CUBULATING MAPPING TORI OF POLYNOMIAL GROWTH FREE GROUP AUTOMORPHISMS

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Abstract. Let $\Phi : F \to F$ be a polynomially-growing automorphism of a finite-rank free group F. Then $G = F \rtimes_{\Phi} \mathbb{Z}$ acts freely on a CAT(0) cube complex.

1. Introduction

The goal of this paper is to prove:

Theorem A. *Let F be a finite-rank free group and let $\Phi : F \to F$ be a polynomially-growing automorphism. Then $G = F \rtimes_{\Phi} \mathbb{Z}$ acts freely on a CAT(0) cube complex.*

We emphasize that the action of G on the cube complex above has exotic properties: it is not in general metrically proper, and the cube complex is not in general locally finite or finite-dimensional.

Theorem A applies to natural examples coming from a diverse and well-studied class of groups, mapping tori of free-group automorphisms. There has been a wave of recent interest in this topic, whose broad appeal is illustrated by the many recent results; see e.g. [AKHR15, AKR15, BBC10, BMMV06, BG10, CL14, Cla15, CP10, DR10, DKL13, DKL15, Kap14, DT16, KL15, Lev09, Lus14, Mac00, Mac02, Rey10, Sch08].

Relative train track maps provide a fundamental tool for studying free-by-cyclic groups (see [BH92, BFH00, BFH05]) and classify them according to the dynamics of the monodromy. After replacing Φ with a power (thus replacing G with a finite-index subgroup), Φ is represented by a so-called *improved relative train track map* in the sense of [BFH00]. In this paper, we focus on groups of the form $G = F \rtimes_{\Phi} \langle t\rangle$ where Φ is *polynomially growing* in the sense that there exists a polynomial h so that $|\Phi^n(f)| \leq h(n)|f|$ for each $f \in F$. We call such a G a *polynomial free-by-$\mathbb{Z}$ group*. The nature of the improved relative train track in the polynomial case shows that G splits as an iterated HNN extension with cyclic edge groups at each stage. Specifically, $G = G_n$, where $G_{i+1} = \langle G_i, e_i \mid e_i t e_i^{-1} = p_i t\rangle$ where $p_i \in G_i$. This sequence can be arranged to terminate with $G_0 = \langle t\rangle$.

This sequence of splittings has been used to prove various results illuminating the geometry of polynomial free-by-$\mathbb{Z}$ groups. For example, in [Mac02], Macura studied the divergence function of these groups, a helpful quasi-isometry invariant, showing that the divergence of G is $\sim x^{r+1}$, where

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r is the order of the (polynomial) growth of Φ. In fact, it is implicit in [Mac02] that G is *thick* in the sense of [BDM09] (this is made explicit in [BH18] using the same splittings).

Several other theorems on the geometry of free-by-$\mathbb{Z}$ groups illustrate the presence or absence of various nonpositive curvature properties. A free-by-$\mathbb{Z}$ group is hyperbolic when it contains no $\mathbb{Z}^2$ subgroup, i.e. when F has no Φ–periodic nontrivial conjugacy class [BF92, Bri00]. However, in the polynomial case, G is never hyperbolic (unless $G \cong \mathbb{Z}$ which happens in the degenerate case where F is trivial). In fact, polynomial free-by-$\mathbb{Z}$ groups need not be CAT(0), or even admit a proper semisimple action on a CAT(0) space, as shown by the following example of Gersten [Ger94]:

$$\langle a, b, c, t \mid a^t = a, b^t = ba, c^t = ca^2 \rangle$$

Nonetheless, polynomial free-by-$\mathbb{Z}$ groups enjoy properties reminiscent of nonpositive curvature: Macura showed that they satisfy a quadratic isoperimetric inequality [Mac00], which was generalized by Bridson-Groves to all free-by-$\mathbb{Z}$ groups [BG10]. Both results use improved relative train track maps.

The story of free-by-$\mathbb{Z}$ groups acting on CAT(0) cube complexes is arduous. If $G = F \rtimes_\Phi \mathbb{Z}$ is hyperbolic then G acts freely and cocompactly on a CAT(0) cube complex [HW13, HW14]. When F has rank 2, the automorphism is represented by a surface group automorphism. In this situation, the mapping torus is either a hyperbolic manifold or a graph manifold with a single block which is virtually a product, so the fundamental group is virtually cocompactly cubulated (the hyperbolic case is done in [Wis11]). Moreover, Button-Kropholler [BK15] have gone further and shown directly that G is the fundamental group of a compact nonpositively curved 2-dimensional cube complex. However, in view of Haglund's semisimplicity theorem [Hag07], Gersten's polynomial free-by-$\mathbb{Z}$ group cannot act freely on a proper CAT(0) cube complex.

In addition to being free-by-$\mathbb{Z}$, Gersten's group is *tubular*, i.e. it splits as a graph of groups with $\mathbb{Z}^2$ vertex groups and cyclic edge groups. There are many tubular groups that act freely on CAT(0) cube complexes, even though they cannot act metrically properly. A necessary and sufficient condition for acting freely was provided in [Wis14], and in particular, Gersten's group satisfies this condition. Recently, Button characterized the free-by-$\mathbb{Z}$ groups that are tubular, and applied the condition from [Wis14] to see that every tubular free-by-$\mathbb{Z}$ group acts freely on a CAT(0) cube complex [But15]. Since tubular groups are thick of order ≤ 1 relative to the $\mathbb{Z}^2$ vertex groups, results of [Mac02, BD14] show that if $G \cong F \rtimes_\Phi \mathbb{Z}$ is tubular, then Φ is linearly-growing.

Theorem A is motivated by many of the above results. Most directly, it generalizes the tubular/linear-growth case to arbitrary polynomial free-by-$\mathbb{Z}$ groups and also provides a non-hyperbolic counterpart to the cubulation of hyperbolic free-by-$\mathbb{Z}$ groups. It also broadens the agenda of [Wis14]: instead of studying a single splitting of a group as a graph of groups with cyclic edge groups, we are instead studying a sequence of iterated cyclic HNN extensions, where the edge groups at each stage can involve stable letters from previous stages. Thus, although the initial framework of the proof of Theorem A is superficially similar to that used for tubular groups, our situation is significantly more complicated.

The main object in this paper is a space D consisting of a nonpositively curved cube complex with a cylinder attached along its boundary circles, which we may assume are local geodesics. The fundamental group of D is G_n, and the cube complex has fundamental group G_{n-1}. Attaching the cylinder performs the HNN extension.

The results of this paper contribute to the problem of cubulating a cyclic HNN extension of a cubulated group. When D is word-hyperbolic, this problem has a positive solution [HW15]. However, this is not the case in general. For instance, the Baumslag-Solitar group $BS(1, 2) \cong$

$\langle e, a \mid eae^{-1} = a^2 \rangle$ cannot act freely on a CAT(0) cube complex [Hag07], and even assuming that there are no bad Baumslag-Solitar subgroups, there is even an example of an HNN extension of a nonpositively curved square complex with a special double cover which cannot act freely on a CAT(0) cube complex [HW15, Example 8.7].

1.1. **Summary of the proof.** Let $G = F \rtimes_\Phi \langle t \rangle$. We first reduce to the case where Φ is represented by an improved relative train track map enabling us to decompose G as an iterated HNN extension $G = G_n \cong \langle G_{n-1}, e_n \mid e_n t e_n^{-1} = p_n t \rangle$, where G_{n-1} is a polynomial free-by-cyclic group and $p_n, t \in G_{n-1}$ (see Remark 2.4). The stable letter e_n comes from the top stratum of the relative train track representative. We induct on n.

The inductive hypothesis is not simply that G_{n-1} acts freely on a CAT(0) cube complex, but that G_{n-1} acts freely on a CAT(0) cube complex $C_{\circledcirc}^{n-1} \times C_{\sharp}^{n-1}$, with the following properties. First, each element of G_n, including t, that will generate an edge-group in the future splitting of G_n acts with translation length 0 on $C_\sharp$, while each of these elements have the same, positive, translation length on $C_{\circledcirc}^{n-1}$. In the base case, this is easily satisfied by taking $C_{\circledcirc}^0$ to be a line with $\langle t \rangle$ acting by translations, and taking $C_\sharp^0$ to be a point.

In the inductive step, we attach cyclinders to the quotient $G^{n-1} \setminus C_\sharp^{n-1}$, performing a multiple HNN extension conjugating $p_i t$ to t for each p_i that already appears in G_{n-1}. We use the equality of the translation lengths of the attaching circles to extend the hyperplanes of $C_{\circledcirc}^{n-1}$ to walls in G_n. Using cubical small-cancellation theory, and discarding some walls, we modify this wallspace structure to get a new cube complex $C_\sharp^n$, equipped with a G^n–action where t fixes a point and each element of G^n that could generate an edge-group of a future splitting (these elements are necessarily hyperbolic on the Bass-Serre tree) also has translation length 0.

We also construct $C_{\circledcirc}^n$, with a G_n–action in which the above hyperbolic elements, and t, have the same positive translation length. The idea is roughly to produce a virtual surjection from G_n to $H \times \langle t \rangle$, where H is a free group, such that all future edge-groups get sent to elements of the form pt^d, for some fixed $d > 0$.

We then apply Lemma 3.4, which provides a (non-free) action of $H \times \langle t \rangle$ on a CAT(0) cube complex where each $\langle p, t \rangle$ acts freely and each pt^d has the same (positive) translation length as pt^d. This lemma is of independent interest, and relies on *canonical completions* for maps of graphs. We thus have a virtual action of G_n on this cube complex, which we promote to an action of G_n, with the desired translation lengths, in a standard way.

1.2. **Problems.** We conclude with problems suggested by the proof of Theorem A.

1.2.1. *Metrically proper actions.* As we are dealing with CAT(0) cube complexes that may not be locally finite or finite-dimensional, we must distinguish between free actions and metrically proper actions. In view of Theorem A, it is reasonable to ask:

Problem 1. Which polynomial free-by-cyclic groups act metrically properly on CAT(0) cube complexes, and, slightly more strongly, which such groups act freely on proper CAT(0) cube complexes? For which polynomially-growing Φ can one produce a free action of G on a finite-dimensional CAT(0) cube complex? Additionally, we wonder if there is a characterization of the polynomially-growing Φ for which $F \rtimes_\Phi \mathbb{Z}$ (virtually) acts freely and cocompactly on a CAT(0) cube complex.

For tubular groups, a criterion for finite-dimensional cubulation was given by Woodhouse [Woo15], which is part of a proof that tubular groups acting freely on locally finite CAT(0) cube complexes are virtually special [Woo16a]. We wonder about the implications between the following statements for a polynomial free-by-$\mathbb{Z}$ group G:

- G acts metrically properly on a CAT(0) cube complex;
- G acts freely on a locally finite CAT(0) cube complex;
- G acts freely on a uniformly locally finite CAT(0) cube complex;
- G is virtually special.

The class of tubular free-by-$\mathbb{Z}$ groups shows that among G with Φ linearly-growing one can already find examples where no metrically proper action exists. Accordingly, an answer to Problem 1 must involve more subtle properties of Φ than the growth. We suspect it has to do with the nature of the splitting of G as an iterated cyclic HNN extension $G_n = \langle G_{n-1}, e_n \mid e_n t e_n^{-1} = p_n t\rangle$. Intuitively, in the situation where the rate of polynomial growth is n, i.e. each $p_i \in G_{i-1}$ involves the stable letter in the splitting of G_{i-1}, we are in a "hyperbolic-like" situation where it is possible that there is an easier cubulation, possibly giving rise to a nicer cube complex and a metrically proper action.

The cubulations should be forced to be nasty if the edge-groups $\langle p_i t\rangle$ appear early in the hierarchy relative to i. Accordingly, consider the following situation: $G_0 \cong F_0 \times \langle t\rangle$ and $G_{i+1} = \langle G_t, e_{i+1} \mid e_{i+1} t e_{i+1}^{-1} = f_i t\rangle$ for $0 \leq i \leq n$, where each $f_i \in F_0$. For the purposes of obtaining a metrically proper action of $G = G_n$ on a cube complex, is this in some sense a worst-case scenario? For which choices of $\{f_i\}$ can one produce such an action?

1.2.2. *Arbitrary mapping tori.* Finally, the methodology used here is totally unrelated to the technique used to cubulate mapping tori of hyperbolic free group automorphisms in [HW14] [HW13]. However, we hope the methods will eventually be combined to resolve:

Problem 2. Show that every free-by-$\mathbb{Z}$ group acts freely on a CAT(0) cube complex.

We do not expect that a free action on a locally infinite CAT(0) cube complex will shed a large amount of light on the structure of the free-by-$\mathbb{Z}$ group, although there are some interesting consequences such as undistortedness of maximal finitely generated free abelian subgroups [Woo16b]. It is nonetheless intriguing that groups in this natural class admit such actions.

Organization of the paper. Section 2 contains background on relative train track maps, actions on CAT(0) cube complexes, cubical small-cancellation theory, and a useful lemma about actions on trees with $\mathbb{Z}$ edge-stabilizers. In Section 3, we construct the required cubical action of $H \times \mathbb{Z}$, and the main lemmas used in the inductive step — i.e. the constructions of $C^n_{\circledcirc}$ and $C^n_{\sharp}$ — are proved in Section 4. Theorem 1 is finally proved in Section 5.

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2. Background

2.1. **Relative train track maps for polynomial automorphisms.** Throughout, F is a finitely-generated free group and $\Phi \in \mathrm{Aut}(F)$.

Lemma 2.1. *Let $G = F \rtimes_{\Phi} \langle t\rangle$, let $f \in F$, let $H = \langle ft\rangle$. Then $H \leq G$ is separable.*

Proof. The subgroup $\langle ft\rangle$ is a retract of G for each $f \in F$: the retraction sends each $h \in F$ to 1 and sends t to ft. Any retract of a residually finite group is separable: indeed, since G is residually finite, the profinite topology on G is Hausdorff, and retracts of Hausdorff spaces are closed. □

Definition 2.2 (Polynomial growth)**.** The automorphism Φ is *polynomially growing* if for each $f \in F$, the word length $|\Phi^k(f)|$ grows polynomially as $k \to \infty$.

We set up our proof of Theorem A using a result of [BFH00] saying that each element of $\mathrm{Aut}(F)$ has a nonzero power represented by an *improved relative train track map*, a homotopy equivalence of graphs with numerous useful properties. The full statement can be found in [BFH00, Theorem 5.1.5], but we need just the following:

Proposition 2.3 (Relative train tracks for polynomially-growing automorphisms)**.** *Let F be a finitely-generated free group and let $\Phi : F \to F$ be a polynomially-growing automorphism. Then there exists $k > 0$, a connected graph Γ, and a homotopy equivalence $f : \Gamma \to \Gamma$, representing Φ^k, with the following properties:*

(1) There is a sequence of (f–invariant) subgraphs $\emptyset = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \ldots \subset \Gamma_r = \Gamma$ such that $\Gamma_i - \mathrm{Int}(\Gamma_{i-1})$ consists of a single (closed) edge E_i for $1 \leq i \leq r$;
(2) $f(E_i) = E_i P_{i-1}$, where P_{i-1} is a (possibly trivial) closed combinatorial path whose edges are in Γ_{i-1} and whose basepoint is fixed by f. Thus $f(v) = v$ for all $v \in \Gamma^0$.

Remark 2.4 (The multiple HNN extension)**.** Our application of Proposition 2.3 will be as follows.

Connecting the Γ_i: First, it will be (notationally) convenient to work with connected Γ_i. Hence let $\bar{\Gamma}$ be obtained from Γ by identifying all of the vertices, so that each edge E of Γ projects to a loop $\bar{E}$, and f descends to a cellular homotopy equivalence $\bar{f} : \bar{\Gamma} \to \bar{\Gamma}$. Moreover, the image $\bar{\Gamma}_i$ of each Γ_i is connected, and $\bar{\Gamma}_i$ is obtained from $\bar{\Gamma}_{i-1}$ by adding $\bar{E}_i$. For each i, we have $\bar{f}(\bar{E}_i) = \bar{E}_i \bar{P}_i$, i.e. $\bar{f}$ satisfies the conclusion of Proposition 2.3.

Let $\bar{\Phi}^k$ be the automorphism of $\pi_1 \bar{\Gamma}$ induced by $\bar{f}$. We claim that $F \rtimes_{\Phi^k} \mathbb{Z}$ embeds in $\pi_1 \bar{\Gamma} \rtimes_{\bar{\Phi}^k} \mathbb{Z}$. Indeed, let $\hat{\Gamma}$ be the graph obtained by choosing an ordering $v_1, \ldots, v_s$ of the vertices of Γ and adding edges $(v_1, v_2), \ldots, (v_{s-1}, v_s)$. The subgraph of $\hat{\Gamma}$ consisting of the union of the new edges is a path. Extend f to a map $\hat{f} : \hat{\Gamma} \to \hat{\Gamma}$ by declaring that $\hat{f}$ sends each of these new edges to itself identically. Then $\hat{\Gamma}$ is homotopy-equivalent to $\bar{\Gamma}$ via a deformation retraction collapsing the new edges, and this homotopy takes $\hat{f}$ to $\bar{f}$. On the other hand, $\hat{f}$ restricts on $\Gamma \subset \hat{\Gamma}$ to f. Now, for each new edge e, the mapping torus of $\hat{f}$ contains an annulus $e \times \mathbb{S}^1$; cutting along the core curve of the annulus, for each such e, induces a graph of groups decomposition of $\pi_1 \bar{\Gamma} \rtimes_{\bar{\Phi}^k} \mathbb{Z}$ whose edge groups are isomorphic to $\mathbb{Z}$ and whose vertex groups are isomorphic to the fundamental group of the mapping torus of f, i.e. $F \rtimes_{\Phi^k} \mathbb{Z}$. Hence $F \rtimes_{\Phi^k} \mathbb{Z} \leq \pi_1 \bar{\Gamma} \rtimes_{\bar{\Phi}^k} \mathbb{Z}$.

Why it is sufficient to work with connected Γ_i: In the proof of Theorem 1, we will see that it suffices to produce a free action of $F \rtimes_{\Phi^k} \mathbb{Z}$ on a CAT(0) cube complex, where $k > 0$ is the power from Proposition 2.3. Hence it suffices to produce such an action for any supergroup of $F \rtimes_{\Phi^k} \mathbb{Z}$, and in particular it is sufficient to work with $\pi_1 \bar{\Gamma} \rtimes_{\bar{\Phi}^k} \mathbb{Z}$.

Moreover, for convenience, we can and shall take Γ_0 to be a single (f–invariant) vertex instead of $\emptyset$ (as in Proposition 2.3).

The final multiple HNN extension: Hence we can and shall assume that all relative train track maps in this paper are have domains Γ with each Γ_i connected. Each stratum from the proposition thus yields a splitting as a multiple HNN extension with $\mathbb{Z}$ edge groups, so after replacing G if necessary by a finite-index subgroup (obtained by replacing Φ by an appropriate positive power), we have a sequence $F_0 \times \langle t \rangle = G_0 \leq G_1 \leq \cdots \leq G_n = G$ of subgroups so that the following holds:

- For $0 \leq i \leq n$, we have $G_i = F_i \rtimes \langle t \rangle$, where F_i is a free group, and $G_0 = F_0 \times \langle t \rangle$.

- For $0 \leq i \leq n-1$, the free group F_i contains a finite multiset $\mathcal{P}_i$ such that, for $1 \leq i \leq n$, the group G_i splits as a graph of groups with G_{i-1} the single vertex group, and $\mathbb{Z}$ edge groups, as follows:
$$G_i = \langle G_{i-1}, e^i_1, \ldots, e^i_{n_i} \mid e^i_j t (e^i_j)^{-1} = p^i_j t,\ 1 \leq j \leq n_i \rangle,$$
where $\{p^i_j : 1 \leq j \leq n_i\} = \mathcal{P}_i \subset F_{i-1}$.
- For all $i \geq 1$, each $p \in \mathcal{P}_{i+1} \subset G_i$ and each pt with $p \in \mathcal{P}_{i+1}$ acts as a hyperbolic isometry of the Bass-Serre tree $\mathcal{T}_i$ for the splitting of G_i described above.

Since we have arranged for Γ_0 to be a single vertex, we can and shall take $F_0 = \{1\}$.

2.2. **Cubical translation length.** Let C be a CAT(0) cube complex. Then $g \in \mathrm{Aut}(C)$ is *(combinatorially) elliptic* if g fixes a 0–cube and *(combinatorially) hyperbolic* if there is a bi-infinite combinatorial geodesic $\widetilde{A}$ in C so that g stabilizes $\widetilde{A}$ in the following way: there exists $\tau \in \mathbb{Z}$ so that, regarding $\widetilde{A}$ as an isometric embedding $\widetilde{A} : \mathbb{R} \to C^1$, we have $g\widetilde{A}(t) = \widetilde{A}(t+\tau)$ for all $t \in \mathbb{R}$. By Corollary 5.2 of [Hag07], the translation length τ is the same for any choice of axis $\widetilde{A}$, and in fact τ is the minimal distance by which g moves a 0–cube. By replacing C by a *cubical subdivision* – i.e. by subdividing C so that the hyperplanes become subcomplexes and each hyperplane is replaced by 2 parallel hyperplanes – we can assume that any fixed group $G \leq \mathrm{Aut}(C)$ has the property that each $g \in G$ is either elliptic or hyperbolic [Hag07]. Indeed, it is shown in [Hag07] that the action of G on C has this property provided G acts *without inversions in hyperplanes* in the sense that whenever $g \in G$ stabilizes a hyperplane, it stabilizes each of the associated halfspaces. Passing to the cubical subdivision of C ensures that the action is without inversions.

Notation 2.5 (Translation length)**.** Let C be a CAT(0) cube complex. If $g \in \mathrm{Aut}(C)$ fixes a point, let $\|g\| = 0$. If $g \in \mathrm{Aut}(C)$ nontrivially stabilizes a combinatorial geodesic $\tilde{A}$ in C, then $\|g\|$ denotes the number of $\langle g \rangle$–orbits of 1–cubes in $\tilde{A}$. Note that if G acts on C without inversions, then $\|g\|$ is well-defined for all $g \in G$. Since we will consider actions on multiple cube complexes, we adopt the following convention on subscripts, where it will not cause confusion: when g is acting on $C_\heartsuit$, then we use the notation $\|g\|_\heartsuit$.

Observation 2.6. Suppose that $C_\heartsuit$ and $C_\diamondsuit$ are CAT(0) cube complexes, and G acts on $C_\heartsuit$ and $C_\diamondsuit$. Let $C = C_\heartsuit \times C_\diamondsuit$. Then, with respect to the product action $G \to \mathrm{Aut}(C)$,
$$\|g\|_C = \|g\|_\heartsuit + \|g\|_\diamondsuit$$
for all $g \in G$. Moreover, $\|g^d\|_\heartsuit = |d|\|g\|_\heartsuit$ for all $g \in G$ and $d \in \mathbb{Z}$.

2.3. **Cutting.** We now recall a special case of the notion of a *wallspace*, introduced in [HP98], and refer the reader to [HW] for a more detailed discussion.

A *wallspace* $(\widetilde{X}, \mathcal{W})$ consists of a metric space $\widetilde{X}$ with a set $\mathcal{W}$ of connected subspaces $\widetilde{W}$, called *walls*, so that $\widetilde{X} - \widetilde{W}$ has exactly two components, called *halfspaces*, with the additional property that for all $x, y \in \widetilde{X}$, there are finitely many $\widetilde{W} \in \mathcal{W}$ such that x, y lie in different components of $\widetilde{X} - \widetilde{W}$ (in which case $\widetilde{W}$ *separates* x, y). An *automorphism* of $(\widetilde{X}, \mathcal{W})$ is an isometry $g : \widetilde{X} \to \widetilde{X}$ so that $g\widetilde{W} \in \mathcal{W}$ for $\widetilde{W} \in \mathcal{W}$; the group of automorphisms of $(\widetilde{X}, \mathcal{W})$ is denoted $\mathrm{Aut}(\widetilde{X}, \mathcal{W})$.

Definition 2.7 (Cutting, parallel)**.** Let $(\widetilde{X}, \mathcal{W})$ be a wallspace and let $g \in \mathrm{Aut}(\widetilde{X}, \mathcal{W})$. Let $A : \mathbb{R} \to \widetilde{X}$ be an embedding with g–invariant image so that $g \circ A : \mathbb{R} \to \widetilde{X}$ is increasing. Then $\widetilde{W} \in \mathcal{W}$ *cuts* g if there exists $x \in \mathbb{R}$ so that $A((x, \infty))$ and $A((-\infty, x))$ lie in different halfspaces associated to $\widetilde{W}$. If no such A exists, then the set of walls cutting g is empty.

We call $g, h \in \mathrm{Aut}(\widetilde{X}, \mathcal{W})$ *parallel* if: each $W \in \mathcal{W}$ cuts g if and only if W cuts h.

The CAT(0) cube complex $C = C(\widetilde{X}, \mathcal{W})$ *dual to* the wallspace $(\widetilde{X}, \mathcal{W})$ was defined in [Sag95]. Denoting by $\widehat{\mathcal{W}}$ the set of halfspaces, the 0–cubes of C are maps $c : \mathcal{W} \to \widehat{\mathcal{W}}$ sending each wall to one of the two associated halfspaces, subject to: $c(\widetilde{W}) \cap c(\widetilde{W}') \neq \emptyset$ for $\widetilde{W}, \widetilde{W}' \in \mathcal{W}$, and for all $x \in \widetilde{X}$, we have $|\{\widetilde{W} : x \notin c(\widetilde{W})\}| < \infty$. The 0–cubes c, c' are joined by a 1–cube if and only if there is a unique wall $\widetilde{W}$ with $c(\widetilde{W}) \neq c'(\widetilde{W})$, and higher-dimensional cubes are added when their 1–skeleta appear.

If G acts on $(\widetilde{X}, \mathcal{W})$, then there is an induced G–action on C defined as follows: for $g \in G$ and $c \in C^0$, we have $(gc)(\widetilde{W}) = g \cdot c(g^{-1}\widetilde{W})$ for all $\widetilde{W} \in \mathcal{W}$. This action clearly takes hyperplanes to hyperplanes, and there is a G–equivariant bijection from $\mathcal{W}$ [respectively $\widehat{\mathcal{W}}$] to the set of hyperplanes [respectively, halfspaces] in C.

We now state two lemmas, the first of which is [Wis14, Lemma 2.1]. The second follows by considering hyperplanes intersecting axes in the dual cube complex.

Lemma 2.8 (Cut-wall criterion)**.** *Let G act on a wallspace $(\widetilde{X}, \mathcal{W})$ and suppose that each $g \in G - \{1_G\}$ is cut by some wall in $\mathcal{W}$. Then G acts freely on $C(\widetilde{X}, \mathcal{W})$.*

Lemma 2.9 (Cut-walls and translation length in $C(\widetilde{X}, \mathcal{W})$)**.** *Let G act on the wallspace $(\widetilde{X}, \widetilde{W})$ and let $g \in G$. Let $\|g\|_{\mathcal{W}}$ be the number of $\langle g \rangle$–orbits of walls that cut g. Then the translation length $\|g\|$ of g on C is defined and $\|g\| = \|g\|_{\mathcal{W}}$.*

We remark that if $g \in G$ has finite order, then g cannot be cut by a wall in $\mathcal{W}$, so $\|g\|_{\mathcal{W}} = 0$. On the other hand, g must fix a point in C (see [Hag07]) so $\|g\| = 0$. In this paper, the groups under consideration are torsion-free, so this situation only arises for $g = 1$.

2.4. **Cubical small-cancellation theory.** We review the following background from [Wis11]. Throughout, $\widetilde{X}$ denotes a CAT(0) cube complex and X a nonpositively-curved cube complex. For a hyperplane $\widetilde{U}$ of $\widetilde{X}$, we denote by $N(\widetilde{U})$ its *carrier*, i.e. the union of all closed cubes intersecting $\widetilde{U}$. We do the same for (immersed) hyperplanes in X. The *systole* $\|X\|$ is the infimal length of an essential combinatorial closed path in X.

2.4.1. *Cubical presentations and pieces.*

Definition 2.10 (Cubical presentation)**.** A *cubical presentation* $\langle X | \{Y_i : i \in I\} \rangle$ consists of connected non-positively curved cube complexes X and $\{Y_i\}$, and local isometries $\{Y_i \to X : i \in I\}$. We use the notation X^* for the cubical presentation above. As a topological space, X^* is X with a cone on each Y_i.

More background on cubical presentations appears in Section 3 of [Wis11]. Since each $Y_i \to X$ is a local isometry, it is π_1–injective, and we have $\pi_1 X^* \cong \pi_1 X / \langle\langle \{\pi_1 Y_i : i \in I\} \rangle\rangle$. For each $i \in I$, the cone over Y_i in X^* lifts to the universal cover $\widetilde{X}^*$, and accordingly $Y_i \to X \to X^*$ lifts to $Y_i \to \widetilde{X}^*$.

Definition 2.11 (Piece)**.** A *cone-piece* of X^* in Y_i is a component of $\widetilde{Y}_i \cap g\widetilde{Y}_j$, where $g \in \pi_1(X)$, excluding the case where $i = j$ and $g \in \mathrm{Stab}(\widetilde{Y}_i)$. A *wall-piece* of X^* in Y_i is a component of $\widetilde{Y}_i \cap N(\widetilde{U})$, where $\widetilde{U}$ is a hyperplane that is disjoint from $\widetilde{Y}_i$. A *piece-path* in Y is a path in a piece of Y.

2.4.2. *Simplified $B(6)$ condition.*

Definition 2.12 ($C'(\alpha)$ cubical presentation)**.** The cubical presentation $X^* = \langle X \mid \{Y_i : i \in I\}\rangle$ satisfies the $C'(\alpha)$ *condition* if $|P| \leq \alpha|S|$ for each geodesic piece-path P and each essential closed path $S \to Y_i$ with P a subpath of S.

The following is a simplified form of the $B(6)$ condition described in [Wis11]; see Definition 5.1 and Remark 5.2.

Definition 2.13 (Simplified $B(6)$)**.** Let $X^* = \langle X \mid \{Y_i : i \in I\}\rangle$ be a cubical presentation satisfying the $C'(1/14)$ condition. Then X^* satisfies the $B(6)$ *condition* if the following hold:

(1) for each $i \in I$, there is an equivalence relation $\sim_i$ on the hyperplanes of Y_i.
(2) for each $i \in I$ and each pair of (possibly equal) hyperplanes U, V of Y_i with $U \sim_i V$, the hyperplanes U, V do not cross or osculate (recall that hyperplanes U, V *osculate* if they are disjoint but have intersecting carriers);
(3) for each $i \in I$, and each $\sim_i$–class $[U]$ of hyperplanes, $W = \cup_{V\in[U]} V$ is a *wall*: the space $Y_i - W$ consists of two subspaces $\overleftarrow{Y}_i, \overrightarrow{Y}_i$ such that $\mathrm{Cl}(\overleftarrow{Y}_i) \cap \mathrm{Cl}(\overrightarrow{Y}_i) = W$;
(4) if $P \to Y_i$ is a path that is the concatenation of at most 7 piece-paths and P starts and ends on the carrier $N(U)$ of a wall then P is path-homotopic into $N(U)$;
(5) $\mathrm{Aut}(Y_i \to X)$ preserves the wallspace structure.

The walls in Y_i Definition 2.13 need not be connected, so differ from the walls discussed in Section 2.3.

The $B(6)$ condition provides a wallspace structure on $\widetilde{X^*}$ as follows: two hyperplanes of $\widetilde{X^*}$ are *elementary equivalent* if they intersect a lift $Y_i \hookrightarrow \widetilde{X^*}$ of some Y_i in equivalent hyperplanes. The transitive closure of this relation is an equivalence relation on the hyperplanes of $\widetilde{X^*}$ and the union of the hyperplanes in each class is a wall. The CAT(0) cube complex dual to this wallspace is the cube complex *associated* to X^*; observe that $\pi_1 X^*$ acts on this cube complex.

2.4.3. *Free action of $\pi_1 X^*$ on the associated cube complex.*

Definition 2.14 (Proximate)**.** Let $\langle X \mid \{Y_i : i \in I\}\rangle$ be a cubical presentation satisfying Definition 2.13.(1),(2),(3). A hyperplane U in Y_i is *proximate* to a 0–cube v of Y_i if there is a 1-cube u dual to U and a path AB that starts with u and ends with v, and where each of A and B is either a 1-cube or lies in a piece. A wall W in Y_i is *proximate* to a v if some hyperplane U of W is proximate to v.

Given $\langle X \mid \{Y_i : i \in I\}\rangle$ as in Definition 2.14, a set $\mathcal{R}$ of hyperplanes of X is *preferred* if each Y_i has the following property: let W be a wall in Y_i. Then either all hyperplanes of W map to a hyperplane of X belonging to $\mathcal{R}$, or no hyperplane of W maps to a hyperplane in $\mathcal{R}$. If $\mathcal{R}$ is a preferred set of hyperplanes in X, then a wall $\widetilde{W}$ of $\widetilde{X}^*$ is *preferred* (with respect to $\mathcal{R}$) if some (and hence any) hyperplane in $\widetilde{W}$ maps to a preferred hyperplane in X. If $\widetilde{W}$ is not preferred, then none of its constituent hyperplanes maps to a preferred hyperplane in X.

The following is Theorem 5.40 in [Wis11]:

Theorem 2.15. *Let $\langle X \mid \{Y_i\}_i\rangle$ be a cubical presentation. Suppose that:*

(1) X^* satisfies the $B(6)$ condition and has short innerpaths *in the sense of* [Wis11, Definition 3.53]*.*
(2) X contains a preferred set $\mathcal{R}$ of hyperplanes.

(3) *The following holds for each* $Y \in \{Y_i\}$. *Let* $\kappa \to Y$ *be a geodesic with endpoints* p, q. *Let* w_1, w_1' *be hyperplanes of* Y *lying in the same wall and mapping under* $Y \to X$ *to hyperplanes in* $\mathcal{R}$. *Suppose that* κ *traverses a* 1*–cube dual to* w_1 *and either* w_1' *is proximate to* q *or* w_1' *is dual to a* 1*–cube traversed by* κ. *Then there is a preferred wall* w_2 *in* Y *that separates* p, q *but is not proximate to* p *or* q.

Let $g \in \pi_1 X^*$. *Then one of the following holds:*

(1) *there exists* $Y \in \{Y_i\}$ *so that* $g \in \mathrm{Aut}(Y)$ *for some lift of* Y *to* $\widetilde{X}^*$;
(2) g *is cut by a preferred wall of* $\widetilde{X}^*$;
(3) g *is the image of some* $\tilde{g} \in \pi_1 X$ *that is not cut by a hyperplane of* $\widetilde{X}$ *mapping to a hyperplane in* $\mathcal{R}$;
(4) $g \in \pi_1 X^*$ *has finite order.*

In our applications, $\langle X \mid \{Y_i\}_i \rangle$ will satisfy the short innerpaths condition by [Wis11, Lemma 3.67] because of the metric small-cancellation condition. Therefore, it is not necessary to define short innerpaths.

We will apply Theorem 2.15 in the case where each Y_i is a specific CAT(0) cube complex mapping by a local isometry to X. We also need Theorem 5.20 in [Wis11], i.e.:

Theorem 2.16. *Let* X^* *be a* $B(6)$ *presentation and let* W *be a wall in* $\widetilde{X}^*$. *If* H_1, H_2 *are hyperplanes in* W, *and* Y *is a cone, then* $H_1 \cap Y$ *and* $H_2 \cap Y$ *lie in the same wall of* Y.

2.5. **Graphs of groups with cyclic edge groups.** The following lemma about groups acting on trees is fundamental in the proof of Theorem A.

Lemma 2.17. *Let* H *act without inversions on a tree* $\mathcal{T}$. *Suppose that for each edge* e *of* $\mathcal{T}$, *the following hold:*

(1) *the edge group* $H_e = \langle h_e \rangle$ *is a maximal* $\mathbb{Z}$ *subgroup;*
(2) *if* $k h_e^p k^{-1} = h_e^q$ *for some* $k \in H, p, q \neq 0$ *then* $k h_e = h_e k$.

Let L *be a combinatorial line in* T *with stabilizer* $K \leq H$. *Then both of the following hold:*

- *every element of* K *either acts as a translation on* L *or fixes* L *pointwise;*
- *one of the following holds:*
  - $K \cong \{1\}$;
  - $K \cong \mathbb{Z}$ *and* K *acts by translations on* L;
  - *each* $k \in K$ *centralizes* h_e *for each edge* e *of* L.

Proof. Denote by $\mathrm{Aut}(L)$ the group of combinatorial automorphisms of L, so that the action of K on L gives a homomorphism $K \to \mathrm{Aut}(L)$.

Suppose that v is a vertex of L with incident edges e, e'. Let $H_e = \langle h_e \rangle$ and $H_{e'} = \langle h_{e'} \rangle$. Suppose $x \in K$ exchanges e, e'. Then $x^2 \in \langle h_e \rangle$, but $x \notin H_e$, contradicting maximality of the edge groups. Hence, since K does not stabilize an edge of L but invert its endpoints, $\mathrm{Im}(K \to \mathrm{Aut}(L))$ is trivial or infinite cyclic, which proves the first assertion.

Note that $\mathrm{Ker}(K \to \mathrm{Aut}(L)) = \cap_{e \in \mathrm{Edges}(L)} H_e$. Suppose that $k \in K$ and $h \in \mathrm{Ker}(K \to \mathrm{Aut}(L)) - \{1\}$. Then for each edge e of L, there exists $n \neq 0$ so that $h = h_e^n$. Since k acts as a translation on L, we have that $[k, h]$ stabilizes e, i.e. there exists $\ell \in \mathbb{Z}$ so that $[k, h_e^n] = h_e^\ell$, i.e. $k h_e^n k^{-1} = h_e^{\ell + n}$. Hence $[k, h_e] = 1$ by hypothesis (2). Hence either k centralizes each h_e or $\mathrm{Ker}(K \to \mathrm{Aut}(L)) = \{1\}$. $\square$

3. Cubical actions of $F \times \mathbb{Z}$ with specified translation lengths

The goal of this section is to prove Lemma 3.4. We need a preparatory definition and lemma.

Definition 3.1 (Virtual translation length)**.** Let G be a group and let $G' \leq G$ be a subgroup of index $m < \infty$ acting on a CAT(0) cube complex C'. Let $g \in G$. Let $r > 0$ be such that $g^r \in G'$, e.g. $r = m!$. The *virtual translation length* of g is $\|g\|_{C'}^{virt} = \frac{1}{r}\|g^r\|_{C'}$. Note that this is independent of the choice of r.

Lemma 3.2. *Let G be a group. Let $G' \leq G$ be a finite-index subgroup acting on a CAT(0) cube complex C'. There exists a CAT(0) cube complex C so that G acts on C and the following holds for all $a, b \in G$. Suppose that $\|h_1ah_1^{-1}\|_{C'}^{virt} = \|h_2bh_2^{-1}\|_{C'}^{virt}$ for all $h_1, h_2 \in G$. Then $\|a\|_C = \|b\|_C$.*

Proof. Let $\{g_1, \ldots, g_m\}$ be a complete set of representatives of left cosets of G' in G. By [Wis11, Lem 7.8], G acts on a CAT(0) cube complex $C = \prod_{g_i} C'_{g_i}$, where each C'_{g_i} is a copy of C'. Moreover, each subgroup $g_iG'g_i^{-1}$ stabilizes the g_i–coordinate and acts C'_{g_i} via the given G'–action on C'.

For any $b \in G$, the translation length $\|b^{m!}\|_C$ is the sum over i of the number of $\langle b^{m!}\rangle$–orbits of hyperplanes in C'_{g_i} that cut $b^{m!}$. For each i, the number of $\langle b^{m!}\rangle$–orbits of hyperplanes in C'_{g_i} that cut $b^{m!}$ is the translation length in C' of $g_ib^{m!}g_i^{-1}$, so

$$\frac{1}{m!}\|b^{m!}\|_C \quad = \quad \frac{1}{m!}\sum_{i=1}^{m}\|g_ib^{m!}g_i^{-1}\|_{C'} = \frac{m}{m!}\|b^{m!}\|_{C'} = m\|b\|_{C'}^{virt}.$$

Similarly, $\|a\|_C = m\|a\|_{C'}^{virt}$. But $m\|b\|_{C'}^{virt} = m\|a\|_{C'}^{virt}$, so the conclusion follows. □

We also need the following special case of [Wis14, Thm 4.5]:

Lemma 3.3. *Let $A \cong \langle a, t \mid [a, t]\rangle$ and let $n \in \mathbb{N}$. Let $\mathcal{P} = \{a^m : 0 < |m| \leq n\}$. Then there exists a free action of A on a CAT(0) cube complex C so that for any $d \geq 1$, we have $\|t^d\|_C = \|pt^d\|_C$ for all $p \in \mathcal{P}$.*

Proof. Consider the action of A on $\mathbb{E}^2$ by affine isometries, with a acting as a unit translation along $(1\ 0)^T$ and t as a unit translation along $(0\ 1)^T$. Let $L_0 \subset \mathbb{E}^2$ be the line through the origin parallel to $(n\ 1)^T$ and let L_1 be the line parallel to $(-n\ 1)^T$. Then L_0, L_1 are geometric walls in $\mathbb{E}^2$, and thus $(\mathbb{E}^2, A \cdot L_0 \cup A \cdot L_1)$ is a geometric wallspace on which A acts. Let C be the dual cube complex.

Let $(x\ y) \in \mathbb{Z}^2$ (so that $(x\ y)$ corresponds to the element $a^xt^y \in A$). Then

$$\|a^xt^y\|_C = \left|\det\begin{pmatrix} x & n \\ y & 1 \end{pmatrix}\right| + \left|\det\begin{pmatrix} x & -n \\ y & 1 \end{pmatrix}\right| = |x - ny| + |x + ny|.$$

For any $x, y \in \mathbb{Z}$, we thus have that $\|a^xt^y\|_C = 0$ only if $x = ny$ and $x = -ny$, which is possible only if $x = y = 0$. Hence the action of A on C is free.

When $x = 0$ and $y = d$, we obtain $\|t^d\|_C = 2dn$. Next, consider the case where $x = m$ for $0 < |m| \leq n$ and $y = d$. Then $\|a^mt^d\|_C = |m - dn| + |m + dn|$. If $m > 0$, this yields $\|a^mt^d\|_C = dn - m + m + dn = 2dn$. (Here, we have used that $|m| \leq n \leq dn$.) If $m \leq 0$, this yields $\|a^mt^d\|_C = dn + |m| + dn - |m| = 2dn$. Hence $\|a^mt^d\|_C = \|t^d\|_C$ for $0 \leq |m| \leq n$, as required. □

We can now state the main lemma of this section.

Lemma 3.4. *Let F be a free group and let $G = F \times \mathbb{Z}$. We regard G as being presented by $\langle F, t \mid [f, t],\ f \in F\rangle$, so each element has the form ft^n for some $f \in F, n \in \mathbb{Z}$. Let $\mathcal{P} \subset F$ be finite. Then for any $d \geq 1$, there exists a CAT(0) cube complex $C_\circledcirc$ and an action of G on $C_\circledcirc$ so that:*

- $\|pt^d\|_{\circledcirc} = \|t^d\|_{\circledcirc}$ *for all* $p \in \mathcal{P}$.
- $\langle p, t\rangle$ *acts freely for all* $p \in \mathcal{P}$.

Proof. Roughly, the idea is to use various canonical completions associated to the $p \in \mathcal{P}$ to dictate virtual retractions from G to various $\mathbb{Z}^2$ subgroups, and apply Lemma 3.3 and Lemma 3.2 to produce, for each p, an action on a CAT(0) cube complex C_p satisfying the conclusion. We finally let $C_{\circledcirc} = \prod_{p\in\mathcal{P}} C_p$.

Simplifying assumptions about $\mathcal{P}$: By replacing $\mathcal{P}$ with a finite superset, we may assume that there are elements $p_1, \ldots, p_n \in F$, generating distinct maximal cyclic subgroups, and natural numbers $n_i \geq 1$, so that $\mathcal{P} = \{p_i^m : 1 \leq i \leq n,\ 0 < |m| \leq n_i\}$.

Representative graphs: Let B be a connected graph with $\pi_1 B$ identified with F, and let S be a circle with one vertex, with $\pi_1 S$ identified with $\langle t\rangle$. We will use the notation p_i to denote both the element $p_i \in F$ and an immersed closed based path in B representing it; we likewise let t denote both the generator of $\pi_1 S$ and a representative embedded closed path.

It suffices to prove the lemma in the case where $\mathcal{P}$ contains at most one element in each conjugacy class, and we shall assume each p_i is a cyclically reduced path. Indeed, conjugate elements will have the same translation length.

Initial finite cover: Let $\widehat{B} \to B$ and $\widehat{S} \to S$ be connected finite regular covers so that:

(1) for each i, each elevation $\hat{p}_i \to \widehat{B}$ of $p_i \to B$ is injective;
(2) there exists $D_0 \geq 1$ so that the degree of $\widehat{p_i^m t^d} \to p_i^m t^d$ is D_0 for each elevation $\widehat{p_i^m t^d} \to \widehat{B} \times \widehat{S}$ of $p_i^m t^d \to B \times S$ and all i, m.

Indeed, let $\widehat{B} \to B$ be a finite regular cover with the first property, which exists by separability of the $\langle p_i\rangle$ in $\pi_1 B$. For each elevation $(p_i^m t^d)'$ of $p_i^m t^d$ to $\widehat{B} \times S$, let δ_{im} be the degree of $(p_i^m t^d)' \to p_i^m t^d$, and let $\widehat{S} \to S$ be a connected D_0–fold cover where $D_0 = \operatorname{lcm}\{\delta_{im}\}$. Hence the degree of $\widehat{p_i^m t^d}$ is D_0 since it is the same as its order in $(\pi_1 B/\pi_1 \widehat{B}) \times (\pi_1 S/\pi_1 \widehat{S})$, since the order D_0 in the second factor is a multiple of the order δ_{im} in the first factor.

Canonical completions: Fix $i \leq n$. By item (1) above, each elevation $\hat{p}_{ij} \to \widehat{B}$ of p_i is embedded. Consider the canonical completion $\mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \to \widehat{B}$, which is a finite cover admitting a retraction $r_{ij} : \mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \to \hat{p}_{ij}$. We also have a canonical completion $\mathsf{C}(\hat{p}_{ij} \to \hat{p}_{ij}) \to \hat{p}_{ij}$. As explained in [Wis12, Section 4.5], the embedding $\hat{p}_{ij} \to \widehat{B}$ induces an embedding $\mathsf{C}(\hat{p}_{ij} \to \hat{p}_{ij}) \to \mathsf{C}(\hat{p}_{ij} \to \widehat{B})$ so that the canonical retraction $\mathsf{C}(\hat{p}_{ij} \to \hat{p}_{ij}) \to \hat{p}_{ij}$ is the restriction of r_{ij}.

Also, $\mathsf{C}(\hat{p}_{ij} \to \hat{p}_{ij}) = \hat{p}_{ij} \sqcup \check{p}_{ij}$, where $\check{p}_{ij}$ is a connected $|\hat{p}_{ij}|(|\hat{p}_{ij}| - 1)$–fold cover of $\hat{p}_{ij}$ on which the canonical retraction map restricts to a map of degree -1.

We now employ $r_{ij} \times \mathrm{id} : \mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \times \widehat{S} \to \hat{p}_{ij} \times \widehat{S}$, whose target is an embedded torus in $\mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \times \widehat{S}$.

Let $(p_i^m t^d)''$be an elevation of $p_i^m t^d$ to $\mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \times \widehat{S}$ that lies in the torus $\hat{p}_{ij} \times \widehat{S}$. Then $(r_{ij} \times \mathrm{id})((p_i^m t^d)'')$ is homotopic to the path $\hat{p}_{ij}^{m'} t^{D_0 d}$, where m' depends on δ_{im} and the degree of $\hat{p}_{ij} \to p_i$.

If $(p_i^m t^d)''$ is homotopic into $\check{p}_i \times \widehat{S}$, then $(r_{ij} \times \mathrm{id})((p_i^m t^d)'')$ is homotopic to the path displayed below. We emphasize that the precise constant D_{mi} is immaterial. We use that $D_{mi} \geq 1$ and D_{mi} is independent of the choice $\hat{p}_{ij}$ of elevation of p_i to $\widehat{B}$ because $\widehat{B} \to B$ is regular. The only important feature of the $\hat{p}_{ij}$ exponent is that it is nonzero. The path is:

$$\hat{p}_{ij}^{\frac{-mD_{mi}}{|\hat{p}_{ij}|-1}} t^{D_0 D_{mi} d} \quad \text{where} \quad D_{mi} = \frac{|\hat{p}_{ij}|}{gcd(m, |\hat{p}_{ij}|-1)}.$$

Otherwise, if $(p_k^m t^d)''$ is an elevation of $p_k^m t^d$ and either $k \neq i$ or $k = i$ but $(p_i^m t^d)''$ is not inside $\mathsf{C}(\hat{p}_{ij} \to \hat{p}_{ij}) \times \widehat{S}$, then $(r_{ij} \times \mathrm{id})((p_k^m t^d)'')$ is homotopic to $t^{D_0 E_{kij} d}$, where E_{kij} depends on the length of the elevations of p_k to $\widehat{B}$ and on the degree of the cover $\mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \to \widehat{B}$.

The finite cover associated to $\hat{p}_{ij}$: Let $D = \mathrm{lcm}\{D_0, \{D_{mi}\}_{mi}, \{E_{kij}\}_{kij}\}$. Let $\ddot{S} \to \widehat{S}$ be a connected D–fold cover. For each i and each elevation $\hat{p}_{ij}$ of p_i to $\widehat{B}$, consider the finite cover $\mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \times \ddot{S} \to \widehat{B} \times S$ and the retraction $r_{ij} \times \mathrm{id} : \mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \times \ddot{S} \to \hat{p}_{ij} \times \ddot{S}$. Then each elevation of each $p_k^m t^d$ to this cover has one of the following two properties:

- it is sent by $r_{ij} \times \mathrm{id}$ to t^{Kd}, where K is a *fixed* integer depending on D_0, D;
- it lies in $\mathsf{C}(\hat{p}_{ij} \to \hat{p}_{ij})$ and is sent by $r_{ij} \times \mathrm{id}$ to a path homotopic to $\hat{p}_{ij}^q t^{Kd}$, where q is one of finitely many nonzero natural numbers.

Indeed, $\ddot{S} \to S$ was chosen precisely so that the induced further covers of the above elevations have the same degrees in the $\ddot{S}$ factor under the above retraction maps.

The cubical action associated to $\hat{p}_{ij}$: By Lemma 3.3, there is an action of $\pi_1(\hat{p}_{ij} \times \ddot{S})$ on a CAT(0) cube complex X_{ij} with the following properties. First, $\langle \hat{p}_{ij}, t^{Kd} \rangle$ acts on X_{ij} freely. Second, t has positive translation length and $\|\hat{p}_{ij}^q t^{Kd}\|_{X_{ij}} = \|t^{Kd}\|_{X_{ij}}$ for all values q appearing above. Let $\rho_{ij} : \pi_1(\hat{p}_{ij} \times \ddot{S}) \to \mathrm{Aut}(X_{ij})$ be this action.

For each i, we have $a_i \in \mathbb{Z}$ so that for each j, there exists $h_{ij} \in G$ so that $\hat{p}_{ij}$ corresponds to $h_{ij} p_i^{a_i} h_{ij}^{-1} \in G$. Consider the action $\rho_{ij} \circ (r_{ij} \times \mathrm{id}) : \pi_1(\mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \times \ddot{S}) \to \mathrm{Aut}(X_{ij})$ on the cube complex X_{ij}. By construction, $\langle h_{ij} p_i^{a_i} h_{ij}^{-1}, t^{Kd} \rangle$ acts on X_{ij} freely, and the translation length of each $h_{ij} p_i^{m a_i} h_{ij}^{-1} t^{Kd}$ coincides with that of t^{Kd}. Moreover, by construction, for the remaining $p_{i'}^m t^d$, the virtual translation length of any conjugate u of $p_{i'}^m t^d$ is the same as that of t^d, since u is sent by $r_{ij} \times \mathrm{id}$ to t^d. Hence we can apply Lemma 3.2 to the action $\rho_{ij} \circ (r_{ij} \times \mathrm{id}) : \pi_1(\mathsf{C}(\hat{p}_{ij} \to \widehat{B}) \times \ddot{S}) \to \mathrm{Aut}(X_{ij})$ on the cube complex X_{ij} to obtain an action ρ'_{ij} of $F \times \mathbb{Z}$ on a CAT(0) cube complex C_{ij} so that each $p_k^m t^d$ has the same translation length as t^d, for $k \leq n$ and $|m| \leq n_k$. Moreover, the freeness of the action on X_{ij} ensures that $\langle h_{ij} p_i^{a_i} h_{ij}^{-1}, t \rangle$ acts freely.

Construction of $C_{\circledcirc}$: We have constructed, for each $i \leq n$, an action $\alpha_i : F \times \mathbb{Z} \to \mathrm{Aut}(C_{p_i})$ on a CAT(0) cube complex C_{p_i} with the following properties:

- $\|p_i^m t^d\|_{C_{p_i}} = \|t^d\|_{C_i}$ for $0 < |m| \leq n_i$;
- $\langle p_i, t \rangle$ acts freely on C_i;
- for all $i \neq j$ and $|m| \leq n_i$, we have $\alpha_i(p_j^m t^d) = \alpha_i(t^d)$.

Indeed, for each i, let C_{p_i} be the cube complex C_{ij} associated to the base elevation $\hat{p}_{ij}$ of p_i, and let $\alpha_i = \rho'_{ij}$.

Now let $C_{\circledcirc} = \prod_{i=1}^n C_{p_i}$ and let $\alpha : F \times \mathbb{Z} \to \mathrm{Aut}(C)$ be the diagonal action induced by the actions $\alpha_1, \ldots, \alpha_n$. Then $\langle p_i, t \rangle$ acts freely on $C_{\circledcirc}$ for each $i \leq n$, because it acts freely on at least one factor. Moreover, for each i and $m \leq n_i$, we have that $\|p_i^m t^d\|_{C_{p_i}} = \|t^d\|_{C_{p_i}}$ and $\|p_i^m t^d\|_{C_{p_j}} = \|t^d\|_{C_{p_j}}$ when $j \neq i$. Hence $\|p_i^m t^d\|_{\circledcirc} = \|t^d\|_{C_{p_i}} + \sum_{j \neq i} \|t^d\|_{C_{p_j}} = \|t^d\|_{\circledcirc}$, as required. □

4. Lemmas supporting the inductive step

The next lemma uses Lemma 3.4 and is applied at each step in the inductive proof of Theorem 1.

Lemma 4.1 (Torus cubulation)**.** *Let G split as a finite graph of groups where each edge group is separable in G. Suppose there is a homomorphism $\psi : G \to \mathbb{Z}$ which is surjective on each edge-group in the splitting of G. Choose t in some edge-group of G with $\psi(t) = 1$. Let $\mathcal{P} \subset G$ be finite, with $\psi(p) = 0$ and p acting hyperbolically on the Bass-Serre tree of the splitting of G, for all $p \in \mathcal{P}$.*

Then there exists an action of G on a CAT(0) cube complex $C_{\circledcirc}$ so that $\|pt\|_{\circledcirc} = \|t\|_{\circledcirc}$ and, moreover, $\|p^m t^n\|_{\circledcirc} > 0$ for all $p \in \mathcal{P}$ unless $m = n = 0$.

Proof. The idea is to build a finite-index subgroup $G'' \leq G$, a quotient $G'' \to F \times \mathbb{Z}$, where F is free, so that a generator of each $\langle pt \rangle \cap G''$ has the same nonzero image in $\mathbb{Z}$ as a fixed generator of $\langle t \rangle \cap G''$, and then apply Lemma 3.4 and Lemma 3.2 to obtain the desired cubical action of G.

The first finite-index subgroup: Let Γ be the underlying graph of the hypothesized splitting of G. Let $\mathcal{P} = \{p_i^{m_j} : 1 \leq i \leq n, j \in J_i\}$, where we have partitioned the elements according to maximal cyclic subgroups.

For each i, let $p_i t$ have normal form $p_i t = a_1 e_1 a_2 e_2 \cdots a_k e_k a_{k+1}$, where $e_1 \cdots e_k$ is a closed path in Γ and each a_j belongs to a vertex group. Suppose for some j that $e_j = e_{j+1}^{-1}$. Then, by separability of the edge-groups in G, there is a finite index subgroup of G containing the terminal edge-group of e_j (which is the initial edge-group of e_{j+1}) but not containing a_{j+1}. Repeating this procedure for each such backtrack and taking intersections of conjugates gives a finite-index normal subgroup $G' \leq G$ so that the following holds for all $i \leq n$ and all m_j: let Γ' be the underlying graph of the induced splitting of G' and let q'_{ij} generate the cyclic subgroup $\langle p_i^{m_j} t \rangle \cap G'$. Then q'_{ij} has normal form projecting to an immersed closed path in Γ'.

Equalizing t–lengths: For each q'_{ij}, let $d_{ij} = \psi(q'_{ij})$, which is nonzero since $\psi(p_i^{m_j} t) = 1$. Let $D = \operatorname{lcm}\{d_{ij}\}$ and let G'' be the kernel of the map $G \xrightarrow{\psi} \mathbb{Z} \to \mathbb{Z}/D\mathbb{Z}$. For each i, j, let q''_{ij} generate $\langle p_i^{m_j} t \rangle \cap G''$. By construction, $\psi(q''_{ij}) = D > 0$ for all i, j. Moreover, if $\hat{q} \in G''$ is conjugate in G to q''_{ij}, then $\psi(\hat{q}) = D$. Indeed, conjugate elements of G have the same ψ–image.

Conclusion: Let Γ'' be the underlying graph of the induced splitting of G''. Consider the map $\rho : G'' \to \pi_1\Gamma'' \times \mathbb{Z}$ induced by $G'' \to \pi_1\Gamma''$ (projection to the underlying graph) and $\psi : G'' \to \mathbb{Z}$. We now regard $\pi_1\Gamma'' \times \mathbb{Z}$ as being presented by $\langle \pi_1\Gamma'', \bar{t} \mid [f, \bar{t}],\ f \in \pi_1\Gamma'' \rangle$. Each q''_{ij} maps to some $\bar{q}''_{ij}\bar{t}^D$, where $\bar{q}''_{ij} \in \pi_1\Gamma'' - \{1\}$ and $D > 0$.

Lemma 3.4 provides a CAT(0) cube complex C' and an action $\alpha : \pi_1\Gamma'' \times \mathbb{Z} \to \operatorname{Aut}(C')$ so that the following hold for any $\hat{q} \in G''$ that is conjugate in G to some q''_{ij}:

- each $\langle \rho(\hat{q}), \bar{t} \rangle$ acts freely;
- $\|\bar{t}^D\|_{C'} = \|\rho(\hat{q})\|_{C'} > 0$.

Hence the action $\alpha \circ \rho$ of G'' on C' has the property that $\|t^D\|_{C'} = \|\hat{q}\|_{C'} > 0$ for all $\hat{q} \in G''$ conjugate in G to some q''_{ij}. We can now apply Lemma 3.2 to the action of G'' on C' to obtain an action of G on a CAT(0) cube complex $C_{\circledcirc}$ with the desired properties. □

The following lemma, about cubulating a multiple HNN extension G of a group H, takes as its input a pair of actions of H on CAT(0) cube complexes, and returns a single action of G on a CAT(0) cube complex. In practice, the pair of actions of H will arise as follows: one will exist by induction, and the other will have been obtained by applying Lemma 4.1.

Lemma 4.2 (Cubulation with turns)**.** *Let the group G decompose as a finite graph of groups with a single vertex group H and $\mathbb{Z}$ edge groups. Let $\mathcal{P} \subset H$ be a finite set, and let $t \in H - \mathcal{P}$ generate an edge-group. Suppose each stable letter e conjugates t to pt for some $p \in \mathcal{P}$ (and each $p \in \mathcal{P}$ appears in this way).*

Suppose that H acts on CAT(0) cube complexes $C_\sharp$ and $C_\circledcirc$ with the following properties:

- *For each $p \in \mathcal{P}$, we have $\|t\|_\sharp = \|pt\|_\sharp = 0$.*
- *For each $p \in \mathcal{P}$, we have $\|pt\|_\circledcirc = \|t\|_\circledcirc$.*
- *The diagonal action of H on $C_\sharp \times C_\circledcirc$ is free.*

Finally, let $\mathcal{Q}$ be a finite subset of G consisting of elements acting on $\mathcal{T}$ hyperbolically. Then G acts on a CAT(0) cube complex $C^1_\sharp$ with the following properties:

- *$\|qt\|_{C^1_\sharp} = 0$ for all $q \in \mathcal{Q}$.*
- *For each $g \in H$, we have $\|g\|_{C^1_\sharp} = 2\|g\|_\sharp$. In particular, $\|t\|_{C^1_\sharp} = 0$.*
- *For every $g \in G$ acting hyperbolically on $\mathcal{T}$, either g is conjugate into some $\langle q, t\rangle, q \in \mathcal{Q}$ (and conjugate into $\langle qt\rangle$ if $[q,t] \neq 1$), or $\|g\|_{C^1_\sharp} > 0$.*

Proof. There are several steps.

The initial tree of spaces: Let $D = H\backslash(C_\sharp \times C_\circledcirc)$, which is a nonpositively-curved cube complex because the H–action on $C_\sharp \times C_\circledcirc$ is free. Let $\ell = \|t\|_{C_\sharp \times C_\circledcirc}$, and note that $\|pt\|_{C_\sharp \times C_\circledcirc} = \ell$ for all $p \in \mathcal{P}$.

For each $p \in \mathcal{P}$, let $\widetilde{S}_p \to C_\sharp \times C_\circledcirc$ be a combinatorial geodesic axis for pt in $C_\sharp \times C_\circledcirc$ and let $S_p \to D$ be the path given by $S_p = \langle pt\rangle\backslash\widetilde{S}_p$. Let S_t be defined analogously for t.

Let B_p be the cylinder $S \times [-\frac{1}{2}, \frac{1}{2}]$, where S is a cycle of length ℓ, and for each p, attach B_p to D by gluing along $S \times \{\pm\frac{1}{2}\}$ via $S \times \{\frac{1}{2}\} \to S_p \to D$ and $S \times \{-\frac{1}{2}\} \to S_t \to D$. Let Z denote the resulting space, and observe that $\pi_1 Z \cong G$. Let $\widetilde{Z} \to Z$ be the universal cover, which decomposes as a tree of spaces with underlying tree $\mathcal{T}$. The vertex spaces are the elevations of D and the edge spaces are strips of the form $\widetilde{S} \times [-\frac{1}{2}, \frac{1}{2}]$.

Walls: For each $p \in \mathcal{P}$, and each edge e of S_p, let m_e^+ be the midpoint of e, which maps to an immersed hyperplane H_e of D. Likewise, for each edge f of S_t, let m_f^- be the midpoint of f, which maps to an immersed hyperplane H_f of D. Since $|S_p| = |S_t| = \ell$, we can choose for each p a bijection $b_p : \text{Edges}(S_p) \to \text{Edges}(S_t)$ and, for each $e \in \text{Edges}(S_p)$, choose a properly embedded arc α_e in B_p joining m_e^+ to $m_{b_p(e)}^-$ and intersecting $S \times \{\pm 1/2\}$ in exactly two points. Declare immersed hyperplanes H, H' of D to be elementary equivalent if they intersect some B_p in points joined by some arc α_e; taking the transitive closure of this relation gives an equivalence relation on the immersed hyperplanes of D. An *immersed wall* $W \to D$ is formed from the disjoint union of the immersed hyperplanes in an equivalence class by attaching each of these arcs α_e as follows: join the endpoints of α_e to the points in immersed hyperplanes to which they map. Each immersed wall is connected, and lifts to a wall in $\widetilde{Z}$ that intersects each vertex space in either $\emptyset$ or a single hyperplane, and intersects each edge space in $\emptyset$ or a single arc.

Let $\mathcal{H}$ be the set of walls in $\widetilde{Z}$ of the above type that contain at least one arc. Let $\mathcal{V}$ be the set of walls of the following two types:

- hyperplanes of vertex spaces that do not intersect edge spaces;
- *vertical* walls (isometric to $\mathbb{R}$) of the form $\rho^{-1}(S \times \{0\})$, where $\rho : \widetilde{Z} \to Z$ is the universal covering map and $S \times \{0\}$ is the core curve of some B_p.

Let $\mathcal{E}_\sharp$ be the set of hyperplanes in $\widetilde{D}$ of the form $H \times C_\circledcirc$, where H is a hyperplane of $C_\sharp$. Let $\mathcal{E}_\circledcirc$ be the set of hyperplanes in $\widetilde{D}$ of the form $C_\sharp \times V$, where V is a hyperplane of $C_\circledcirc$. Since t and each pt, and all of their conjugates, act elliptically on $C_\sharp$, each wall $W \in \mathcal{H}$ intersects each vertex space in $\widetilde{Z}$ in a unique hyperplane which is a translate of an element of $\mathcal{E}_\circledcirc$. The remaining walls of $\widetilde{Z}$, i.e. the elements of $\mathcal{V}$, are either vertical, or are translates of elements of $\mathcal{E}_\sharp$, and are in particular confined to single vertex spaces (in the latter case) or edge spaces (in the former).

Let $\mathcal{V}^-$ and $\mathcal{V}^+$ be two copies of the set of walls in $\mathcal{V}$, so that we have G–equivariant bijections $b^\pm : \mathcal{V} \to \mathcal{V}^\pm$, each of which is the identity when $\mathcal{V}^\pm$ is viewed as a copy of $\mathcal{V}$. Since each $V \in \mathcal{V}$ has a product neighbourhood in $\widetilde{Z}$, we can perturb so that $b^\pm(V)$ are parallel, disjoint geometric walls.

Auxiliary cube complex X**:** Let $\widetilde{X}$ be the CAT(0) cube complex dual to the wallspace $(G, \mathcal{H} \sqcup \mathcal{V}^- \sqcup \mathcal{V}^+)$. Note that G acts on $\widetilde{X}$, and moreover this action is free. Indeed, each $\mathcal{T}$–hyperbolic element of G is cut by a vertical wall. Each elliptic element is cut by some wall, because the action of H on $C_\sharp \times C_\circledcirc$ is free and each wall intersects each vertex space in at most one hyperplane of the vertex space (and each hyperplane of a vertex space extends to a wall in $\widetilde{Z}$).

Note that $\widetilde{X}$ decomposes as a tree of spaces whose underlying tree is again $\mathcal{T}$, whose vertex spaces are isomorphic to $C_\sharp \times C_\circledcirc$, and whose edge-spaces are convex subcomplexes.

We use the notation $\mathcal{H}$ to refer to the hyperplanes of $\widetilde{X}$ corresponding to walls of $\widetilde{Z}$ belonging to $\mathcal{H}$, and do likewise for $\mathcal{V}^\pm$. Note that each hyperplane $V^+ \in \mathcal{V}^+$ is parallel to, and osculates with, a hyperplane V^-, where $V^\pm = b^\pm(V)$ for some wall $V \in \mathcal{V}$. In particular, the vertical walls come in parallel pairs whose elements osculate. (Here, hyperplanes in $\widetilde{X}$ are *parallel* if they cross exactly the same hyperplanes.)

The cubical presentation of G**:** Let $\eta : \widetilde{X} \to \mathcal{T}$ be the G–equivariant map sending each vertex-space to the corresponding vertex, and, for each edge-space V, collapsing the hyperplane-carrier $V \times [-\frac{1}{2}, \frac{1}{2}]$ to the corresponding edge (which we identify with $[-\frac{1}{2}, \frac{1}{2}]$) in the obvious way. For each $q \in \mathcal{Q}$, let $\widetilde{Y}_q = \eta^{-1}(L_q)$, where L_q is the axis in $\mathcal{T}$ for the hyperbolic element qt. Note that $\widetilde{Y}_q$ is a convex subcomplex of $\widetilde{X}$ decomposing as a tree of spaces with underlying tree $L_q \cong \mathbb{R}$ and vertex spaces the translates of $C_\sharp \times C_\circledcirc$ corresponding to vertices of L_q. Hence the inclusion $\widetilde{Y}_q \to \widetilde{X}$ as a convex subcomplex descends to a local isometry $\widetilde{Y}_q \to X$, and we define X^* to be the resulting cubical presentation $\langle X \mid \{\widetilde{Y}_q : q \in \mathcal{Q}\}\rangle$. Since each $\widetilde{Y}_q$ is CAT(0), we have $\pi_1 X^* \cong \pi_1 X \cong G$.

Computing $\mathrm{Aut}(\widetilde{Y}_q)$**:** The group $\mathrm{Aut}(\widetilde{Y}_q)$ can be identified with $\mathrm{Stab}_G(\widetilde{Y}_q)$, which we denote by K. We claim first that either $K = \langle qt \rangle$ or, if $[q,t] = 1$, then $K = \langle q, t \rangle$ and $K \cong \langle q, t \mid [q,t] \rangle \cong \mathbb{Z}^2$.

Indeed, the map $G \to \mathrm{Aut}(\mathcal{T})$ induced by η restricts to an action $K \to \mathrm{Aut}(L_q)$. Note that $qt \in K$, so $K \ncong \{1\}$. Applying Lemma 2.17 shows that either $K \cong \mathbb{Z}$, in which case $K \cong \langle qt \rangle$ (since qt generates a maximal cyclic subgroup), or each element of K centralises the generator of any edge group corresponding to an edge in L_q.

In the latter case, the image of K in $\mathrm{Aut}(L_q)$ coincides with the image of $\langle qt \rangle$, and the kernel is equal to the intersection of the edge groups occurring along L_q. Hence $\mathrm{Ker}(K \to \mathrm{Aut}(L_q)) = \bigcap_{n \in \mathbb{Z}} \langle t \rangle^{(qt)^n} = \langle t \rangle$, since qt and t commute. Hence K is generated by q and t, as required.

Metric small-cancellation and short inner paths: Next, we claim that $\langle X \mid \{\widetilde{Y}_q\}\rangle$ satisfies the $C'(1/n)$ cubical small-cancellation condition for any $n \in \mathbb{N}$. Indeed, since each relator is simply connected, there are no essential closed paths in any $\widetilde{Y}_q$, so any metric small-cancellation condition is trivially satisfied. Hence, by [Wis11, Lemma 3.67], $\langle X \mid \{\widetilde{Y}_q\}\rangle$ has short inner paths.

Declaring walls in $\widetilde{Y}_q$: Recall that $\mathcal{V}^{\pm}$ is a collection of hyperplanes of $\widetilde{X}$ that are either edge-spaces, or are contained in a single vertex space and do not intersect any of the incident edge spaces. Hence, for each $V \in \mathcal{V}^{\pm}$ intersecting $\widetilde{Y}_q$, we have that $V \subset \widetilde{Y}_q$ and $\{(qt)^n V : n \in \mathbb{Z}\}$ is a set of pairwise disjoint hyperplanes.

Fix $q \in \mathcal{Q}$ and consider $\widetilde{Y}_q \subset \widetilde{X}$. We declare an equivalence relation on the hyperplanes of $\widetilde{Y}_q$ as follows. First, if $\bar{H} = H \cap \widetilde{Y}_q$ is a hyperplane, where $H \in \mathcal{H}$, then H is declared to be unique in its equivalence class. Next, let M be a large integer, to be specified below. Let $V \in \mathcal{V}$ be a (combinatorial) hyperplane contained in $\widetilde{Y}_q$, which corresponds to a pair V^-, V^+ of parallel osculating hyperplanes in $\widetilde{Y}_q$. We declare $(qt)^n V^{\pm} \sim (qt)^{M+n} V^{\mp}$ for $n \in \mathbb{Z}$. The equivalence relation $\sim$ determines walls in $\widetilde{Y}_q$; each wall is the union of one or two hyperplanes.

We claim that this system of walls in $\widetilde{Y}_q$ is K–invariant. Indeed, if $K \cong \langle qt \rangle$, then this holds by construction.

Hence suppose that q, t commute and $K = \langle q, t \rangle$. Since the $\sim$–class of each $H \in \mathcal{H}$ consists of a single hyperplane, and $\mathcal{H}$ is G–invariant, it is clear that $q^n t^m H$ is a wall for each m, n. Next, let $V \in \mathcal{V}$. If V is a vertical wall, then $tV^+ = V^+$ and $t(qt)^M V^- = (qt)^M tV^- = (qt)^M V^-$, so the class $\{V^+, (qt)^M V^-\}$ is preserved by t. On the other hand, $qV^+ = qtV^+$, since t fixes all edges of L_q, and $q(qt)^M V^- = q^{M+1} t^M V^- = (qt)(qt)^M V^-$, so the equivalence class $\{V^+, (qt)^M V^-\}$ is sent by q to $\{qtV^+, (qt)(qt)^M V^-\}$, which is another $\sim$–class.

Now suppose that V lies in some vertex space $\widetilde{D}$. Without loss of generality, one of the edges of L_q incident to the vertex corresponding to $\widetilde{D}$ has edge group $\langle t \rangle$. Consider the action of t on $\widetilde{D} \cong C_{\sharp} \times C_{\circledcirc}$. Recall that t acts trivially on $C_{\sharp}$, and that V must be a hyperplane of the form $W \times C_{\circledcirc}$ for some hyperplane W of $C_{\sharp}$ (since $V \in \mathcal{V}$). Hence $tV = V$. We can now argue exactly as above to see that t stabilises the equivalence class $\{V^+, (qt)^M V^-\}$. Thus the wallspace structure on $\widetilde{Y}_q$ provided by $\sim$ is K–invariant, as required by the $B(6)$ condition.

Verifying $B(6)$: We now verify that the cubical presentation $\langle X \mid \{\widetilde{Y}_q\}_q \rangle$ satisfies the $B(6)$ condition from Definition 2.13. Indeed, we have already verified Definition 2.13.(1),(3),(5), and it is easy to see that (2) holds whenever M is sufficiently large to ensure that hyperplanes in the same $\sim$–class are separated by a hyperplane (e.g. by virtue of being separated by an edge space). We have also already verified the $C'(1/14)$ condition required by the definition of $B(6)$.

Hence it suffices to check that if $P \to \widetilde{Y}_q$ is a path that decomposes as the concatenation of at most 7 piece-paths, and P starts and ends on the carrier $N(U)$ of a wall U, then P is path-homotopic into U. Since $\widetilde{Y}_q$ is simply connected, this amounts to showing that, if our initial choice of M was sufficiently large, and U is formed from two hyperplanes $V^+, (qt)^M V^- \in \mathcal{V}^+ \sqcup \mathcal{V}^-$, and P travels from V^+ to $(qt)^M V^-$, then P cannot be the concatenation of at most 7 pieces.

First, observe that any hyperplane $\widetilde{W}$ of $\widetilde{X}$ that is not contained in $\widetilde{Y}_q$, but whose carrier intersects $\widetilde{Y}_q$, has the property that $\mathcal{N}(\widetilde{H}) \cap \widetilde{Y}_q$ is contained in a single vertex space of $\widetilde{X}$.

Next, we claim that there exists a constant N_0 so that each piece-geodesic in $\widetilde{Y}_q$ projects to a path in L_q of length at most N_0. Indeed, let $q' \in \mathcal{Q}$ and $g \in G$ be such that $\widetilde{Y}_q$ contains a piece of $g\widetilde{Y}_{q'}$, and let p be a path in this piece. Then the projection $\bar{p}$ of p to L_q is contained in the intersection $L_q \cap gL_{q'}$ in $\mathcal{T}$, so it suffices to bound this intersection. Since $\mathrm{Aut}(\widetilde{Y}_q)$ acts as $\langle qt \rangle$ coboundedly on the line L_q (by Lemma 2.17), and the same is true of the $\mathrm{Aut}(\widetilde{Y}_{q'})^g$–action on $gL_{q'}$ (with qt replaced by $gq'tg^{-1}$), and $\mathcal{Q}$ is finite, there exists a constant N_0 so that either $\mathrm{diam}(L_p \cap L_q) \leq N_0$, or $L_p \cap L_q$ is unbounded, in which case $L_p = L_q$ because L_p, L_q are convex in $\mathcal{T}$. In the latter case,

$\widetilde{Y}_q = g\widetilde{Y}_{q'}$, by definition, contradicting that there is a piece between these two subcomplexes (the existence of a piece requires them to be distinct). Now let M be chosen so that for all $q' \in \mathcal{Q}$ and all vertices $v \in L_{q'}$, we have $\mathsf{d}_{\mathcal{T}}((qt)^M v, v) \gg 7M_0$. Given this choice, we see that no P traveling from V^+ to $(qt)^M V^-$ can decompose as the concatenation of at most 7 pieces. This completes the verification of the $B(6)$ condition.

Verifying the proximality hypothesis: We now declare *preferred* walls and verify that the hypotheses of Theorem 2.15 are satisfied. For each q, a wall in $\widetilde{Y}_q$ is *preferred* if it arises from a $\sim$–class containing two hyperplanes, i.e. if its constituent hyperplanes belong to $\mathcal{V}^{\pm}$ (as opposed to $\mathcal{H}$). It remains to check that the third hypothesis of the theorem holds.

Fix $q \in \mathcal{Q}$ and let $\kappa \to \widetilde{Y}_q$ be a geodesic with endpoints x, y. Let $V^-, (qt)^M V^+$ be preferred hyperplanes in the same wall, and suppose that κ traverses a 1–cube dual to V^-. Suppose, moreover, that either κ traverses a 1–cube dual to $(qt)^M V^-$, or $(qt)^M V^-$ is proximate to y. We claim that there is a preferred wall in $\widetilde{Y}_q$ that separates x, y but is not proximate to either x or y.

Indeed, first suppose that $(qt)^M V^-$ is proximate to y. Then our bound of M_0 on the projection of piece-paths to L_q ensures that $\mathsf{d}_{\mathcal{T}}(y, (qt)^M V) \leq 2M_0 + 1$, while we can make our initial choice of M so that $\mathsf{d}_{\mathcal{T}}(V^-, (qt)^M V^+) > 100M_0$ (here, given subsets A, B of $\widetilde{X}$, we denote by $\mathsf{d}_{\mathcal{T}}(A, B)$ the distance in $\mathcal{T}$ between the projections of A, B). Hence $\mathsf{d}_{\mathcal{T}}(x, y) > 90M_0$. In particular, there exists a vertical (combinatorial) hyperplane V_1 so that the parallel, osculating vertical hyperplanes $V_1^{\pm}$ separate x, y, and $\mathsf{d}_{\mathcal{T}}(V_1^{\pm}, x) > 80M_0$ and $\mathsf{d}_{\mathcal{T}}(V_1^{\pm}, y) \in [5M_0, 10M_0]$. Consider the wall U arising from the equivalence class $\{V_1^+, (qt)^M V_1^-\}$. Then $(qt)^M V^-$ lies at $\mathcal{T}$–distance at least $90M_0$ from y, and also from x, and does not intersect κ. Thus U separates x, y and is not proximate to either (because proximality would require U to lie at $\mathcal{T}$–distance at most $2M_0 + 1$ from one of x or y, which contradicts our choice of V_0.

Next, suppose that $(qt)^M V^-$ is not proximate to y, so that κ traverses 1–cubes dual to both V^+ and $(qt)^M V^-$. Then, again, we have $\mathsf{d}_{\mathcal{T}}(x, y) \geq 90M_0$ and we can argue exactly as above.

Applying Theorem 2.15: We have verified all of the hypotheses of Theorem 2.15, which thus gives the following conclusion about the action of G on the cube complex $C^0_\sharp$ dual to the walls in $\widetilde{X}^*$ arising from the above $\sim$ relation. Let $g \in G$ be an infinite-order element. Then one of the following holds:

- g was not cut by a preferred hyperplane of $\widetilde{X}$. In this situation, g is necessarily $\mathcal{T}$–elliptic, and $\|g\|_{C^0_\sharp} = \|g\|_{C_\circledcirc}$.
- g is elliptic on $\mathcal{T}$, and is cut by a preferred wall. Since each wall intersects each vertex space in a unique hyperplane, and our construction doubled the hyperplanes in $\mathcal{V}$ by subdivision, we have $\|g\|_{C^0_\sharp} = \|g\|_{C_\circledcirc} + 2\|g\|_{C_\sharp}$.
- g is conjugate into $\langle q, t\rangle$ for some $q \in \mathcal{Q}$. (In the case where $[q, t] \neq 1$, we actually have that g is conjugate into $\langle qt\rangle$.) In this case, by construction and Theorem 2.16, g is not cut by any preferred wall, either because we have turned all preferred walls away from g (in the case where g is hyperbolic) or, if g is elliptic, because g must be conjugate to t, which acts trivially on $C_\sharp$.
- g is hyperbolic, and is cut by a preferred wall.

Conclusion: Let $C^1_\sharp$ be the restriction quotient of $C^0_\sharp$ obtained by cubulating the wallspace whose underlying space is $C^0_\sharp$ and whose walls are the preferred walls. The above discussion shows that the action of G on $C^0_\sharp$ descends to an action on $C^1_\sharp$ such that each g has translation length as described in the statement; this completes the proof. □

5. Proof of Theorem 1

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $G = F \rtimes_\Phi \langle t\rangle$, where $\Phi : F \to F$ is a polynomial-growth automorphism. Replacing G with a finite-index subgroup (but keeping our notation), we can assume that Φ has an improved relative train track representative, by Proposition 2.3. Passing to a finite-index subgroup is permitted since, if G has a finite-index subgroup acting freely on a CAT(0) cube complex, then G also admits such an action.

Hence, as discussed in Remark 2.4, we can assume (effectively, by passing to a supergroup of a finite index subgroup) that G admits a decomposition as a sequence of iterated $\mathbb{Z}$ HNN extensions, as follows. First, we have $G_0 \le G_1 \le \ldots \le G_n = G$, where $G_0 = \langle t\rangle$. Next, we have $\{1\} = F_0 \le F_1 \le \ldots \le F_n = F$, where $G_0 = F_0 \times \langle t\rangle$, and the following holds. For each $i \le n$, we have

$$F_i \rtimes \langle t\rangle \cong G_i = \langle G_{i-1}, e^i_1 \ldots, e^i_{n_i} \mid p^i_j t = e^i_j t (e^i_j)^{-1}\rangle,$$

where each $p^i_j \in F_{i-1} \le G_{i-1}$ acts elliptically on the associated Bass-Serre tree $\mathcal{T}_i$, where the edge groups are conjugates of t and $p^j_i t$. Let $\mathcal{P}_i \subset F_{i-1}$ be the multiset of p^i_j. Moreover, $F_i \subset G_i$ contains $\mathcal{P}_{i+1}$, and for each $p \in \mathcal{P}_{i+1}$, the elements p, pt act hyperbolically on $\mathcal{T}_i$.

We now argue by induction on i that G_i acts freely on a CAT(0) cube complex. In fact, to support the induction, we will prove a stronger claim at each step.

Base case, $i = 0$: In the base case, $F_0 = \{1\}, G_0 = \langle t\rangle$, and $\mathcal{P}_1 = \emptyset$ (i.e. the generators of F_1 that we will add at the next stage all commute with t). Let $C^0_\circledcirc$ be the standard tiling of $\mathbb{R}$ by 1–cubes, on which G_0 acts freely, with t acting as a unit translation. Then, vacuously, we have $\|p\|_\circledcirc > 0$ and $\|t\|_\circledcirc = \|pt\|_\circledcirc$ for all $p \in \mathcal{P}_1$. Moreover, $\|t\|_\circledcirc > 0$. Let $C^0_\sharp$ be a single point, with G_0 acting trivially. Then, vacuously, each nontrivial $g \in F_0$ not conjugate into $\langle p\rangle,\ p \in \mathcal{P}_1$ has positive translation length on $C^0_\sharp$ (since there are no such g), and each $p \in \mathcal{P}_1$ has translation length 0, again vacuously.

Let $C_0 = C^0_\circledcirc \times C^0_\sharp$, with G_0 acting diagonally. Then each nontrivial element of G_0 has positive translation length on at least one of the factors, namely $C^0_\circledcirc$, so the action is free. Furthermore, $\|pt\|_0 = \|t\|_0$ for each $p \in \mathcal{P}_1 = \emptyset$.

Inductive step: Suppose by induction that G_i acts on a CAT(0) cube complex $C^i_\sharp$ so that $\|t\|_{C^i_\sharp} = \|tp^i_j\|_{C^i_\sharp} = 0$ for all $j \le n_i$ and every element not conjugate into $\langle t\rangle$ or some $\langle p^i_j, t\rangle$ has positive translation length. Then Lemma 4.1 provides an action of G_i on a CAT(0) cube complex $C^i_\circledcirc$ so that $\|p^{i+1}_j t\|_{C^i_\circledcirc} = \|t\|_{C^i_\circledcirc} > 0$ and $\|p^{i+1}_j\| > 0$ for all $j \le n_{i+1}$.

Indeed, in order to apply Lemma 4.1, we must verify that G_i satisfies all of the hypotheses of that lemma. Since G_i is of the form $F_i \rtimes_{\Phi_i} \langle t\rangle$, where $\Phi_i : F_i \to F_i$ is an automorphism (obtained by restricting some positive power of Φ to $F_i \le F$), we have a surjection $\Phi : G_i \to \mathbb{Z}$ with $\Phi(t) = 1$ and $\operatorname{Ker}\Phi = F_i$. In particular, $p^{i+1}_j \in \operatorname{Ker}\Phi$ for all j. Moreover, G_i splits as a graph of groups, as in Remark 2.4, whose edge groups are of the form $\langle t\rangle$ and $\langle p^i_j t\rangle, j \le n_i$, where each $p^i_j \in F_{i-1}$. By Lemma 2.1, these subgroups are separable in G_i. Hence we can apply Lemma 4.1.

By construction, the diagonal action of G_i on $C^i_\sharp \times C^i_\circledcirc = C_i$ is free, since each nontrivial element acts hyperbolically on at least one of the factors. Lemma 4.2 now provides an action of G_{i+1} on a CAT(0) cube complex $C^{i+1}_\sharp$ with the following properties:

- For all p^{i+1}_j, we have that $\|p^{i+1}_j t\|_{C^{i+1}_\sharp} = 0$.
- For all $\mathcal{T}_{i+1}$–elliptic $g \in G_{i+1}$, we have $\|g\|_{C^{i+1}_\sharp} = 2\|g\|_{C^i_\sharp}$. In particular, $\|t\|_{C^{i+1}_\sharp} = 0$.
- If $g \in G_{i+1}$ is $\mathcal{T}_{i+1}$–hyperbolic and is not conjugate into $\langle p^{i+1}_j, t\rangle$ for any j, then $\|g\|_{C^{i+1}_\sharp} > 0$.

Now, our earlier discussion shows that we may apply Lemma 4.1 to G_{i+1} to produce an action of G_{i+1} on a CAT(0) cube complex $C^{i+1}_\circledcirc$ so that the diagonal action of G_{i+1} on $C_{i+1} = C^{i+1}_\sharp \times C^{i+1}_\circledcirc$ is free. In particular, G_n acts freely on the CAT(0) cube complex C_n. This completes the proof. □

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