using assumption (3) (BGIT) of the definition of composite projection graph, we see that in X the geodesic $[x, x']$ contains a point at distance 1 from s , and fixed by Γ_s .

There are several cases. The first one is when all points x, y, z, t, x' are active for s . Recall that $d_s(x, x') > \Theta_{short}$, and by the triangle inequality and our assumption on Θ_{rot} , at least one of $d_s(x, y), d_s(y, z), d_s(z, t)$ or $d_s(t, x')$ is larger than the constant C from BGIT (and, in the setting of the moreover part, the pair giving a large projection can be chosen not in $\{(v_i, w_i)\}$).

Let (u, u') (not in $\{(v_i, w_i)\}$) be a pair among the aforementioned ones such that $d_s(u, u') > C$. By BGIT, it follows that $[u, u']$ contains a point at distance ≤ 1 from s , hence fixed by γ_s . Replacing u' and all the points in (x, y, z, t, x') after the position of u' by their image by γ_s produces new isometric lifts of the segments, in such a way that the endpoint $\gamma_s x'$ now differs from x by $\gamma_s \gamma$, see Figure 1. Also, in the setting of the moreover part, notice that we replaced each $[v_i, w_i]$ by an N -translate. The induction hypothesis allows us to conclude.

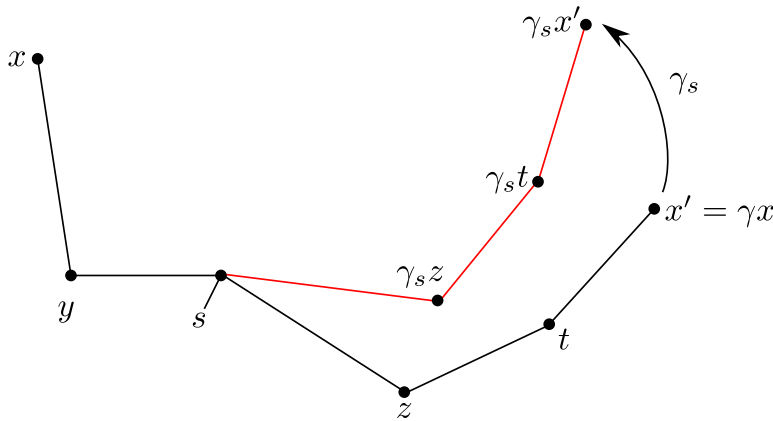


FIGURE 1. The case that s is adjacent to a vertex on $[y, z]$. In all cases, s is adjacent to some vertex on one of the sides (possibly one of x, y, z, t, x').

The second case is when not all x, y, z, t, x' are active for s . The argument is similar. Let u be the first point in the tuple that is inactive for s . This implies that γ_s fixes u . Changing the lifts of all the elements after u by their images by γ_s does not change the property that the lifts of segments are isometric. The conclusion is the same. \square

4.2. Projecting. The following lemma says, informally, that directions with small projection angles in X are preserved by p .

Lemma 4.4. *Suppose that $x, y \in X^{(0)}$ have the property that $d_s(x, y) < \Theta_{rot}/10$ whenever the quantity is defined. Then $p|_{[x, y]}$ is isometric, for any geodesic $[x, y]$.*

Proof. Assume there is a shorter path from $\bar{x} = p(x)$ to $\bar{y} = p(y)$, for some x, y as in the statement. Lift this shorter path as a geodesic segment $[y, x']$. There exists $\gamma \in N$ such that $\gamma x = x'$. Again we proceed by induction on $(\alpha(\gamma), n(\gamma))$, this time to prove that $d_X(y, x') = d_X(y, x)$ (thus falsifying that $d_{X/N}(\bar{x}, \bar{y}) < d_X(x, y)$), for any pair x, y as in the statement. If $\alpha(\gamma) = 0$ then $\gamma = 1$ and it is obvious.

Assume that $\alpha(\gamma) > 0$. Let (s, γ_s) be as in Corollary 3.6. If x is s -inactive, then $\gamma_s x = x$, and we can apply γ_s to both geodesics and conclude by the induction hypothesis.

Otherwise, $d_s(x, \gamma x) > \Theta_{short}$, and in particular there exists s' on $[x, x']$ at distance 1 from s . Since $d_s(x, y)$ is assumed to be small (if defined) we also have that $s' \in [y, x']$ (actually this is also true if $d_s(x, y)$ is not defined: in that case, $s' = y$). Thus, one can change the lift of $[\bar{y}, x]$ as $[y, \gamma_s x']$, while keeping it an isometric lift. One concludes by induction hypothesis, which applies to $\gamma_s \gamma$. \square

4.3. Properties of $G \curvearrowright X$ persisting for $G/N \curvearrowright X/N$. We can now prove the various statements referenced in the proof of Theorem 2.1, thereby completing the proof of that theorem.

First, lifting geodesic triangles from X/N to X using Proposition 4.3, we deduce:

Corollary 4.5. *X/N is δ -hyperbolic, where δ is the hyperbolicity constant of X .*

Next, we investigate survival of loxodromic (resp. WPD) elements of G as loxodromic (resp. WPD) elements of G/N :

Proposition 4.6. *Assume that $\phi \in G$ is a loxodromic isometry of X , and that there is $x_0 \in X$ such that for all $n \in \mathbb{Z}$ for all $s \in X$, we have $d_s(x_0, \phi^n x_0) < \Theta_{rot}/10$ whenever it is defined. Then, $\bar{\phi} \in G/N$ is loxodromic on $\bar{X} = X/N$. Moreover, if ϕ has the WPD property, then so does $\bar{\phi}$.*

Proof. Lemma 4.4 guarantees that $d_{\bar{X}}(\bar{x}_0, \bar{\phi}^n \bar{x}_0)$ grows linearly, thus $\bar{\phi}$ is loxodromic.

By the WPD property, we may take n such that $d(x_0, \phi^n x_0)$ is so large that, for a certain k_0 , at most k_0 elements of G move the pair $(x_0, \phi^n x_0)$ at distance $\leq 100\delta$ from itself. Call $x_n = \phi^n x_0$.

Assume that $\{\bar{\phi}_i, i = 1, \dots, k\}$ is a collection of distinct elements that move each of \bar{x}_0, \bar{x}_n within distance 100δ from itself. Notice that each

$\bar{\phi}_i[\bar{x}_0, \bar{x}_n]$ has a lift with small projections as in the moreover part of Proposition 4.3, namely $\phi'_i[x_0, x_n]$ for some ϕ'_i in the preimage of $\bar{\phi}_i$ (where we choose $[\bar{x}_0, \bar{x}_n] = p([x_0, x_n])$). Hence, there is a lift of a quadrilateral $(\bar{x}_0, \bar{\phi}_i \bar{x}_0, \bar{\phi}_i \bar{x}_n, \bar{x}_n)$ of the form $(x_0, \phi_i x_0, \phi_i x_n, x_n)$, for some ϕ_i in the preimage of $\bar{\phi}_i$. (We took an N -translate of the quadrilateral from Proposition 4.3 to make sure that one side is $[x_0, x_n]$.)

By definition of k_0 , if $k > k_0$, then there are $i \neq j$ such that $\phi_i = \phi_j$. Projecting in G/N , $\bar{\phi}_i = \bar{\phi}_j$, which is a contradiction. This forces $k \leq k_0$. This holds for all sufficiently large n . By, for example, the proof of [DGO17, Proposition 5.31], this is sufficient to ensure the WPD property for $\bar{\phi}$. \square

Finally, we see that non-elementarity persists:

Lemma 4.7. *Suppose that there exist independent loxodromic isometries $\phi, \psi \in G$ of X , and $x_0 \in X$ such that for all $n, m \in \mathbb{Z}$ for all $s \in X$, $d_s(\phi^n x_0, \psi^m x_0) < \Theta_{rot}/10$ whenever it is defined. Then the action of G/N on X/N is non-elementary.*

Proof. By Proposition 4.6, $\bar{\phi}$ and $\bar{\psi}$ are loxodromic on X/N . Moreover, by Lemma 4.4, p restricted to $\langle \phi \rangle x_0 \cup \langle \psi \rangle x_0$ is an isometry, so that $\bar{\phi}$ and $\bar{\psi}$ are in fact independent. \square

4.4. Compatibility with stabilisers. The following proposition is not used in the proof of Theorem 2.1, but it is useful in applications. It describes the structure of stabilisers of X/N , which turn out to be exactly what one expects.

Proposition 4.8. *Under the Standing Assumption 4.1, the following holds. For any vertex v of X ,*

$$\text{Stab}(v) \cap N = \langle \{\Gamma_w \cap \text{Stab}(v) : w \in \mathbb{Y}_*\} \rangle = \langle \{\Gamma_w : w \in \mathbb{Y}_* \setminus \text{Act}(v)\} \rangle.$$

Proof. To see the second equality, observe that if $w \notin \text{Act}(v)$ then $\Gamma_w \subset \text{Stab}(v)$. If $\Gamma_w \cap \text{Stab}(v)$ is non-trivial, then since Γ_w consists only of large rotations, w must be v -inactive (in other words $w \notin \text{Act}(v)$). In fact, if w is v -active, then, for any nontrivial $\gamma_w \in \Gamma_w$, $d_w(v, \gamma_w v)$ is defined, and hence nonzero by Corollary 1.8. Thus $\gamma_w v \neq v$.

The right-hand-side is contained in the left-hand-side. Let us prove the other inclusion. Take $\gamma \in \text{Stab}(v) \cap N$. We want to show that $\gamma \in \langle \{\Gamma_w : w \in \mathbb{Y}_* \setminus \text{Act}(v)\} \rangle$. We proceed by induction on $(\alpha(\gamma), n(\gamma))$.

If $\alpha(\gamma) = 0$, then γ is trivial. If $\alpha(\gamma) > 0$, let (s, γ_s) be as in Corollary 3.6. If s is v -inactive, then $\gamma_s \gamma v = v$, and the induction hypothesis applies to $\gamma_s \gamma$. In particular, γ is also in $\langle \{\Gamma_w : w \in \mathbb{Y}_* \setminus \text{Act}(v)\} \rangle$.

On the other hand, we cannot have $d_s^{\triangleleft}(v, \gamma v) > \Theta_{short}$, since this contradicts $v = \gamma v$. \square

5. APPLICATIONS TO MAPPING CLASS GROUPS

Let $\Sigma_{g,p}$ denote the genus- g oriented surface with p punctures, and let $\mathcal{C}(\Sigma_{g,p})$ denote its curve graph.

Proposition 5.1. *There exists K such that the following holds: $\mathcal{C}(\Sigma_{g,p})$ is an $MCG(\Sigma_{g,p})$ -composite projection graph. For any $k \in \mathbb{Z} \setminus \{0\}$, there is a composite rotating family on the above composite projection graph such that, for each curve y , we have $\Gamma_y = \langle T_y^{kK} \rangle$, where T_y is the Dehn twist about y .*

Proof. This follows the discussion in [Dah18, Section 3] exactly, so we just describe the data here and refer the reader to [Dah18] for the explanation of why this data determines a composite projection graph and rotating family.

Composite projection graph: $\mathcal{C}(\Sigma_{g,p})$ is hyperbolic [MM99], and $MCG(\Sigma_{g,p})$ acts by simplicial automorphisms. For each (isotopy class of) curve y , let $\text{Act}(y)$ be the set of curves x that intersect y , i.e. the set of vertices of $\mathcal{C}(\Sigma_{g,p})$ distinct from, and not adjacent to, y . Note that $x \in \text{Act}(y)$ if and only if $y \in \text{Act}(x)$. The $MCG(\Sigma_{g,p})$ -invariant colouring $\mathcal{C}(\Sigma_{g,p}) = \bigsqcup_{i=0}^m \mathbb{Y}_i$ is described in [Dah18] and is derived from the colouring in [BBF15]. (There is a finite-index normal subgroup $G_0 \leq MCG(\Sigma_{g,p})$ that preserves each colour, and the colours correspond to the cosets of G_0 , so that $MCG(\Sigma_{g,p})$ acts on the set of colours.) Given a curve y , and curves x, z intersecting y , the distance $d_y^\pi(x, z)$ is defined via subsurface projection, and satisfies the properties from Definition 1.1 by results from [MM00, Beh06], see the discussion in [Dah18].

Composite rotating family: The discussion in [Dah18] provides an integer $K > 0$ such that the following holds for all $k \in \mathbb{Z} \setminus \{0\}$. For each curve y , let T_y be the Dehn twist about y . Let $\Gamma_y = \langle T_y^{kK} \rangle$. Then the subgroups Γ_y , $y \in \mathcal{C}(\Sigma_{g,p})^{(0)}$, form a composite rotating family on the $MCG(\Sigma_{g,p})$ -composite projection system discussed above. \square

Theorem 5.2. *Let $\Sigma_{g,p}$ be a finite-type surface, with either $g > 0$ or $p > 3$. Then there exists a positive integer K_0 so that for all non-zero multiples K of K_0 , the group $MCG(\Sigma_{g,p})/DT_K$ is acylindrically hyperbolic, where DT_K is the normal subgroup generated by all K^{th} powers of Dehn twists.*

Proof. By Theorem 2.1, the group $MCG(\Sigma_{g,p})/DT_K$ acts non-elementarily on $\mathcal{C}(\Sigma_{g,p})/DT_K$ with loxodromic WPD elements. Hence $MCG(\Sigma_{g,p})/DT_K$ is acylindrically hyperbolic, by [Osi16, Theorem 1.2]. \square

Remark 5.3. A similar theorem also holds for quotients of mapping class groups by powers of Dehn twists around curves of one specified topological type (we allow \mathbb{Y}_*^τ to be a proper subset of \mathbb{Y}_* in the definition of a composite rotating family on a composite projection graph).

5.1. Relative hyperbolicity and relative quasiconvexity. We will see that, in low-complexity, quotients of mapping class groups by powers of Dehn twists are in fact hyperbolic groups. To prove this, we will use relative hyperbolicity.

We will use the following definition of relative hyperbolicity (following Proposition 4.28 of [DGO17]):

Definition 5.4. Let G be a group and let \mathcal{H} be a collection of subgroups of G . Let $\mathcal{S} \subset G$ be a finite set that is closed under taking inverses. Suppose that $\mathcal{S} \sqcup \bigsqcup_{H \in \mathcal{H}}$ generates G , and let $\text{Cay}(G, \mathcal{S} \sqcup \bigsqcup_{H \in \mathcal{H}})$ be the Cayley graph with respect to this generating set.

(Note that, if $s \in \mathcal{S}$ is also contained in some $H \in \mathcal{H}$, and $g \in G$, we regard g and gs as being joined by two edges labelled by s , one of which is in a the graph $g\Gamma_H$ defined below.)

For each $H \in \mathcal{H}$, let Γ_H be the Cayley graph of H with respect to the generating set H , so that Γ_H is a diameter-1 subgraph of $\text{Cay}(G, \mathcal{S} \sqcup \bigsqcup_{H \in \mathcal{H}})$ for each $H \in \mathcal{H}$. Define a metric d_H on H as follows: given vertices $x, y \in H$, a combinatorial path σ in $\text{Cay}(G, \mathcal{S} \sqcup \bigsqcup_{H \in \mathcal{H}})$ is *admissible* if σ does not traverse an edge of Γ_H . Then $d_H(x, y)$ is the infimum of the lengths of admissible paths from x to y . We say that G is *hyperbolic relative to \mathcal{H}* if $\text{Cay}(G, \mathcal{S} \sqcup \bigsqcup_{H \in \mathcal{H}})$ is a hyperbolic graph and the metric d_H is proper for each $H \in \mathcal{H}$.

We also require the notion of (strong) relative quasiconvexity. In fact, we will take as the definition the characterisation provided by Theorem 4.13 of [Osi06]:

Definition 5.5. Let G be hyperbolic relative to a collection \mathcal{H} of subgroups. The subgroup $Q \leq G$ is *strongly relatively quasiconvex* if:

- Q is generated by a finite set Y ;
- letting d_Y be the word-metric on Q with respect to Y , the inclusion $(Q, d_Y) \rightarrow \text{Cay}(G, \mathcal{S} \sqcup \bigsqcup_{H \in \mathcal{H}})$ is a quasi-isometric embedding.

We will also use the following well-known fact (see [CC07, Theorem 5.1]):

Lemma 5.6. *Let G act cocompactly on the graph X . Let $\{v_i\}$ be a (necessarily finite) collection of representatives of the G -orbits of the vertices of X . Then G is finitely generated relative to $\{\text{Stab}(v_i)\}$, and any orbit map defines a quasi-isometry between $\text{Cay}(G, \mathcal{S} \sqcup \bigsqcup_{v_i} \text{Stab}(v_i))$ and X , for any finite relative generating set \mathcal{S} .*

5.2. Vertex links in the curve graph. Let $(g, p) \in \{(0, 5), (1, 2)\}$, and let $\gamma \in \mathcal{C}(\Sigma_{g,p})^{(0)}$. The link $lk(\gamma)$ of γ in $\mathcal{C}(\Sigma_{g,p})$ is a discrete set that can be identified with the vertex set of the curve graph of the only component of $\Sigma_{g,p} - \gamma$ which is not homeomorphic to $\Sigma_{0,3}$. We equip $lk(\gamma)$ with the metric induced by this identification of $lk(\gamma)$ with a subset of the curve graph of the aforementioned complementary component. Denote by L_γ the metric space with underlying set $lk(\gamma)$, endowed with the metric just described.

For a vertex $x \in \mathcal{C}(\Sigma_{g,p})$, with $x \neq \gamma$, we denote by $r(x)$ the entrance point in $lk(\gamma)$ of any geodesic from x to γ .

The following is an easy consequence of the Bounded Geodesic Image Theorem, [MM00, Theorem 3.1].

Lemma 5.7. *There exists C so that the following holds. Let x, y, s be curves. If some geodesic $[x, y]$ in $\mathcal{C}(\Sigma_{g,p})$ from x to y does not contain s , then $d_{L_s}(r(x), r(y)) \leq C$.*

Roughly speaking, the lemma says that there is a Lipschitz retraction of the complement of γ onto $lk(\gamma)$. We will use such retraction to prove that the relative metric is finite by, again roughly speaking, starting with a path in the complement of γ and constructing a path in $lk(\gamma)$ which is not much longer.

5.3. Hyperbolic and relatively hyperbolic quotients. Fix $(g, p) \in \{(0, 4), (0, 5), (1, 0), (1, 1), (1, 2)\}$. Let $K \in \mathbb{Z} \setminus \{0\}$, and let DT_K be the normal subgroup generated by all K^{th} powers of Dehn twists. Let $G(g, p, K) = MCG(\Sigma_{g,p})/DT_K$, and let $X_K = \mathcal{C}(\Sigma_{g,p})/DT_K$.

Theorem 5.8. *Let (g, p) be as above. Then there exists a positive integer K_0 so that for all sufficiently large multiples K of K_0 , the following hold.*

- (1) *Suppose $(g, p) \in \{(0, 4), (1, 0), (1, 1)\}$. Then $G(g, p, K)$ is non-elementary hyperbolic.*
- (2) *Suppose $(g, p) = (0, 5)$ (resp. $(g, p) = (1, 2)$). Then $G(g, p, K)$ is hyperbolic relative to an infinite index subgroup H virtually isomorphic to $G(0, 4, K)$ (resp. relative to two infinite index subgroups H_1, H_2 , one virtually isomorphic to $G(0, 4, K)$ and one virtually isomorphic to $G(1, 1, K)$). In particular, $G(g, p, K)$ is non-elementary hyperbolic.*

Moreover, letting \mathcal{H} be the set of peripheral subgroups arising in the second case, we have that for each relative generating set \mathcal{S} , there is a G_K -equivariant quasi-isometry $\text{Cay}(G_K, \mathcal{S} \cup \bigcup_{H \in \mathcal{H}} H) \rightarrow X_K$.

Proof. The curve graph $\mathcal{C}(\Sigma_{g,p})$ is an $MCG(\Sigma_{g,p})$ -composite projection graph, and large powers of Dehn twists define a composite rotating family on it. Thus, there exists K_0 so that for sufficiently large $K \in K_0\mathbb{Z}$, the graph X_K is hyperbolic by Theorem 2.1. Moreover, the action of $G(g, p, K)$ on X_K is non-elementary, since the action of $MCG(\Sigma_{g,p})$ on $\mathcal{C}(\Sigma_{g,p})$ is non-elementary. In particular, $G(g, p, K)$ contains elements acting on X_K loxodromically.

Hyperbolicity in lowest-complexity cases: Suppose that $(g, p) \in \{(0, 4), (1, 0), (1, 1)\}$. Then the action of $G(g, p, K)$ on X_K has finite vertex stabilisers. Indeed, each vertex stabiliser is a quotient of a vertex stabiliser of $\mathcal{C}(\Sigma_{g,p})$, which in our case is virtually generated by a Dehn twist. By Lemma 5.6 and cocompactness of the $G(g, p, K)$ -action on X_K , $G(g, p, K)$ is $G(g, p, K)$ -equivariantly quasi-isometric to X_K and is thus hyperbolic. (Since $MCG(\Sigma_{g,p})$ is hyperbolic in this case, hyperbolicity of $G(g, p, K)$ can also be deduced from [Del96].)

Stabilisers: Suppose from now on that (g, p) is as in the second case. By Proposition 4.8, all infinite vertex stabilisers for the action of $G(g, p, K)$ on X_K are of the form specified by the statement. Specifically, for $(g, p) = (0, 5)$ there is only one topological type of curves (yielding exactly one conjugacy class of stabilisers in X_K), with stabiliser virtually isomorphic to a central extension of $MCG(\Sigma_{0,4})$ by a cyclic subgroup generated by a Dehn twist.

Proposition 4.8 guarantees that the image of such a stabiliser in $G(g, p, K)$ is also obtained by modding out powers of Dehn twists.

For $(g, p) = (1, 2)$ the situation is similar, except that there are two topological types of curves (one non-separating, with complement $\Sigma_{0,4}$, and one separating with complement $\Sigma_{0,3} \sqcup \Sigma_{1,1}$).

Relative hyperbolicity: By Lemma 5.6, $\text{Cay}(G(g, p, K), \mathcal{S} \cup \mathcal{H})$ is equivariantly quasi-isometric to X_K , where \mathcal{S} is any fixed finite generating set, and \mathcal{H} is a union of conjugacy representatives of stabilisers. Since the action of $G(g, p, K)$ on X_K has a loxodromic element, the stabilisers must have infinite index. We now have to prove that the relative metric on each stabiliser, H , is proper.

Recall from Subsection 5.2 that, for γ a curve on $\Sigma_{g,p}$, we defined a metric space L_γ with underlying set the link $lk(\gamma)$ of γ (regarded as a vertex of $\mathcal{C}(\Sigma_{g,p})$). Moreover, L_γ is naturally isometric to the vertex set of the curve graph of the (only non- $\Sigma_{0,3}$ component of the) complement of γ . Similarly, in view of the discussion above about stabilisers, the link of $\bar{\gamma}$ in X_K can be made into a metric space $L_{\bar{\gamma}}$ naturally isometric to the vertex set of the quotient of the curve graph of a complexity-1 surface by the action of the subgroup generated by K -th powers of Dehn twists supported on said surface.

Lemma 5.9. *Let γ be a curve. Let $\bar{x}, \bar{y} \neq \bar{\gamma}$ be adjacent vertices of X_K . Let $r(\bar{x}) = r_{\bar{\gamma}}(x), r(\bar{y}) = r_{\bar{\gamma}}(y)$ be the entrance points in $lk(\bar{\gamma})$ of geodesics from \bar{x}, \bar{y} to $\bar{\gamma}$. Then $d_{L_{\bar{\gamma}}}(r(\bar{x}), r(\bar{y})) \leq C$, where C is as in Lemma 5.7.*

Proof. For \bar{z} a vertex of X_K , we choose any geodesic $\alpha_{\bar{z}}$ from \bar{z} to $\bar{\gamma}$. By Proposition 4.3, we can lift the geodesic triangle formed by $\alpha_{\bar{x}}, \alpha_{\bar{y}}$ and an edge connecting \bar{x}, \bar{y} to $\mathcal{C}(\Sigma_{g,p})$, and for convenience we arrange that one of the vertices of the lift is γ .

By Lemma 5.7, we see that, in $\mathcal{C}(\Sigma_{g,p})$, the entrance points of the lifts of $\alpha_{\bar{x}}, \alpha_{\bar{y}}$ in $lk(\gamma)$ are close to each other as measured in L_γ , which implies that the same holds for $r(\bar{x}), r(\bar{y})$ in $L_{\bar{\gamma}}$. \square

We now claim that, for any fixed $x_0 \in lk(\bar{\gamma})$, there exists M with the following property. Suppose that we can write some $h \in H = \text{Stab}(\bar{\gamma})$ as $h = s_1 h_1 \dots s_n$, where $s_i \in S, h_i \in H$ and $s_1 h_1 \dots s_i \notin H$ for all $i \leq n-1$. Then $d_{L_{\bar{\gamma}}}(x_0, h x_0) \leq M n$. Since $L_{\bar{\gamma}}$ is equivariantly quasi-isometric to H (see the argument for the hyperbolicity of $G(0, 4, K)$ and $G(1, 1, K)$ above), this concludes the proof of relative hyperbolicity.

We can assume that S is symmetric, and that $S \cap H = \emptyset$. We will use the maps r_γ (for any γ) from the lemma, where we assume that the relevant geodesics are chosen equivariantly.

Let $L_0 = \max_{s \in S} \max_{\bar{\gamma}'} d_{L_{\bar{\gamma}'}}(r(x_0), r(s x_0))$. Notice that L_0 is indeed attained because for any given s (of which there are finitely many), there are only finitely many $\bar{\gamma}'$ for which $d_{L_{\bar{\gamma}'}}(r(x_0), r(s x_0))$ can exceed C times the

distance between x_0 and sx_0 , namely those occurring along a geodesic from x_0 to sx_0 .

By equivariance, we have $d_{L_{\bar{\gamma}}}(r(gx_0), r(gsx_0)) \leq L_0$ for any $g \in G(g, p, K)$. Moreover, if $g \notin H$ and $h' \in H$, then gx_0 and ghx_0 are connected by two edges not containing $\bar{\gamma}$, so that $d_{L_{\bar{\gamma}}}(r(gx_0), r(ghx_0)) \leq 2C$. Hence

$$d_{L_{\bar{\gamma}}}(x_0, hx_0) = d_{L_{\bar{\gamma}}}(r(x_0), r(hx_0)) \leq L_0n + 2(n-1)C,$$

as required.

Finally, in case (2), hyperbolicity of $G(g, p, K)$ follows since a group hyperbolic relative to hyperbolic subgroups is hyperbolic by [Osi06, Corollary 2.41]. \square

5.4. Residual properties of $MCG(\Sigma_{g,p})$. Let K_0, K be as in the statement of Theorem 5.8 and let $G(g, p, K)$ and X_K be as above. Let $\Psi_K : MCG(\Sigma_{g,p}) \rightarrow G(g, p, K)$ be the quotient map.

Lemma 5.10. *Let $x \in MCG(\Sigma_{g,p}) \setminus \{1\}$. Then $\Psi_K(x) \neq 1$ for all sufficiently large $K \in K_0\mathbb{Z}$.*

Proof. We consider three cases.

- If x is pseudo-Anosov, then it acts loxodromically on $\mathcal{C}(\Sigma_{g,p})$, so by Proposition 4.6, $\Psi_K(x) \neq 1$ for sufficiently large K .
- If x has finite order, then $x \notin DT_K$ since each element of DT_K has infinite order. This follows by transfinite induction using Theorem 3.1.(2),(4).
- The remaining possibility is that x is reducible. In this case, there exists $n > 0$, depending only on (g, p) , so that x^n stabilises a simple closed curve γ of $\Sigma_{g,p}$. Let U, V be the components of $\Sigma_{g,p} \setminus \gamma$ (if γ is non-separating, we take $V = \emptyset$ and $U = \Sigma_{g,p} \setminus \gamma$). Let $H \leq MCG(\Sigma_{g,p})$ be the stabiliser of γ . Let $W = U \sqcup V$, which is a (possibly disconnected) subsurface of $\Sigma_{g,p}$ of complexity strictly less than that of $\Sigma_{g,p}$.

The action of H on W gives an exact sequence $1 \rightarrow Z \rightarrow H \xrightarrow{\phi} \widehat{A}$, where Z is central in H and is the cyclic subgroup generated by the Dehn twist about γ , and \widehat{A} has a finite-index subgroup A such that $A \leq MCG(U)$ if $V = \emptyset$ and $A \leq MCG(U) \times MCG(V)$ otherwise.

Let $H' \leq H$ be the finite-index subgroup $H' = \phi^{-1}(A)$. Let $\Psi_K^U : MCG(U) \rightarrow G_K(U)$ be the quotient obtained by killing K^{th} powers of Dehn twists in $MCG(U)$, and define $\Psi_K^V : MCG(V) \rightarrow G_K(V)$ analogously (if $V \neq \emptyset$). Let $\Psi'_K : A \rightarrow \bar{A}$ be the restriction of Ψ_K^U to A if $V = \emptyset$ and the restriction of $\Psi_K^U \times \Psi_K^V$ to A otherwise.

Define a homomorphism $\bar{\phi} : \Psi_K(H') \rightarrow \bar{A}$ by $\bar{\phi}(\Psi_K(g)) = \Psi'_K(\phi(g))$. To see that this is well-defined, it suffices to show that $\Psi'_K(\phi(n)) = 1$ whenever $n \in DT_K \cap H'$. By Proposition 4.8, we have $n = yq$, where y is a power of the Dehn twist about γ and q is the product of powers of Dehn twists about curves in W . Moreover, $\phi(q) = \phi(yq) =$

$\phi(n) \in A$, since $n \in H'$. Hence $\phi(n)$ is the product of powers of Dehn twists in U and V , and lies in A , so $\Psi'_K(\phi(n))$ is defined and $\Psi'_K(\phi(n)) = 1$, as required.

Choose $m \geq 1$ so that $x^{mn} \in H'$. Write $x^{mn} = zw$, where $z \in Z$ and w is supported on W , with either $x^{mn} = z$ or $\phi(w) \neq 1$.

If $x^{mn} = z$, then $\Psi_K(x^{mn}) = \Psi_K(z) \neq 1$ for sufficiently large K , by Proposition 4.8.

Otherwise, $\phi(x^{mn}) = \phi(w) \neq 1$. Moreover, $\phi(w) \in A$, since $\phi(w) = \phi(x^{mn})$ and $x^{mn} \in H'$. Hence either $\phi(w) = a \in MCG(U)$ (if $V = \emptyset$) or $\phi(w) = (a, b) \in MCG(U) \times MCG(V)$. In either case, we can assume $a \neq 1$, so by induction on complexity, $\Psi_K^U(a) \neq 1$, and hence $\Psi'_K(\phi(w)) \neq 1$, for all sufficiently large K , as required. But $\Psi'_K(\phi(w)) = \bar{\phi}(\Psi_K(x^{mn}))$, so $\Psi_K(x^{mn}) \neq 1$, and hence $\Psi_K(x) \neq 1$.

(In the base case, U is an annulus or pair of pants. When U is a pair of pants, Ψ_K is the identity. When U is an annulus, $MCG(U)$ has a finite-index normal subgroup generated by a single Dehn twist, and the lemma clearly holds.)

By the Nielsen-Thurston classification, any x is of one of the above three types, so the lemma holds. \square

This, and Theorem 5.8, are already sufficient to prove:

Corollary 5.11. *Suppose $(g, p) \in \{(0, 4), (0, 5), (1, 0), (1, 1), (1, 2)\}$. Then $MCG(\Sigma_{g,p})$ is fully residually non-elementary hyperbolic.*

Proof. By Theorem 5.8, $G(g, p, K)$ is non-elementary hyperbolic, and by Lemma 5.10, any finite subset of $MCG(\Sigma_{g,p}) \setminus \{1\}$ is mapped injectively to $G(g, p, K)$ for all sufficiently large K . \square

For convenience, whenever g, p are fixed, we write G_K to mean $G(g, p, K)$. We now study images of convex-cocompact subgroups in G_K .

Proposition 5.12. *Let (g, p) be as in Section 5.3, let K_0, K be as in Theorem 5.8, and let (G_K, \mathcal{H}) be the relatively hyperbolic structure from Theorem 5.8. Let $Q \leq MCG(\Sigma_{g,p})$ be a convex-cocompact subgroup. Then:*

- (1) *For all sufficiently large K , the quotient map Ψ_K is injective on Q .*
- (2) *For all sufficiently large K , $\Psi_K(Q)$ is strongly relatively quasiconvex in (G_K, \mathcal{H}) , and hence quasiconvex in G_K .*
- (3) *For all $x \in MCG(\Sigma_{g,p}) \setminus Q$ and all sufficiently large K , we have $\Psi_K(x) \notin \Psi_K(Q)$.*

Proof. Consider the action of Q on $\mathcal{C}(\Sigma_{g,p})$ arising as the restriction of the action of $MCG(\Sigma_{g,p})$. Fix a vertex $v_0 \in \mathcal{C}(\Sigma_{g,p})$. Fix $h \in Q$, and let s be a curve. Then $d_s(v_0, hv_0) \leq C$ whenever the quantity is defined, for some uniform constant C : this is contained in the proof of [KL08, Theorem 7.4], see [DT15, Lemma 5.1]. Choosing K sufficiently large (in terms of C) and applying Lemma 4.4 implies that geodesics $[v_0, hv_0] \in \mathcal{C}(\Sigma_{g,p})$ map

isometrically to geodesics in X_K joining the images of v_0, hv_0 . This implies assertion (1).

Now, fix a finite generating set \mathcal{Y} of Q . Since geodesics $[v_0, hv_0]$ map isometrically to geodesics in X_K , we see that the orbit map $\Psi_K(h) \mapsto \Psi_K(h)v_0$ is a quasi-isometric embedding $\Psi_K(Q) \rightarrow X_K$ whose constants depend on \mathcal{Y} but are independent of K .

By Theorem 5.8, X_K is G_K -equivariantly quasi-isometric to $\text{Cay}(G_K, \mathcal{S} \cup \mathcal{H})$ for any finite relative generating set \mathcal{S} , so $\Psi_K(Q) \rightarrow \text{Cay}(G_K, \mathcal{S} \cup \mathcal{H})$ is a quasi-isometric embedding. Choosing \mathcal{S} to be a finite generating set of G_K , we can pull back the quasi-isometric embedding $Q \rightarrow \text{Cay}(G_K, \mathcal{S} \cup \mathcal{H})$ under the Lipschitz map $\text{Cay}(G_K, \mathcal{S}) \rightarrow \text{Cay}(G_K, \mathcal{S} \cup \mathcal{H})$ to get a quasi-isometric embedding $Q \rightarrow \text{Cay}(G_K, \mathcal{S})$, proving that $\Psi_K(Q)$ is quasiconvex in the hyperbolic group G_K . This proves assertion (2).

Let $x \in MCG(\Sigma_{g,p}) \setminus Q$. Given K , let \bar{v}_0 be the image of v_0 in X_K . Let $\Delta = d_{\mathcal{C}(\Sigma_{g,p})}(v_0, xv_0)$. Then there exists Δ' , depending only on Δ and the generating set of Q , such that for all sufficiently large K , the set of $h \in Q$ such that $d_{X_K}(\Psi_K(h)\bar{v}_0, \bar{v}_0) \leq \Delta$ is contained in the set $\{h_i\}$ of elements of Q of word-length at most Δ' . This is because any orbit map $Q \rightarrow \Psi_K(Q) \rightarrow X_K$ is a quasi-isometric embedding with constants independent of K .

Suppose that $\Psi_K(x) \in \Psi_K(Q)$. Since $d_{X_K}(\Psi_K(x)\bar{v}_0, \bar{v}_0) \leq \Delta$, we have $\Psi_K(xh_i^{-1}) = 1$ for some i . For sufficiently large K , the map Ψ_K is injective on Q , so $x = h_i^{-1}$, contradicting that $x \notin Q$. This proves assertion (3). \square

Theorem 5.13. *Assume that all hyperbolic groups are residually finite. Let $(g, p) \in \{(0, 4), (0, 5), (1, 0), (1, 1), (1, 2)\}$. Then any convex-cocompact subgroup $Q < MCG(\Sigma_{g,p})$ is separable in $MCG(\Sigma_{g,p})$.*

Proof. Let G_K, Ψ_K and X_K be as in Section 5.3. Fix $x \in MCG(\Sigma_{g,p}) \setminus Q$. Using Theorem 5.8 and Proposition 5.12, we can choose K so that G_K is hyperbolic, $\Psi_K(Q)$ is quasiconvex in G_K , and $\Psi_K(x) \notin \Psi_K(Q)$. If every hyperbolic group is residually finite, then by [AGM09, Theorem 0.1], for any hyperbolic group, all of its quasiconvex subgroups are separable. In particular, G_K has a finite quotient separating x from $\Psi_K(Q)$. Hence there is a finite quotient of $MCG(\Sigma_{g,p})$ separating x from Q , as required. \square

REFERENCES

- [AAS07] James W. Anderson, Javier Aramayona, and Kenneth J. Shackleton. An obstruction to the strong relative hyperbolicity of a group. *J. Group Theory*, 10(6):749–756, 2007.
- [AGM09] Ian Agol, Daniel Groves, and Jason Fox Manning. Residual finiteness, QCERF and fillings of hyperbolic groups. *Geom. Topol.*, 13(2):1043–1073, 2009.
- [Ago13] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.
- [AL17] Aurelien Alvarez and Vincent Lafforgue. Actions affines isométriques propres des groupes hyperboliques sur des espaces ℓ^p . *Expo. Math.*, 35(1):103–118, 2017.
- [BBF15] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara. Constructing group actions on quasi-trees and applications to mapping class groups. *Publ. Math. Inst. Hautes Études Sci.*, 122:1–64, 2015.

- [BBFS17] Mladen Bestvina, Ken Bromberg, Koji Fujiwara, and Alessandro Sisto. Acylindrical actions on projection complexes. *arXiv:1711.08722*, 2017.
- [BDM09] Jason Behrstock, Cornelia Druţu, and Lee Mosher. Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity. *Math. Ann.*, 344(3):543–595, 2009.
- [Beh06] Jason A. Behrstock. Asymptotic geometry of the mapping class group and Teichmüller space. *Geom. Topol.*, 10:1523–1578, 2006.
- [BHS15] Jason Behrstock, Mark F Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces II: combination theorems and the distance formula. *arXiv preprint arXiv:1509.00632*, 2015.
- [BHS17a] J. Behrstock, M. F. Hagen, and A. Sisto. Quasiflats in hierarchically hyperbolic spaces. *arXiv:1704.04271*, 2017.
- [BHS17b] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups. *Geom. Topol.*, 21(3):1731–1804, 2017.
- [CC07] Ruth Charney and John Crisp. Relative hyperbolicity and Artin groups. *Geom. Dedicata*, 129:1–13, 2007.
- [Dah18] François Dahmani. The normal closure of big Dehn twists, and plate spinning with rotating families. *Geom. Topol.*, 22:4113–4144, 2018.
- [Del96] Thomas Delzant. Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Math. J.*, 83(3):661–682, 1996.
- [DG18] François Dahmani and Vincent Guirardel. Recognizing a relatively hyperbolic group by its Dehn fillings. *Duke Math. J.*, 167(12):2189–2241, 2018.
- [DGO17] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.*, 245(1156):v+152, 2017.
- [DT15] Matthew Durham and Samuel J Taylor. Convex cocompactness and stability in mapping class groups. *Algebraic & Geometric Topology*, 15(5):2837–2857, 2015.
- [DT19] François Dahmani and Nicholas Touikan. Deciding isomorphy using Dehn fillings: the splitting case. *Invent. Math.*, 2019. to appear.
- [FM02] Benson Farb and Lee Mosher. Convex cocompact subgroups of mapping class groups. *Geom. Topol.*, 6:91–152, 2002.
- [GM08] D. Groves and J. F. Manning. Dehn filling in relatively hyperbolic groups. *Israel J. Math.*, 168:317–429, 2008.
- [Ham05] Ursula Hamenstädt. Word hyperbolic extensions of surface groups. *arXiv preprint math/0505244*, 2005.
- [KL08] Autumn E Kent and Christopher J Leininger. Shadows of mapping class groups: capturing convex cocompactness. *Geometric and Functional Analysis*, 18(4):1270–1325, 2008.
- [LM07] Christopher J. Leininger and D. B. McReynolds. Separable subgroups of mapping class groups. *Topology Appl.*, 154(1):1–10, 2007.
- [MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.
- [MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.
- [Nic13] Bogdan Nica. Proper isometric actions of hyperbolic groups on L^p -spaces. *Compos. Math.*, 149(5):773–792, 2013.
- [Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.
- [Osi07] D. V. Osin. Peripheral fillings of relatively hyperbolic groups. *Invent. Math.*, 167(2):295–326, 2007.
- [Osi16] D. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.

- [Rei06] Alan W. Reid. Surface subgroups of mapping class groups. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 257–268. Amer. Math. Soc., Providence, RI, 2006.
- [Yu05] Guoliang Yu. Hyperbolic groups admit proper affine isometric actions on l^p -spaces. *Geom. Funct. Anal.*, 15(5):1144–1151, 2005.

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