ACYLINDRICAL HYPERBOLICITY OF CUBICAL SMALL CANCELLATION GROUPS

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Abstract. We provide an analogue of Strebel’s classification of geodesic triangles in classical $C'(\frac{1}{2})$ groups for groups given by Wise’s cubical presentations satisfying sufficiently strong metric cubical small cancellation conditions. Using our classification, we prove that, except in specific degenerate cases, such groups are acylindrically hyperbolic.

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Introduction

A cubical presentation of a group is a natural high-dimensional generalization of both a “classical” and a “graphical” presentation of a group in terms of generators and relators. The theory of cubical presentations, and especially the cubical small cancellation theory developed by Wise [Wis, Section 3], has begun to play a significant role in geometric group theory following spectacular solutions of the virtual Haken conjecture by Agol and of Baumslag’s conjecture on one-relator groups with torsion by Wise (for example, cubical small cancellation theory is used in the original proof of the malnormal virtually special quotient theorem [Wis, Theorem 12.2], which is in turn used in the resolution of the virtual Haken conjecture, via [Ago13, Theorem A.1]). A classical presentation of a group $G$ consists of a wedge $X$ of circles and a collection of combinatorial immersions $Y_i \to X$ of circles so that the presentation complex $X^*$ formed from $X$ by coning off the various $Y_i$ satisfies $\pi_1 X^* \cong G$. The 1–skeleton $\text{Cay}(X^*)$ of the universal cover $\tilde{X}^*$ of $X^*$ is a Cayley graph of $G$ with respect to the generating set implicit in the choice of $X$. A graphical presentation is a natural generalization of this: $X$ is allowed to be an arbitrary graph, and each $Y_i \to X$ becomes an immersion of graphs.

In [Wis], it is observed that allowing even more flexibility in the choice of $X$ can lead to more tractable “presentations” for a given group. This leads to the notion of a cubical presentation: $X$ is now a nonpositively curved cube complex and each $Y_i$ is a connected nonpositively curved cube complex equipped with a local isometry $Y_i \to X$. The presentation complex $X^*$ is defined analogously, and there is a generalized Cayley graph $\text{Cay}(X^*)$ which is the cubical part of the

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universal cover of $X^*$, i.e. the cover of $X$ corresponding to $\ker(\pi_1 X \to \pi_1 X^*)$. The analogy with classical presentations is clear: the cube complex $X$ is a kind of “high-dimensional generating set”, the CAT(0) cube complex $\tilde{X}$ is the “high-dimensional tree” taking the place of the free group on the generating set in the classical case, and $\text{Cay}(X^*)$ corresponds to a Cayley graph.

One can then impose cubical small cancellation conditions, in which “generalized overlaps” between the various $Y_i$ (i.e. shadows of $Y_i$ on $Y_j$, as propagated through the intervening cubes) are small in the appropriate metric sense. In this setting, there are powerful tools – specifically, the ladder theorem and the cubical Greendlinger lemma/diagram trichotomy (see Section 2) – that allow one to extract considerable geometric and algebraic information about a group from a small cancellation cubical presentation. The small cancellation conditions of interest in this paper are the cubical $C'(\alpha)$ conditions, for $\alpha > 0$. These say that $|P| < \alpha \|Y_i\|$ for all $P, i$, where $\|Y_i\|$ denotes the length of a shortest essential closed combinatorial path in $Y_i$ and $|P|$ is the length of the geodesic piece $P$. A piece is, roughly, a path in the “generalized overlap” between distinct elevations to $\tilde{X}$ of the various $Y_i$, or between such elevations and hyperplane carriers in $\tilde{X}$. See Definition 2.5.

One advantage of passing to cubical presentations is that many groups that do not admit classical presentations satisfying strong small cancellation conditions nonetheless admit cubical presentations with these properties. For example, if $G$ is the fundamental group of a nonpositively curved cube complex $X$, then $G$ admits a cubical presentation with no relators, and therefore satisfies arbitrarily strong cubical small cancellation conditions; on the other hand, $G$ does not in general satisfy strong classical small cancellation conditions, as can be seen by considering, for instance, right-angled Artin groups. Later in this introduction, we list more examples of cubical small cancellation groups.

Classifying triangles. Our first result is geometric. We classify geodesic triangles in $\text{Cay}(X^*)$ in terms of the disc diagrams that they bound in $\tilde{X}^*$. This is a cubical analogue of Strebel’s classification of geodesic triangles in $C'(\frac{1}{6})$ groups (Theorem 43 of [Str90]), which says that any geodesic triangle bounds a disc diagram of one of a small number of specific combinatorial types:

**Theorem A** (Classification of triangles). There exists $\alpha > 0$ so that the following holds. Let $X$ be a nonpositively curved cube complex, let $\mathcal{I}$ be a (possibly infinite) index set and let $\{Y_i \to X : i \in \mathcal{I}\}$ be a set of local isometries. Suppose that the cubical presentation $(X | \{Y_i\}_{i \in \mathcal{I}})$ satisfies the cubical $C'(\alpha)$ condition. Let $X^*$ be the presentation complex and $x, y, z$ be 0–cells of the universal cover $\tilde{X}^*$. Then there exists a geodesic triangle $\Delta$ in $\tilde{X}^*$, with corners $x, y, z$, so that $\Delta$ is the boundary path of a disc diagram $D \to X^*$ of one of 9 types; in particular, $D$ is the union of 3 padded ladders. Moreover, any other geodesic triangle with corners $x, y, z$ is square-homotopic to such a $\Delta$.

The precise (longer) statement is Theorem 2.18, which explains exactly what the “9 types” of disc diagram are; a padded ladder is a disc diagram of the type in Figure 2. The $\alpha$ required in our proof is $\frac{1}{11}$. Conceptually, our theorem says that any geodesic triangle bounds a disc diagram which is square-homotopic (fixing corners) to a disc diagram which is a “thickened tripod”.

Theorem A reduces to existing results in special cases when $X$ or $\mathcal{I}$ are restricted.

- If $\mathcal{I} = \emptyset$, then Theorem A says that any three vertices in a CAT(0) cube complex determine a geodesic tripod, which is a consequence of the fact that CAT(0) cube complexes are exactly the simply connected cube complexes whose 1–skeleta are median graphs [Che00].
- If $X$ is a wedge of circles and each $Y_i$ is an immersed circle, then $(X | \{Y_i\})$ is a classical $C'(\frac{1}{11})$ presentation, and the original Strebel classification for classical $C'(\frac{1}{6})$ groups applies, and follows from Theorem A.
• If \( \dim X = 1 \) (i.e. \( X \) is a graph) and each \( Y_i \to X \) is an immersion of graphs, then we have graphical presentations. In this setting, there is a classification of triangles that holds under weaker small cancellation conditions than are required in the cubical setting. Indeed, the classification of triangles is completely combinatorial, and Strebel's proof actually applies in the setting of the \((3, 7)\)-\emph{diagrams} used by Gruber–Sisto in their proof of acylindrical hyperbolicity for graphical small cancellation groups \[GS14\]; this combinatorial observation was made by Gruber \[Gru15, \text{Remark 3.11}\]. While the result about \((3, 7)\)-\emph{diagrams} suffices for graphical small cancellation groups, one cannot extend it directly to disk diagrams over cubical presentations since the presence of squares means that such diagrams need not satisfy the \((3, 7)\) condition.

However, it is easy to construct examples of small-cancellation cubical presentations covered by Theorem A but not by the classical or graphical small-cancellation conditions, and one cannot deduce Theorem A from the corresponding purely cubical or graphical results. Some explicit examples of cubical small-cancellation groups to which the theorem applies are discussed below. There is no requirement for the cubical presentation in Theorem A to have finitely many relators.

**Applications of the classification.** One can imagine applications of Theorem A to the thorough investigation of cubical small cancellation groups analogous to applications of Strebel’s classification of triangles in classical small-cancellation theory (e.g. conformal dimension of the boundary \[Mac12\], growth tightness \[Sam02\], the rapid decay property \[AD12\] etc.).

In this paper, we focus on acylindrical hyperbolicity, inspired by the corresponding result for graphical small cancellation groups \[GS14\]. A group \( G \) is \emph{acylindrically hyperbolic} if it admits a nonelementary acylindrical action on a hyperbolic space (acylindricity generalizes \emph{uniform} properness). The notion of acylindrical hyperbolicity, due to Osin \[Osi16\], unifies several generalizations of relative hyperbolicity \[BF02, DGO17, Ham08, Sis18\] and provides a class of groups with many strong properties: if \( G \) is acylindrically hyperbolic, then \( G \) is SQ-universal, contains normal free subgroups, and is \( C^*\)-simple if and only if it has no finite normal subgroup \[DGO17\]; \( G \) contains Morse elements and thus all asymptotic cones of \( G \) contain cut-points \[Sis14\]; the bounded cohomology of \( G \) has infinite dimension in dimensions 2 \[HO13\] and 3 \[FPS15\]; every commensurating endomorphism of \( G \) is an inner automorphism \[AMS16\], etc.

The class of acylindrically hyperbolic groups is now known to be vast, see e.g. \[Bow08, Osi16, DGO17, MO15, Osi15, BF10, GS14, BHS17, PS17\]. Our second result adds Wise’s cubical small cancellation groups to this notable list:

**Theorem B** (Acylindrical hyperbolicity from cubical small cancellation). Let \( X \) be a compact nonpositively curved cube complex. Then there exists a constant \( L = L(X) \) so that the following holds. Let \( \alpha_0 = \min\{\frac{1}{144}, \frac{1}{7}\} \) and let \( \alpha \in [0, \alpha_0] \).

Let \( \langle X \mid \{Y_i\}_{i \in \mathcal{I}} \rangle \) be a (possibly infinite) uniform \( C'(\alpha) \) cubical presentation with each \( Y_i \) compact. Let \( X^* \) denote the presentation complex. Then one of the following holds:

1. \( \pi_1 X^* \) is finite or two-ended;
2. each \( Y_i \) is contractible, \( \pi_1 X^* = \pi_1 X \), and the universal cover \( \tilde{X} \) of \( X \) contains a convex \( \pi_1 X^* \)-invariant subcomplex splitting as the product of unbounded cube complexes;
3. \( \pi_1 X^* \) is acylindrically hyperbolic.

We do not require \( \mathcal{I} \) to be finite, which is why we impose the uniform \( C'(\alpha) \) condition (see Definition \([2.5]\)).

**Remark 1** (Variations on hypotheses). Almost all of the proof of Theorem B makes use of only the \( C'(\frac{1}{144}) \) hypothesis. The role of \( L(X) \) is just in the proof of Lemma 5.7. That lemma, and hence the theorem, would also hold under a slightly different hypothesis: \( \langle X \mid \{Y_i\}_{i \in \mathcal{I}} \rangle \) satisfies the uniform \( C'(\frac{1}{144}) \) condition, and each \( Y_i \) has systole at least \( 7L(X) \).
Remark 2 (The constant $L(X)$). The constant $L(X)$ is defined as follows. Let $C\tilde{X}$ be the contact graph of $\tilde{X}$, which is the intersection graph of the set of hyperplane carriers; this was defined, and shown to be hyperbolic, in [Hag14]. Let $L$ be the set of $\tilde{g}\in \pi_1X$ that act on $C\tilde{X}$ as loxodromic isometries. From results in [CS11] and [Hag13], $L\neq \emptyset$ if and only if $\tilde{X}$ does not contain a convex $\pi_1X$–invariant subcomplex decomposing as the product of unbounded CAT(0) cube complexes. The proof shows that we can take $L(X) = \inf_{\tilde{g}\in L} \inf_{\tilde{a}\in X^{(0)}} d_{\tilde{X}}(\tilde{a}, \tilde{g}\tilde{a})$. So, $L(X) < \infty$ unless $\tilde{X}$ contains a product subcomplex of the above type. Moreover, since we are considering combinatorial translation length, $L(X) \geq 1$. Note that $L(X)$ depends only on the cube complex $X$.

As explained in Subsection 5.3, this shows that when $X$ is a wedge of circles, our proof works under the uniform $C'(\frac{1}{144})$ condition, and the same holds when $X$ is the Salvetti complex of a right-angled Artin group. When $X$ is a graph, $L(X)$ can be taken to be the girth of $X$.

Remark 3 (The purely cubical case). When $I = \emptyset$, Theorem B recovers known results. In [BHS17], it is shown that under natural extra hypotheses, the action of $\pi_1X$ on $C\tilde{X}$ is acylindrical, and $\tilde{X}$ is unbounded in the absence of an invariant product subcomplex. Even without the extra hypotheses, any $g\in \pi_1X$ acting loxodromically on $H$ actually acts as a WPD element in the sense of [BF02], by [BHS17 Proposition 5.1]. Together with results in [Hag13] characterizing the loxodromic isometries of $C\tilde{X}$, and a result of Osin connecting WPD elements to acylindricity [Osi16], this implies the virtually cyclic/product/acylindrically hyperbolic trichotomy of Theorem B in the case where $I = \emptyset$. This trichotomy (in the purely cubical case) also follows from the Caprace-Sageev rank rigidity theorem [CS11] and general results about groups acting on CAT(0) spaces and containing rank one elements [Osi16, Sis18].

Remark 4 (Classical and graphical cases). The comparison with the acylindrical hyperbolicity result of Gruber-Sisto, for graphical small cancellation groups (as formulated in [Gru15]), is interesting; our results about cubical small cancellation groups do not follow from corresponding results about graphical small cancellation presentations, since the latter viewpoint does not fully account for high-dimensional cubes. (See also Remark 7 for a discussion of an alternate approach using rotating families, and why it does not quite work in our setting.)

On the other hand, restricting Theorems A and B to the case where $\dim X = 1$ and each $Y_i$ is a graph, one does not reprove the results of [GS14] or [Str90] in full generality, since the cubical $C'(\frac{1}{144})$ condition is more restrictive than the conditions needed in the classical and graphical cases (which are the classical $C'(\frac{1}{6})$ and the graphical $Gr(7)$ conditions, respectively).

Remark 5 (Acylindrical action on a quasi-tree). Combining Theorem B with a recent result of Balasubramanya [Bal17] shows that any group covered by Theorem B either satisfies one of the first two conclusions or acts acylindrically and non-elementarily on a quasi-tree.

On the proof of Theorem B. Theorem B is proved roughly as follows. First, we create a hyperbolic $\pi_1X^*$–space $H$ by coning off each hyperplane carrier $N(H)$ and each relator $Y_i$ in the generalized Cayley graph $\text{Cay}(X^*)$. This procedure is a common generalization of the constructions used in the purely cubical case (where the space obtained from coning off the hyperplane carriers is quasi-isometric to the contact graph) and in the graphical case (where the space is obtained from coning off each relator graph $Y_i$ by attaching the complete graph on its vertices, as in [GS14]). Next, we apply Theorem A to show that if $g\in \pi_1X^*$ acts loxodromically on $H$, then $g$ is a WPD isometry of $H$, and therefore Osin’s theorem tells us that $\pi_1X^*$ is acylindrically hyperbolic or virtually cyclic. Proving the hyperbolicity of $H$ also uses Theorem A.

It remains to find loxodromic isometries of $H$. This is done in Section 5. First, we show that if $\tilde{g}\in \pi_1X$ acts loxodromically on the contact graph of $\tilde{X}$, and axes of $\tilde{g}$ have suitably
bounded interaction with elevations of relators \( Y_i \), then the image \( g \in \pi_1 X^1 \) of \( \tilde{g} \) is loxodromic on \( \mathcal{H} \). This is accomplished in Lemma 5.3, Lemma 5.5, and Lemma 5.6. The “suitably bounded interaction” hypothesis is made precise in Definition 5.4: \( \tilde{g} \) must be \textit{asystolic}. Up to this point, we only require uniform \( C'(\frac{1}{|\mathcal{H}|}) \) condition.

Finally, in Lemma 5.7, we show that if \( \tilde{g} \in \mathcal{L} \) realises the minimal translation length, i.e. the translation length of \( \tilde{g} \) is \( L(X) \), then \( \tilde{g} \) is asystolic, and hence \( g \in \pi_1 X^1 \) is loxodromic on \( \mathcal{H} \). The remainder of the proof is essentially an application of results in [CS11] and [Hag13] characterising when \( \mathcal{L} \neq \emptyset \). Lemma 5.7 is where we use the \( \alpha \leq \frac{1}{\pi_L (X)} \) part of the small cancellation condition.

When the set of relators \( Y_i \) is nonempty, there are technical difficulties that are not present in the purely cubical case. This is as one would expect: the difference in complication between the proofs of acylindrical hyperbolicity of \( \pi_1 X^1 \) and \( \pi_1 X \) is analogous to the difference in complication between the proof that a free group is hyperbolic and the proof that a \( C'(\frac{1}{4}) \) group is hyperbolic, with the added complication that the geometry of \( \tilde{X} \) is generally much more sophisticated than that of a tree.

One way in which this manifests is the necessity of using \( L(X) \) in our small cancellation hypothesis. We need to control not only the honest intersection between an axis \( \overline{A} \) of \( \tilde{g} \) and each elevation \( \tilde{Y}_i \) of each relator, but also the “generalised intersection”, i.e. the number of hyperplanes that cross both \( \tilde{Y}_i \) and \( \overline{A} \). (In other words, we need to control the diameter of the image of \( \overline{A} \) under closest-point projection to \( \tilde{Y}_i \).) We also need to know that we can choose elements with controlled overlaps with relators in such a way that they are loxodromic on the contact graph. The challenge is to control all of this with some parameter that \textit{only} depends on \( X \), and not on the various \( Y_i \). The right parameter turns out to be our constant \( L(X) \). These worries are completely absent in the 1–dimensional case, because all nontrivial elements of a free group are loxodromic on the contact graph of the Cayley tree.

**Remark 6** (No proof by cubulation). Cubical small cancellation theory is partly motivated by the fact that groups satisfying strong classical small cancellation conditions act nicely on CAT(0) cube complexes [Wis04] Theorem 1.2. This generalizes in various ways to cubical presentations: if \( \langle X \mid \{Y_i\} \rangle \) satisfies the \textit{generalized} \( B(6) \) condition, one can often cubulate the corresponding group; see, for instance, [Wis] Theorem 5.42.

It is tempting to try to prove Theorem B using this approach, together with the above-mentioned results about acylindrical hyperbolicity of groups acting on cube complexes. However, there are various problems with this approach. For example, the generalized \( B(6) \) condition requires each \( Y_i \) to have a wallspace structure, compatible with the local isometry \( Y_i \to X \), generalizing the wallspace structure on a circle in which each wall is a pair of antipodal points. (Compare with the \textit{lacunary walling} condition on graphical presentations from [AO14].)

No cubical \( C'(\alpha) \) small cancellation condition implies the generalized \( B(6) \) condition, and indeed there are groups that are covered by Theorem B but which do not admit an action on a CAT(0) cube complex with no global fixed point. This can already be seen in the 1–dimensional case: Proposition 7.1 of [OW07] yields, for any \( \alpha > 0 \), a graphical presentation \( \langle X \mid Y \rangle \), where \( X \) is a graph and \( Y \to X \) an immersed graph, satisfying the graphical (hence 1–dimensional cubical) \( C'(\alpha) \) condition, with the additional property that the group thus presented has Kazhdan’s property (T), and thus cannot act fixed point-freely on a CAT(0) cube complex [NR98].

**Remark 7** (No proof by rotating families). One can imagine an alternate approach to Theorem B using [DGO17] Proposition 5.33. Specifically, if \( X \) is a nonpositively curved cube complex, then \( \pi_1 X \) acts on \( CX \) in such a way that any loxodromic element is WPD [BHS17] Proposition 5.1]. From Caprace-Sageev rank rigidity, and the consequences discussed in [Hag13]
Section 5], it is possible to conclude that either $\overline{X}$ is a nontrivial product (up to passing to an invariant subcomplex), or such a loxodromic element exists.

At least in cases where this action is acylindrical (which include all cases where $X$ is virtually special [BHS17], and indeed all known compact $X$ [HS16]), one is then tempted to proceed as follows.

Form a new graph $\hat{H}$ from $\overline{X}$ by coning off each hyperplane carrier and each elevation $\overline{Y}_i$ of each $Y_i$. (Up to quasi-isometry, this is the same as starting with $C\overline{X}$ and coning off the subgraph corresponding to the set of hyperplanes crossing $\overline{Y}_i$, for each $\overline{Y}_i$.)

In $\hat{H}$, on which $\pi_1X$ acts, each conjugate of each $\pi_1Y_i \leq \pi_1X$ fixes a cone-point: this gives a rotating family. One might then hope to apply [DGO17, Proposition 5.33] to conclude.

However, no metric cubical small-cancellation condition is sufficient to allow this. Indeed, if $H$ is a hyperplane that crosses $\overline{Y}_i$, then paths in the carrier of $H$ do not count as wall-pieces in $\overline{Y}_i$, and thus the small-cancellation condition places no restriction on these pieces. In particular, there may be infinite-order elements of $\pi_1Y_i$ that stabilise vertices at distance 2 in $\hat{H}$ from the cone-point corresponding to $\overline{Y}_i$. This makes it difficult to expect that one could use the techniques in [DGO17] to obtain Theorem B in its current formulation (i.e. without narrowing the class of group presentations under consideration).

**Examples of cubical small cancellation groups.** We list here some examples of cubical small cancellation groups to which Theorem [X] and Theorem [B] apply. We have earlier mentioned classical and graphical small cancellation presentations to which our results apply, as well as the case where $X$ is a nonpositively curved cube complex and $\mathcal{I} = \emptyset$.

1. Classical/RAAG hybrid: let $X$ be the Salvetti complex of a right-angled Artin group $A$, with presentation graph $\mathcal{G}$, and let $\{g_i\}_{i \in I}$ be a collection of independent elements, none of which is supported on a proper join in $\mathcal{G}$ (i.e. each $g_i$ is a rank one isometry of $\overline{X}$). More generally, choose $\{(g_i)\}_{i \in I}$ to be a malnormal collection of cyclic subgroups, each of which has a convex cocompact core $\overline{Y}_i$ in $\overline{X}$. Then for each $i$ there exists $n_i > 0$ so that, letting $Y_1 = \langle g_i^{n_i} \rangle \setminus \overline{Y}_i$, the cubical presentation $\langle X \mid \{Y_i\} \rangle$ is a $C'(1/117)$ presentation. Furthermore, instead of cyclic subgroups, one could use appropriately chosen purely loxodromic subgroups as described in [KMT17], which are necessarily free.

2. More generally, let $X$ be a compact nonpositively curved cube complex. Let $\{Y_i \to X\}$ be a collection of local isometries of cube complexes so that the resulting cubical presentation $\langle X \mid \{Y_i\} \rangle$ satisfies the (uniform) cubical $C'(\alpha)$ condition for some $\alpha > 0$. Suppose that each $Y_i$ has residually finite fundamental group. Thus, for any $n \in \mathbb{N}$, there is a finite cover $\overline{Y}_i \to Y_i$ with $\|\overline{Y}_i\| \geq n\|Y_i\|$. Thus the related cubical presentation $\langle X \mid \{\overline{Y}_i\} \rangle$ satisfies the (uniform) cubical $C'(n\alpha)$ condition. So, for sufficiently large $n$, it satisfies $C'(\alpha_0)$, where $\alpha_0 = \alpha_0(X)$ is as in Theorem [B].

3. Given letters $x, y$ and $m \geq 1$, let $(x, y)^m$ denote the first half of the word $(xy)^m$. Consider the Artin group

$$A = \langle a_1, a_2, \ldots, a_n \mid (a_i, a_j)^{m_{ij}} = (a_j, a_i)^{m_{ij}} \text{ whenever } i \neq j \rangle.$$  

(Note that we follow the convention of letting $m_{ij} = \infty$ to indicate that there is no relation between $a_i, a_j$.) Let $\hat{A} = \langle a_1, a_2, \ldots, a_n \mid [a_i, a_j] \text{ whenever } m_{ij} = 2 \rangle$ be the underlying right-angled Artin group, $X$ be its Salvetti complex, and $\Gamma$ be its presentation graph (a graph with a vertex for each $a_i$ and with an edge from $a_i$ to $a_j$ when $m_{ij} = 2$). That is, $A$ is a quotient of $\hat{A}$ obtained by adding the relations $(a_i, a_j)^{m_{ij}} = (a_j, a_i)^{m_{ij}}$ when $m_{ij} \neq 2$. Observe that each element $g_{ij} = (a_i, a_j)^{m_{ij}}(a_j, a_i)^{-m_{ij}}$ of $\hat{A}$, with $m_{ij} \neq 2$, is a rank one isometry of $X$, since, if it is supported in a join in the presentation graph $\Gamma$ of $\hat{A}$, then it is supported in a factor of that join. Hence there is a convex
subcomplex $\tilde{Y}_{ij}$ of $\tilde{X}$ that is cocompactly stabilized by $\langle g_{ij} \rangle$, and which is just the convex hull of a combinatorial $g_{ij}$–axis. Let $Y_{ij}$ be the quotient of $\tilde{Y}_{ij}$ by the $\langle g_{ij} \rangle$–action, so that $\langle X \mid Y_{ij} \text{ whenever } 2 < m_{ij} < \infty \rangle$ is a cubical presentation for the Artin group $A$. Clearly, $Y_{ij}$ has systole $2m_{ij}$, so in order to impose a condition on $m_{ij}$ ensuring that this presentation satisfies the cubical $C'(\frac{1}{144})$ condition, we need only investigate the cone-pieces and wall-pieces.

If $\tilde{\mathcal{P}}$ is a cone-piece between $\tilde{Y}_{ij}$ and $\tilde{Y}_{k\ell}$, then $|\tilde{\mathcal{P}}| = 1$. On the other hand, if $a_i, a_j$ lie in the link of some $a_k$ in $\Gamma$, then any geodesic in $\tilde{Y}_{ij}$ is a wall-piece, but otherwise wall-pieces have length $\leq 1$. Hence suppose that $A$ satisfies the following:

- for all $i \neq j$, either $m_{ij} = 2$ or $m_{ij} = \infty$ or $m_{ij} > 72$;
- for all $i \neq j$ such that there exists $k$ with $m_{ik} = m_{jk} = 2$, we have either $m_{ij} = 2$ or $m_{ij} = \infty$.

Then the above cubical presentation for $A$ is $C'(\frac{1}{144})$ and our acylindricity theorem applies to $A$. There is a related recipe in Section 20 of [Wis], for building $C(6)$ cubical presentations of Artin groups (cf. [AS83]) but it is harder to see when these are $C'(\frac{1}{144})$.

In a similar manner, up to some limited worries about torsion, one should be able to produce Coxeter groups that virtually admit cubical $C'(\frac{1}{144})$ presentations; the base cube complex should be the Davis complex of the underlying right-angled Coxeter group.

**Outline of the paper.** In Section 1, we recall background on acylindrical hyperbolicity and WPD elements. Section 2 contains a discussion of cubical presentations, disc diagrams, and the parts of cubical small cancellation theory needed in the proof of the classification of triangles, Theorem A, which also occurs in this section. The proof uses the theory developed in [Wis]. The key tools are the ladder theorem [Wis, Theorem 3.42], the Greendlinger lemma/diagram trichotomy for cubical small cancellation presentations [Wis, Theorem 3.45], as well as the splitting lemma from [Wis, Section 3.h], which is a system for assigning angles to corners of 2–cells in a disc diagram over a cubical presentation.

In Section 3, we give a list of conditions on a cubical small cancellation group $G$ acting on a hyperbolic space $\mathcal{H}$ sufficient to ensure that $G$ contains an element $g$ acting on $\mathcal{H}$ as a WPD element. Specifically, we use Theorem A to show that any $g \in G$ acting loxodromically on a space $\mathcal{H}$ satisfying the given conditions acts as a WPD element. In Section 4, we produce such a space $\mathcal{H}$, formed from a generalized Cayley graph $\text{Cay}(X^*)$ by coning off both the relators and the hyperplane carriers. Theorem A is also used here to check that $\mathcal{H}$ is hyperbolic. Finally, in Section 5, we explain when one can find elements of $G$ acting loxodromically on $\mathcal{H}$.

We assume basic knowledge of CAT(0) and nonpositively curved cube complexes and cubical presentations; we refer the reader to [Wis] for most of the background. Most of the material that we will need from [Wis] is restated below, although for some more technical points we will refer the reader to [Wis] with various citations.

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## 1. Acylindrical hyperbolicity and WPD elements

The notion of an *acylindrically hyperbolic group* was defined in [Osi16], as follows:

**Definition 1.1** (Acylindrical action, acylindrical hyperbolicity). Let $(X, d)$ be a metric space and let $G$ act on $X$ by isometries. Then the action is *acylindrical* if for each $\epsilon > 0$, there exists
Let $X$ be Gromov-hyperbolic and let $G$ act by isometries on $X$. The action of $G$ is elementary if the limit set of $G$ on $\partial X$ has at most two points. If $G$ acts non-elementarily and acylindrically on a hyperbolic space, then $G$ is acylindrically hyperbolic.

When $G$ is a cubical small cancellation group, we will construct an explicit action of $G$ on a hyperbolic space $\mathcal{H}$, but this will not necessarily be the action that witnesses acylindrical hyperbolicity. Instead, the action will be such that $G$ contains a WPD isometry of $\mathcal{H}$.

**Definition 1.2** (WPD element [BP02]). Let $G$ act by isometries on the space $X$. Then $h \in G$ is a WPD element if for each $\epsilon > 0$ and each $x \in X$, there exists $M \in \mathbb{N}$ so that

$$|\{g \in G : d(x, gx) \leq \epsilon, d(h^M x, gh^M x) \leq \epsilon\}| < \infty.$$ 

In [Osi16], Osin showed that if $G$ is not virtually cyclic and acts on a hyperbolic space $\mathcal{H}$, and some $g \in G$ acts on $\mathcal{H}$ as a loxodromic WPD element, then $G$ is acylindrically hyperbolic. This is instrumental in the proof of Theorem B.

2. Triangles in Cubical Small Cancellation Groups

In this section, $X$ denotes a connected nonpositively curved cube complex with universal cover $\tilde{X}$. When doing geometry in $\tilde{X}$, we never use the CAT(0) metric and instead only use the usual graph metric on $\tilde{X}$ in which each 1–cube has length 1 and a combinatorial path is geodesic if and only if it contains at most one edge intersecting each hyperplane of $\tilde{X}$. We fix a (possibly infinite) index set $I$, and for each $i \in I$, let $Y_i \to X$ be a local isometry of connected cube complexes. Each $Y_i$ is necessarily nonpositively curved. Following [Wis], the associated cubical presentation is $(X \mid \{Y_i\}_{i \in I})$ and the corresponding cubical presentation complex $X^*$ is formed as follows. For each $i \in I$, let $C(Y_i)$ be the relator on $Y_i$, i.e. the space formed from $Y_i \times [0, 1]$ by collapsing $Y_i \times \{1\}$ to a point. This space has an obvious cell-structure so that $Y_i \sim Y_i \times \{0\} \hookrightarrow C(Y_i)$ is a combinatorial embedding. For each $i \in I$, we attach $C(Y_i)$ to $X$ along $Y_i \times \{0\}$ using the above local isometry. The resulting complex is $X^*$. The group of our interest is defined by $G = \pi_1 X^*$. We say that $(X \mid \{Y_i\}_{i \in I})$ is a cubical presentation for $G$.

The universal cover $\tilde{X}^*$ of $X^*$ is a nonpositively curved cube complex with cones attached. Let $\text{Cay}(X^*)$ be the part of $\tilde{X}^*$ consisting only of cubes (i.e. the complement of the open cones). This is the generalized Cayley graph of $G$ with the given cubical presentation. Note that there are covering maps $\tilde{X} \to \text{Cay}(X^*) \to X$; the generalized Cayley graph is the nonpositively curved cube complex obtained by taking the cover of $X$ corresponding to the kernel of $\pi_1 X \to \pi_1 X^*$.

**Remark 2.1.** (Classical and graphical presentations) If $X$ is a wedge of circles and each $Y_i$ is an immersed combinatorial circle, then $(X \mid \{Y_i\}_{i \in I})$ is a group presentation in the usual sense (each $C(Y_i)$ is a disc) and $\text{Cay}(X^*)$ is the associated Cayley graph of $G$. As mentioned in [Wis] Examples 3.3, if $X$ is a graph and each $Y_i$ is an immersed graph, then the above cubical presentation is a graphical presentation in the sense of [RS87] [Gro03] [Oll06].

**Remark 2.2** (Elevations). The local isometries $Y_i \to X$ lift to local isometries $\tilde{Y}_i \to \text{Cay}(X^*)$ (in fact, under the small cancellation conditions we shall soon be assuming, the latter maps are embeddings [Wis Section 4]). We use the term elevation to refer to a lift $\tilde{Y}_i \to \tilde{X}$ of the map $\tilde{Y}_i \to Y_i \to X$, where $\tilde{Y}_i \to Y_i$ is the universal covering map. Since $Y_i \to X$ is a local isometry, it is $\pi_1$–injective and $\tilde{Y}_i \to \tilde{X}$ is a combinatorial embedding with convex image.

**Notation 2.3** (Carriers and neighbourhoods). Let $X$ be a nonpositively curved cube complex and let $H \to X$ be an immersed hyperplane. The (abstract) carrier of $H$ is $N(H) = H \times \mathbb{R}$. Let $x, y \in X$ for which $d(x, y) \geq R$, we have

$$|\{g \in G : d(x, gx) \leq \epsilon, d(y, gy) \leq \epsilon\}| \leq N.$$
\([-\frac{1}{2}, \frac{1}{2}]\), equipped with the product cubical structure, where \(H\) has a nonpositively curved cubical structure coming from \(X\) and \([-\frac{1}{2}, \frac{1}{2}]\) is a 1–cube. The map \(H \to X\) extends to a cubical map \(N(H) \to X\) (see e.g. \([\text{Wis} \; \text{Section} \; 2.9]\)). When \(X\) is CAT(0), this map is an embedding whose image is a convex subcomplex \(N(H)\), the carrier of the hyperplane \(H\). In this case, \(N(H)\) is just the union of the closed cubes that intersect \(H\).

Given an arbitrary metric space \(M\) and a subspace \(H\), we denote by \(\mathcal{N}_r(H)\) the closed \(r\)–neighbourhood of \(H\) in \(M\). The similarity in notation is justified by the fact that, when \(X\) is a CAT(0) cube complex and \(H\) is a hyperplane, \(\mathcal{N}(H) = \mathcal{N}_\frac{\pi}{2}(H)\).

We now review background about cubical small cancellation theory, following \([\text{Wis}]\).

**Definition 2.4** (Abstract cone-piece, abstract wall-piece, cone-piece, wall-piece, piece). Given a CAT(0) cube complex \(\bar{X}\), and convex subcomplexes \(U, V\), let \(\text{Proj}(U \to V)\) be the subcomplex of \(V\) defined as follows. First, a closed 1–cube \(e\) of \(V\) is in \(\text{Proj}(U \to V)\) if \(e\) is dual to a hyperplane intersecting \(U\). Then add any cube of \(V\) whose 1–skeleton appears.

Let \(\langle X \mid \{Y_i\}_{i \in I} \rangle\) be a cubical presentation. Let \(A, B \in \{Y_i\}_{i \in I}\). An abstract cone-piece of \(B\) in \(A\) is a component of \(\text{Proj}(\bar{B} \to \bar{A})\), where \(\bar{B}, \bar{A}\) are components of the preimages of \(B, A\) respectively, under the covering map \(\bar{X} \to X\), satisfying \(\bar{B} \neq \bar{A}\). If \(H\) is a hyperplane of \(X\) not intersecting \(B\), then, likewise, an abstract wall-piece of \(H\) in \(B\) is a component of \(\text{Proj}(\mathcal{N}(_H) \to \bar{B})\). A cone-piece is a nontrivial path in an abstract cone-piece, and a wall-piece is a nontrivial path in an abstract wall-piece, and a piece is a path which is either a cone-piece or a wall-piece.

In the case where \(X\) is a wedge of circles and each \(Y_i\) is a loop, all wall-pieces are trivial, and cone-pieces correspond to pieces in the sense of classical small-cancellation theory.

**Definition 2.5** (\(C'(\alpha)\) condition, uniform \(C'(\alpha)\) condition). The cubical presentation \(\langle X \mid \{Y_i\}_{i \in I} \rangle\) satisfies the cubical \(C'(\alpha)\) small cancellation condition if the following holds for all \(i \in I\): \(\text{diam}(P) < \alpha \|Y_i\|\) for all abstract pieces \(P\) in \(Y_i\), where \(\|Y_i\|\) denotes the infimum of the lengths of essential closed paths in \(Y_i\). In this case, we say that \(\langle X \mid \{Y_i\}_{i \in I} \rangle\) is a \(C'(\alpha)\) presentation and \(G = \pi_1 X^*\) is a \(C'(\alpha)\) group.

Note that if \(|I| < \infty\), then the \(C'(\alpha)\) condition yields a uniform bound on the length of all pieces, namely \(\alpha \max_{i \in I} \|Y_i\|\).

In Section 5 we will use the stronger uniform \(C'(\alpha)\) condition. The cubical presentation \(\langle X \mid \{Y_i\}_{i \in I} \rangle\) satisfies the uniform \(C'(\alpha)\) condition, which requires that \(\text{diam}(P) \leq \alpha \|Y_i\|\) for all \(i\), whenever \(P\) is an abstract piece (not necessarily in \(Y_i\)). This condition is needed to maintain an upper bound on the sizes of pieces, needed, for example, in the proof of Lemma 5.3.

Another way to phrase the uniform condition is to let \(Y = \bigsqcup_{i \in I} Y_i\), so that the local isometries \(Y_i \to X\) induce a local isometry \(Y \to X\). Then the uniform \(C'(\alpha)\) condition for \(\langle X \mid \{Y_i\}_{i \in I} \rangle\) is equivalent to the (non-uniform) cubical \(C'(\alpha)\) condition for the presentation \(\langle X \mid \{Y\} \rangle\) (except allowing disconnected relators). This should be compared to the small cancellation conditions in \([\text{GSt} \; \text{Section} \; 2.2]\): in both cases, infinitely many relations can be encoded in a single cube complex (a possibly infinite, disconnected graph in the graphical case, \(Y\) here), and it’s the systole of that complex that is used in the uniform small-cancellation condition.

**Definition 2.6** (Disc diagram, boundary path). A disc diagram is a compact, contractible 2–dimensional cell complex equipped with a fixed embedding in \(\mathbb{R}^2\). We regard \(S^2\) as \(\mathbb{R}^2 \cup \{\infty\}\), so that \(S^2\) is obtained from \(D\) by attaching a 2–cell containing \(\infty\). The attaching map of this 2–cell is the boundary path \(\partial_p D\) of \(D\).

Given a cubical presentation \(\langle X \mid \{Y_i\}_{i \in I} \rangle\) and a closed path \(P \to X\) that is nullhomotopic in \(X^*\), van Kampen’s lemma provides a disc diagram \((D, \partial_p D) \to (X^*, X)\) whose boundary path \(\partial_p D = P\).
The 2–cells of such a diagram are either squares (mapping to 2–cubes of $X \subseteq X^*$) or 2–simplices mapping to cones over the various $Y_i$. Since $P$ avoids cone-points, the 2–simplices of $D$ are partitioned into classes: for each vertex of $D$ mapping to a cone-point in $X^*$, the incident 2–simplices are arranged cyclically around the vertex to form a subspace $C$ of $D$ which is equal to the cone on its boundary path (a path in $D$ mapping to $X$). The subspace $C$ is a cone-cell.

In practice, we ignore the subdivision of cubes intersecting and a midcube of a $0$–cube, any dual curve has its two ends either on $1$–cells, so that the cone-cell $C$ has no common non-trivial subpath.

The complexity of $D$ is the pair $(c, s)$, where $c$ is the number of cone-cells and $s$ is the number of squares. Taking the complexity in lexicographic order, we always consider diagrams $(D, \partial_p D) \to (X^*, X)$ which are minimal in the sense that the complexity of $D$ is lexicographically minimal among diagrams with boundary path $\partial_p D$. This implies that for each cone-cell $C$, the path $\partial_p C \to D \to \bar{Y} \to X$ is essential.

**Remark 2.7** (Dual curves and hexagon moves). Let $D \to X$ be a square diagram. A dual curve in $D$ is a path which is the concatenation of midcubes of squares of $D$ that starts and ends on $\partial_p D$, where a midcube of a square $[-\frac{1}{2}, \frac{1}{2}]^2$ is obtained by restricting exactly one coordinate to 0 and a midcube of a 1–cube is its midpoint. If $X$ is a nonpositively curved cube complex, then each dual curve maps to a hyperplane. If $K$ is a dual curve in $D$, then the union of all closed cubes intersecting $K$ is its carrier (in analogy to the definition for a hyperplane).

More generally, if $D \to X^*$ is a disc diagram, then one can define dual curves as above, but any dual curve has its two ends either on $\partial_p D$ or on the boundary path of a cone-cell of $X^*$.

A *hexagon move* is a homotopy of the diagram $D \to X^*$ that fixes the boundary path and the cone-cells and their boundary paths, while modifying the square part of $D$. Specifically, if $s_1, s_2, s_3$ are squares in $D$ arranged cyclically around a central vertex $v$, forming a hexagonal subdiagram $E$ of $D$, then $X$ must contain a 3–cube $c$ with a corner at the image of $v$ formed by the images of $s_1, s_2, s_3$. The (hexagonal) boundary path of $E$ maps to a combinatorial path in $c$, and we can replace $E$ by a diagram $E'$ formed from the other 3 squares on the boundary of $c$; this yields a new diagram $D' \to X^*$, with the same boundary path as $D$, formed by replacing $E$ by $E'$. This modification is a hexagon move. Hexagon moves are used to reduce area in various ways; detailed accounts can be found in e.g. [Wis, Wis12].

**Definition 2.8** (External cone-cell, internal cone-cell, internal path). The cone-cell $C$ of the disc diagram $D$ is external if $\partial_p C = QS$, where $Q$ is a non-trivial subpath of $\partial_p D$ (i.e. containing at least one 1–cell) and $S$ is an internal path in the sense that no 1–cell of $S$ lies on $\partial_p D$. The cone-cell $C$ is internal if $\partial_p C$ and $\partial_p D$ have no common non-trivial subpath.

**Remark 2.9** (Rectification and angling). Given a disc diagram $(D, \partial_p D) \to (X^*, X)$, one can *rectify* $D$, to produce a rectified diagram $\bar{D}$, by removing some internal open 1–cells, so that $\bar{D}$ is subdivided into cone-cells, *rectangles* which are obtained from square ladders by deleting the internal open 1–cells, and complementary regions called shards. See [Wis Section 3.f] for more discussion of rectified diagrams. We will not require further details here.

After rectifying $D$, each corner in each of the resulting 2–cells is assigned an *angle* according to one of several possible schemes. We follow the *split-angling* defined in [Wis Section 3.h]. Specifically, if $v$ is a vertex of the rectified diagram $D$, and $c$ is an edge in the link of $v$ (i.e. a corner of a 2–cell at the vertex $v$), then we assign an angle $\angle(c)$ according to rules discussed in [Wis Section 3.h]. Since we will just be using consequences of these angle assignments, rather than the exact (long) definition, we refer the reader to [Wis Section 3.h]. Suffice it to say that:

- the angle $\angle(c)$ is always $\pi/2, \pi, 2\pi/3, 3\pi/4, \text{ or } 0$;
- the choice of angle is made in a way that guarantees nonpositive curvature at vertices and shards, in the sense described momentarily.

**Remark 2.10** (Defects and curvature). We now briefly review some notions related to curvature, from [Wis Section 3.g], that we will require below. Given a rectified disc diagram $\bar{D}$,
we assign an angle $\angle(c)$ – a real number – to each corner $c$ of each 2-cell (i.e. to each 1-cell of each vertex-link). In our setting, we always assume that this is done using the split-angling convention.

The defect $d(c)$ at the corner $c$ is $d(c) = 2\pi - \angle(c)$. The curvature $\kappa(v)$ at a vertex $v$ of $\bar{D}$ is $\kappa(v) = 2\pi - \sum \angle(c) - \pi \chi(Lk(v))$, where $Lk(v)$ is the link of $v$ and the sum is taken over the 1-cells $c$ of $Lk(v)$. The curvature $\kappa(f)$ at a 2-cell $f$ of $\bar{D}$ is $\kappa(f) = 2\pi - \sum d(c)$, where $c$ varies over the corners of $f$.

We will need the following theorem, which follows immediately from the “combinatorial Gauss-Bonnet Theorem” as stated in [MW02, Theorem 4.6]; very similar statements can be found in [Bri48, Ger87, BB96].

**Theorem 2.11** (Gauss-Bonnet for diagrams). Let $\bar{D} \to X^*$ be a rectified disc diagram. Then

$$\sum_f \kappa(f) + \sum_v \kappa(v) = 2\pi,$$

where $f$ varies over the 2-cells of $\bar{D}$ and $v$ varies over the 0-cells of $\bar{D}$.

In the case where $X$ is a wedge of circles and each $Y_i$ is an immersed circle, i.e. $X^*$ is an ordinary presentation complex, disc diagrams are ordinary van Kampen diagrams, all rectangles are single edge, and rectifying has no effect on the diagram. In this case, all 2-cells of $D$ are cone-cells, the split-angling continues to ensure that the curvature at each vertex is nonpositive, and the condition on the curvature of shards is vacuous.

**Definition 2.12** (Generalized corner, spur, shell). A (positively-curved) shell $C$ in the disc diagram $D$ is an external cone-cell whose curvature is positive; the boundary path of a shell has the form $QS$, where the outer path $Q$ is a subpath of the boundary path of $D$, and the inner path $S$ has no open 1-cell on $\partial p D$. A spur in $D$ is a vertex $v$ in $\partial p D$ so that the incoming and outgoing 1-cells of $\partial p D$ map to the same 1-cell of $X$, i.e. $v$ is the second vertex in a subpath of $\partial p D$ of the form $ee^{-1}$, where $e \to X$ is a 1-cell. A generalized corner is a path $ef$ in $D$, where each of $e, f$ is an edge, so that the dual curves emanating from $e, f$ cross inside a square $s$ of $D$, as shown in Figure 1.

![Figure 1](image)

**Figure 1.** $ef$ and $ab$ are generalized corners of the shaded squares. $ef$ lies on the boundary of $D$, while $ab$ lies on the boundary of a cone-cell.

**Remark 2.13** (Pushing generalized corners to the boundary). If $ef$ is a generalized corner of a square $s$, and $ef$ lies along $\partial p D$, and the subdiagram bounded by the carriers of the dual curves emanating from $e, f$ is a square diagram, then we can perform a series of hexagon moves (see [Wis, Section 2.e]) to homotope $D$, fixing its boundary path, so that there is a square with
boundary path $efe'f'$, i.e. we can move squares to the boundary. In [Wis], this procedure is called “shuffling”.

The same situation could occur, except with $ef$ lying on the boundary of some cone-cell $C$ mapping to a relator $Y$. In this case, we can again shuffle until the square $s$ has two consecutive edges on $\partial_p C$. Convexity of $Y$ allows us to “absorb” the square $s$ into $C$, lowering complexity of $D$. (In the current version of [Wis], what we call a generalised corner is called a cornsquare, described in Definition 2.5 of loc. cit.)

**Definition 2.14** (Padded ladder, ladder, cut-vertex). A padded ladder is a disc diagram $D \to X^*$ (or $\tilde{X}^*$) with the following structure. First, there is a sequence $C_1, \ldots, C_n$, where each $C_i$ is a cone-cell or vertex of $D$, so that $C_i, C_k$ lie in distinct components of $D - C_j$ whenever $i < j < k$. The diagram $D$ is an alternating union of these vertices and cone-cells with a sequence of subdiagrams $R_0, \ldots, R_n$ called pseudorectangles, so that:

1. The path $\partial_p D$ is a concatenation $P_1P_2^{-1}$, where each of $P_1, P_2$ starts on $R_0$ and ends on $\tilde{R}_n$.
2. We have $P_1 = \nu_0\alpha_0\rho_1 \cdots \alpha_n\rho_n$ and $P_2 = \rho_0\gamma_1\gamma_1 \cdots \gamma_n\nu_n\mu_{n+1}$.
3. We have $\partial_p C_i = \mu_i\alpha_i\nu_i^{-1}\mu_i^{-1}$.
4. We have $\partial_p R_i = \nu_i\rho_i\mu_i^{-1}\rho_i^{-1}$.
5. Each $R_i$ is a square diagram, i.e. contains no cone-cells.
6. For each $i$, any dual curve in $R_i$ emanating from $\rho_i$ ends on $\gamma_i$ and vice versa. Hence any dual curve emanating from $\nu_i$ ends on $\mu_i$ and vice versa.
7. For each $i$, no two dual curves emanating from $\mu_i$ cross.

See Figure 2 for a picture illustrating the notation. We say that $R_i$ is horizontally degenerate if $|\mu_{i+1}| = |\nu_i| = 0$ and vertically degenerate if $|\rho_i| = |\gamma_i| = 0$. When $C_i$ is a vertex, we call it a cut-vertex of the padded ladder $D$. If $R_0, R_n$ are vertically degenerate, then $D$ is a ladder.

![Figure 2. A padded ladder.](image)

(A padded ladder is a special case of what Jankiewicz calls a generalized ladder in [Jan17]; the definition of ladder here is equivalent to that in [Wis Definition 3.41]).

We require the following three crucial facts, due to Wise. These are tailored to our specific situation; the statements in [Wis] are more general.

**Theorem 2.15** (Ladder theorem). Let $(X \mid \{Y_i\})$ be a cubical $C'(\frac{1}{12})$ presentation. Let $D \to X^*$ be a minimal disc diagram such that the corresponding rectified diagram has exactly two positively-curved cells along $\partial_p D$. Then $D$ is a ladder.

**Proof.** This follows by combining [Wis Theorem 3.42] with [Wis Theorem 3.31]. See also [Wis Examples 3.h.(3)]. □
Proof. Apply [Wis, Theorem 3.31] and [Wis, Theorem 3.45]. □

The next theorem follows directly from Lemma 3.68 of [Wis]. In fact, it holds under weaker small-cancellation conditions (see [Wis, Lemma 3.68] or [Jan17]), but we will not require this.

Theorem 2.17 (Short inner paths). Let \( \langle X \mid \{Y_i\}_i \rangle \) be a cubical \( C'(\frac{1}{144}) \) presentation. Let \( D \to X^* \) be a disc diagram and let \( C \) be a shell in \( D \) with boundary path \( QS \), with \( Q \) a maximal common subpath of \( \partial_p C \) and \( \partial_D D \), and \( S \) an internal path. Suppose that \( QS \) is essential in the relator to which \( C \) maps, and that \( S \) is of minimal length among all paths \( S' \to Y \) that are homotopic rel endpoints in \( Y \) to \( S \). Finally, suppose that the total curvature contribution from \( C \) is \(< \pi \). Then \( |S| \to |Q| \).

2.1. The classification theorem. Using the above fundamental results, we produce a cubical small cancellation version of Strebel’s classification of triangles in classical small cancellation groups [Str90]. An exposed square in a disc diagram \( D \) is a square with two consecutive edges on \( \partial_D D \). A tripod is a triangle diagram with no cone-cells or squares. We can now state our classification of triangles:

Theorem 2.18 (Classification of triangles in cubical \( C'(\frac{1}{144}) \) groups). Let \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) be a cubical presentation satisfying the \( C'(\frac{1}{144}) \) condition. Let \( \alpha, \beta, \gamma \to \text{Cay}(X^*) \) be combinatorial geodesics so that \( \alpha \beta \gamma \) is a geodesic triangle. Then there exists a disc diagram \((D, \partial_D D) \to (X^*, X)\) with boundary path \( \alpha' \beta' \gamma' \to \text{Cay}(X^*) \to X^* \) lying in \( X \), so that the following hold. First, \( \alpha \to X \) and \( \alpha' \to X \) co-bound a bigon \( B \to X \) (i.e. they are square-homotopic) and the same is true of \( \beta, \beta' \) and \( \gamma, \gamma' \). Second, \( D \) is of one of the following types.

1) (3-shell generic:) \( D \) has exactly three external cone-cells, \( C_1, C_2, C_3 \), respectively containing the points \( \alpha'_i \cap \beta'_i \cap \gamma'_i \cap \alpha'_i \). There is exactly one cone-cell \( M \) that intersects \( \alpha'_i, \beta'_i, \) and \( \gamma'_i \). Moreover, \( D \) is the union of three ladders, \( L_1, L_2, L_3 \) so that \( L_i \cap L_j = M \) for all \( i, j \). In particular, every cone-cell except \( M \) intersects exactly two of the geodesics \( \alpha, \beta, \gamma \).

2) (3-shell tripod:) \( D \) has exactly three external cone-cells, \( C_1, C_2, C_3 \), respectively containing the points \( \alpha'_i \cap \beta'_i \cap \gamma'_i \cap \alpha'_i \). Every other cone-cell intersects exactly two of the geodesics \( \alpha'_i, \beta'_i, \gamma'_i \). In this case, \( D \) is the union of 3 (possibly padded) ladders \( L_1, L_2, L_3 \) and a tripod triangle \( P_1 P_2 P_3 \to X \) so that \( L_i \) intersects the other two ladders in the path \( P_i \).

3) (2-shell generic:) Same as 3-shell generic, except exactly one of \( C_1, C_2, C_3 \) is a spur or exposed square instead of a cone-cell.

4) (2-shell tripod:) Same as 3-shell tripod, except exactly one of \( C_1, C_2, C_3 \) is a spur or exposed square instead of a cone-cell.

5) (1-shell generic:) Same as 2-shell generic, except exactly two of \( C_1, C_2, C_3 \) are spurs or exposed squares.

6) (1-shell tripod:) Same as 2-shell tripod, except exactly two of \( C_1, C_2, C_3 \) are spurs or exposed squares.

7) (No-shell generic:) Same as 3-shell generic, except \( C_1, C_2, C_3 \) are all spurs or exposed squares.

8) (No-shell tripod:) Same as 3-shell tripod except \( C_1, C_2, C_3 \) are all spurs or exposed squares. This includes the case where \( \alpha' \beta' \gamma' \) is nullhomotopic in \( X \), in which case \( D \) is a tripod.
(9) (Degenerate triangle:) \( D \) is a single vertex or cone-cell, or \( D \) is a ladder. In this case, at least one of \( \alpha, \beta, \gamma \) is trivial.

The diagram \( D \rightarrow X^* \) is a standard diagram for the triangle \( \alpha \beta \gamma \). The eight non-degenerate cases are shown in Figure 3.

\begin{figure}
\includegraphics[width=\textwidth]{figure3.png}
\caption{The cases from Theorem 2.18 are shown, clockwise from the top left: cases (1),(3),(5),(7),(8),(6),(4), (2). The features of positive curvature are labelled, and the constituent ladders in some cases are labelled. Any of the spurs may instead be exposed squares and vice versa. The ladder case is not shown.}
\end{figure}

Remark 2.19 (Media and small cancellation parameters). The standard diagram depends only on the endpoints of the geodesics \( \alpha, \beta, \gamma \). Just as it is usual in CAT(0) cube complexes, fixing their endpoints, in order to minimize the area of diagrams, here we are not married to particular geodesics, just to square-homotopy classes rel endpoints. In particular, if \( \alpha \beta \gamma \) bounds a disc diagram in \( X \), then \( D \) is a tripod. When \( I = \emptyset \), Theorem 2.18 just says: any three 0-cubes in a CAT(0) cube complex determine a geodesic tripod.

More generally, as illustrated by Figure 3, Theorem 2.18 should be interpreted as saying that the vertices of the triangle have a “median” which is either a vertex or a cone-cell, and there is a geodesic triangle connecting the given three points, each of whose sides passes within a wall-piece of the “median”. In other words, given 0-cells \( a, b, c \in \text{Cay}(X^*) \), the “convex hulls” of the three possible pairs mutually coarsely intersect.

At the other extreme, when \( X \) is a graph and each \( Y_i \) is an immersed circle, Theorem 2.18 generalizes a weak version of Strebel’s classification of triangles [Str90, Theorem 43]; specifically, Theorem 2.18 provides the same classification as Strebel’s result, but, because the proof must work in the more general context of cubical presentations, we require stronger metric small cancellation conditions than Strebel needs in the classical setting.

Proof of Theorem 2.18. This is essentially a meticulous application of Theorem 2.16, Theorem 2.15 and Theorem 2.17 following and followed by appropriately chosen square homotopies; the main point is a curvature computation to eliminate the possibility of internal cone-cells (in the Strebel classification in the case of classical \( C'(\frac{1}{2}) \) condition, one of the primary features is that the disc diagrams do not have internal cells). This computation, which will use a slightly modified version of the proof of Theorem 3.29 of [Wis], is why we need the \( C'(\frac{1}{14}) \) condition.

Choosing \( \alpha', \beta', \gamma' \) and constructing \( D \): Given a geodesic \( P \rightarrow \text{Cay}(X^*) \), let \( [P] \) be the set of geodesics \( Q \) that have the same endpoints as \( P \) and the additional property that \( PQ^{-1} \) bounds a disc diagram containing no cone-cells, i.e. there is a disc diagram \( E \rightarrow X \) whose boundary path is \( PQ^{-1} \rightarrow \text{Cay}(X^*) \rightarrow X^* \).

Choose a disc diagram \( D \rightarrow \tilde{X}^* \) so that \( \partial_D = \alpha' \beta' \gamma' \), where \( \alpha' \in [\alpha], \beta' \in [\beta], \gamma' \in [\gamma] \). Choose \( D \) so that the complexity is minimal among all disc diagrams with the given properties.
Abusing notation slightly, we now temporarily regard $D$ as the rectified diagram from Remark 2.9. This means that certain square ladders are regarded as single (rectangular) 2–cells, cone-cells are regarded as single cells, and the remaining parts of the diagram are 2–cells (formed by ignoring non-boundary 1–cells in certain square subdiagrams) called shards. Angles are assigned to corners according to the split-angling discussed above.

**Applying the ladder theorem:** By Theorem 2.15 either $D$ is a ladder, so assertion (9) holds, and we are done, or $D$ has at least 3 “features of positive curvature” – spurs, generalized corners, or shells – along $\partial_p D$. We assume the latter.

**Applying the Greendlinger lemma:** By Theorem 2.16 $D$ has at least 3 features of positive curvature along the boundary, each of which is a shell, a spur, or a generalized corner.

We may assume that all generalized corners along $\partial_p D$ are actually squares with corners on $\partial_p D$. Indeed, let $s$ be a square in $D$ with a generalized corner on $\partial_p D$, so the dual curves $K_1, K_2$ intersecting $s$ end at consecutive 1–cubes $e_1, e_2$ on $\partial_p D$. By a sequence of hexagon moves (shuffling, in the sense of [Wis]), we modify $D$ – without changing $\partial_p D$ or increasing complexity – so that $s$ lies along the boundary, i.e. $e_1, e_2$ are consecutive 1–cubes of $s$.

**No positive curvature along geodesics (after square homotopy):** Let $s$ be a square of $D$ so that $\partial_p s$ and $\partial_p D$ have a common subpath $e_1 e_2$. For $i \in \{1, 2\}$, let $e'_i$ be the 1–cube of $s$ opposite $e_i$. If $e_1 e_2$ is a subpath of one of the three constituent geodesics of $\partial_p D$ (say, $\alpha'$), then we can modify $\alpha'$ in its square-homotopy class by replacing $e_1 e_2$ by $e'_2 e'_1$, resulting in a new diagram with the same number of cones and fewer squares. This contradicts our minimality assumption. Hence any square $s$ with a corner on the boundary lies at the transition from $\alpha$ to $\beta$, or $\beta$ to $\gamma$, or $\gamma$ to $\alpha$.

Now suppose that $C$ is a positively-curved shell in $D$ whose outer path $P$ is a subpath of one of the named geodesics, say $\alpha'$, and whose inner path we denote $S$. Now, $\kappa(C) = 2\pi - \sum_\varepsilon \vartheta(c)$, where $\varepsilon$ varies over the corners of $C$. The definition of the split-angling (see [Wis, Section 3]) says that $\vartheta(c) = 0$ at each corner $c$ formed by a pair of 1–cells on $\partial_p C$ both lying on $\partial_p D$. Since $\alpha'$ is a geodesic, it cannot be the case that $|S| = 0$, since otherwise we could replace $P$ by $S$ to shorten $\alpha'$. Thus, $C$ has two corners, $c_1, c_2$, at the 0–cells where $P, S$ meet. Each of these has angle $\pi/2$ by definition of the split-angling (see Figure 64 of [Wis]) and thus $\vartheta(c_1) = \vartheta(c_2) = \pi/2$. Thus, $\kappa(C) = \pi - \sum_\varepsilon \vartheta(c)$, where $\varepsilon$ varies over the corners at non-endpoint vertices of $S$. Hence, if $\sum_\varepsilon \vartheta(c) > 0$, the short inner paths condition, Theorem 2.17 shows that replacing $P$ by $S$ yields a strictly shorter path joining the endpoints of $\alpha'$, a contradiction. On the other hand, if $\sum_\varepsilon \vartheta(c) = 0$, then Lemma 3.68 of [Wis] implies that $S$ can be written as the concatenation of at most 7 pieces. The small cancellation assumption then implies that $|S| < |P|$, contradicting that $\alpha$ is a geodesic. Hence any such shell $C$ has outer path $P = AB$, where $A$ is a nontrivial terminal subpath of $\alpha'$, $\beta'$, or $\gamma'$ and $B$ is a nontrivial initial subpath of $\beta', \gamma'$, or $\alpha'$.

Finally, since $\alpha', \beta', \gamma'$ are geodesic, a spur of the form $ee^{-1}$ cannot occur along any of $\alpha', \beta', \gamma'$ so the only spurs consist of overlaps between $\alpha', \beta'$ or $\beta', \gamma'$ or $\gamma', \alpha'$.

Hence, $D$ has exactly three features of positive curvature along the boundary, which are subdiagrams $C_1, C_2, C_3$. For each $i$, $\partial_p C_i = \partial I$, where $O$ is a subpath of $\partial_p D$ and $I$ is an internal path, and $O$ has at least one 1-cube on each of two distinct subpaths $\alpha', \beta', \gamma'$ of the boundary.

**No internal cone-cells:** Let $C$ be a cone-cell of $D$. Recall that $C$ is internal if its boundary path intersects $\partial_p D$ in a set containing no 1–cube. We claim that, if $C$ is internal, then $C$ contributes less than $-4\pi$ to the total curvature. We argue almost exactly as in the proof of [Wis, Theorem 3.29]. Since our presentation satisfies $\lambda'(\frac{1}{2})$, any decomposition of $\partial_p C$ as a concatenation of pieces has more than 72 pieces. Grouping these into triples gives $k > 24$ groups. Each group has total defect at least $\pi/4$ (exactly as in the proof of [Wis, Theorem 3.29]), so the curvature contribution from $C$ is $2\pi - k\pi/4$. If $2\pi - k\pi/4 \geq -4\pi$, we obtain $k \leq 24$, a contradiction. Hence the curvature at $C$ is less than $-4\pi$. 
Suppose that there are \( n \geq 0 \) internal cone-cells. Let \( v \) be a 0–cube of \( D \). Then the curvature contribution from \( v \) is:

(1) at most 0 if \( v \) is internal or not contained in a 2–cell and not a spur;
(2) exactly \( \pi \) if \( v \) is a spur;
(3) exactly \( \frac{\pi}{2} \) if \( v \) is the corner of a square along \( \partial_p D \).

Because we are in the rectified diagram, natural candidates for internal features of positive curvature, i.e. 0–cubes with three incident cyclically-arranged squares – have been relegated to the insides of shards, and do not actually contribute any positive curvature in the split-angling.

Let \( f \) be a 2–cell of \( D \) (a cone-cell, rectangle, or a shard of the corresponding rectified diagram [Wis]). Then the curvature contribution is:

(1) at most 0 if \( f \) is a rectangle or shard;
(2) less than \(-4\pi\) if \( f \) is an internal cone-cell;
(3) at most \(2\pi\) if \( f \) is a shell.

Hence, our three features of positive curvature contribute a total of at most \( 6\pi \) of curvature, while the sum of the remaining curvatures is \(< -4n\pi\). This contradicts Theorem [2.11] unless \( n = 0 \). Hence there are no internal cone-cells.

No shortly-external cone-cells: A cone-cell \( C \) in \( D \) is shortly external if its boundary path has the form \( QI \), where \( I \) is internal and \( Q \) is a subpath of \( \alpha', \beta', \) or \( \gamma' \). Note that \(|Q| \leq |I|\) since \( \alpha', \beta', \gamma' \) are geodesics. Hence, by \( C'(\frac{1}{11}) \), the path \( I \) contains more than \( 72 = \cdot 24 \) transitions between pieces and thus the total angle-defect along \( I \) is more than \( 6\pi \), so that the curvature contribution from \( C \) is less than \(-4\pi\), so, as above, the Gauss-Bonnet theorem ensures that there are no shortly-external cells in \( D \).

At this point, we have completed the curvature computations in the proof, and now regard \( D \) as an ordinary (not rectified) diagram. This amounts to filling in the shards and rectangles with their constituent squares as in the original diagram.

Analysis of the cone-cells: Let \( C \) be a cone-cell in \( D \). Suppose that for some \( \delta \in \{\alpha', \beta', \gamma'\} \), there is a subpath \( \delta' = PQR \) of \( \delta \), where:

- the paths \( P, R \) are subpaths of the boundary path of \( C \),
- the terminal vertex of \( P \) and initial vertex of \( R \) subtend a subpath \( Q' \) of \( \partial_p C \), such that
- the path \( QQ' \) bounds a subdiagram \( E \) of \( D \) between \( C \) and \( \partial_p D \).

If \( E \) contains no cone-cell, then \( E \) is a square diagram between the relator \( Y_i \) to which \( C \) maps and the geodesic \( Q \), so by local convexity of \( Y_i \) in \( X \), we have that \( E \rightarrow X \) factors through \( Y_i \rightarrow X \). Hence \( E \) could have been absorbed into the cone-cell \( C \), whence minimality of the complexity of \( D \) ensures that \( E \) is trivial, i.e. \( Q = (Q')^{-1} \). Moreover, we may assume that \( Q, Q' \) have no common 1–cell, by considering a minimal example.

Thus, assume \( C \) is innermost, so that any cone-cell \( C_0 \) in \( E \) embeds and has connected intersection with \( Q \), and assume that \( E \) contains such a cone-cell \( C_0 \). Then \( C_0 \) is not internal in \( E \), for then it would be internal in \( D \). Moreover, \( C_0 \) cannot be a shell with outer path on \( Q \), because then it would either be an already considered illegal feature of positive curvature or a shortly-external cone-cell in \( D \). If \( C_0 \) is a shell, it must therefore have outer path (within \( E \)) of the form \( TU \), where \( T \) is a terminal subpath of \( Q \) and \( U \) an initial subpath of \( Q' \). But then \( C_0 \) is shortly-external in \( D \), which is impossible. Hence any cone-cell \( C_0 \) in \( E \) has boundary path of the form \( TI_1U_1I_2U_2 \cdots I_kU_kI_{k+1} \), where:

- each \( I_i \) is internal to \( E \),
- each \( U_i \) is a subpath of \( Q' \),
- and \( T \) is a subpath of \( Q \).

But then \( C_0 \) is either a positively curved shell in \( D \) or shortly-external in \( D \), neither of which is possible. Thus \( E \) is a square diagram, which was dealt with above.
We conclude that for each cone-cell $C$ of $D$, the path $\partial_p C$ has connected intersection with each of $\alpha', \beta', \gamma'$. Moreover, $\partial_p C$ intersects at least two of the paths $\alpha', \beta', \gamma'$.

**Dividing into cases:** We are now in the following situation: $D$ is not a ladder, and has precisely 3 features of positive curvature along its boundary path, which are subdiagrams $C_1, C_2, C_3$. The subdiagram $C_1$ has boundary path $ABI$, where $A$ is a nontrivial terminal subpath of $\alpha'$, $B$ is a nontrivial initial subpath of $\beta'$, and $I$ is a (possibly trivial) path. Moreover, either $C_1$ is a single cone-cell (a shell) or $C_1$ is a spur, $|I| = 0$, and $A$ is an edge and $B = A^{-1}$, or $C_1$ is a square, $|A|, |B| \geq 1$, and $|I| \leq 2$. The same description holds for $C_2$ (with $\beta', \gamma'$ replacing $\alpha', \beta'$) and $C_3$ (with $\gamma', \alpha'$ replacing $\alpha', \beta'$).

Moreover, every cone-cell $C$ of $D$ not in $\{C_1, C_2, C_3\}$ has connected intersection with each of $\alpha', \beta', \gamma'$ and intersects at least 2 of these paths. We call $C$ a median-cell if $C$ intersects all three of these paths, and a tail-cell otherwise. We emphasise that if $C$ is a median cell, then it has nonempty, connected intersection with each of $\alpha', \beta', \gamma'$. Hence, if $C$ is a median-cell, then $C$ separates $D$ into three complementary components, each disjoint from one of the paths $\alpha', \beta', \gamma'$. Hence $C$ is the unique cone-cell of $D$ intersecting each of $\alpha', \beta', \gamma'$. (We note that there may be other disc diagrams with the same boundary path, containing a different median-cell.)

We now divide into cases. First, if $D$ contains a median-cell, then we are in one of the generic cases, i.e. we will show that one of $\{1, \bar{5}, 3, 9\}$ holds, according to how many of $\{C_1, C_2, C_3\}$ are spurs or shells. Otherwise, we will show that one of $\{8, \bar{6}, \bar{4}, 2\}$ holds. This will complete the proof.

**The generic cases:** Suppose that $D$ has a (unique) median-cell $M$ and let $i \in \{1, 2, 3\}$. Let $\delta, \delta' \in \{\alpha', \beta', \gamma'\}$ be the parts of the boundary path of $D$ that intersect $C_i$. Let $\partial_p M = AP_1 BP_2 CP_3$, where $A, B, C$ are respectively subpaths of $\alpha', \beta', \gamma'$ and $P_1, P_2, P_3$ are internal paths. Write $\alpha' = \alpha'A\alpha', \beta' = \beta'B\beta'$, $\gamma' = \gamma'C\gamma'$. Consider the subdiagram $L_1$ bounded by $A\alpha'\beta'BP_2CP_3$. The ladder theorem (Theorem 2.15) and our above analysis of the possible features of positive curvature in (the rectification of) $D$ shows that $L_1$ is a ladder. The ladders $L_2, L_3$ are constructed analogously.

**The tripod cases:** Suppose there is no median. Then we have a subdiagram $T$ of $D$ with boundary path $AP_1 BP_2 CP_3$, where $A$ is a subpath of $\alpha'$, $B$ a subpath of $\beta'$, $C$ a subpath of $\gamma'$, and $P_1, P_2, P_3$ internal subpaths that lie on innermost cone-cells in $D$ or, if they do not exist, spurs or exposed squares in $\{C_1, C_2, C_3\}$. By construction, $T$ is a possibly degenerate square diagram, and by convexity of relators and minimality, for each path $Q \in \{A, B, C, P_1, P_2, P_3\}$, no two dual curves in $T$ emanating from $Q$ can cross. Moreover, no dual curve travels from $Q$ to $Q$ or to the next named subpath, for otherwise we could reduce complexity. Some possibilities are shown in Figure 4.

![Figure 4](image.png)

**Figure 4.** Some possibilities for the internal square subdiagram in the tripod cases.

For convenience, we lift $T$ to a diagram $T \rightarrow \tilde{X}$ (the CAT(0) cube complex $\tilde{X}$, not the generalized Cayley graph). Here, analysis of the dual curves shows that $T$ decomposes as required; the analysis is indicated in Figure 5. First, consider dual curves in $T$ traveling from $A$ to $B$, $B$ to $C$, or $C$ to $A$. Taking the union of all carriers of such dual curves yields rectangles.
attached to $P_1, P_2, P_3$. Now consider the subdiagram that remains. It is a hexagon bounded by subpaths of $A, B, C$ and parts of carriers of dual curves. Dual curves in the subdiagram must travel from a subpath of $A, B$, or $C$ to the antipodal dual-curve carrier. Dual curves emanating from the same “syllable” of the boundary path do not cross, and we conclude, as at right in Figure 5, that this subdiagram is a “corner of a subdivided cube”. It is now easy to deduce the padded ladder decomposition of $D$. (Various parts of the picture may be degenerate, as suggested in Figure 4.)

□

3. Detecting WPD elements using the classification of triangles

In this section, we adopt the following assumptions and conventions:

1. $\langle X \mid \{Y_i\}_{i \in I} \rangle$ is a cubical presentation satisfying the $C'(\frac{1}{144})$ condition, $X^*$ is the presentation complex, and $\tilde{X}^*$ is the universal cover. We assume that $X$ is locally finite, but we do not assume $X$ is uniformly locally finite.

2. Denote by $d$ the graph metric on $\text{Cay}(X^*)^{(1)}$, and by $d_{\tilde{X}}$ the graph metric on $\tilde{X}^{(1)}$.

3. Let there be a $\delta$–hyperbolic graph $H$ and a coarsely surjective map $\Pi : \text{Cay}(X^*) \to 2^H$ so that:

   a) There exists $\epsilon \geq 0$ so that $\text{diam}(\Pi(x)) \leq \epsilon$ for all $x \in \text{Cay}(X^*)$.

   b) We have $d_H(\Pi(x), \Pi(y)) \leq d(x, y)$ whenever $x, y \in \text{Cay}(X^*)^{(0)}$.

   c) If $Y_i \subset \text{Cay}(X^*)$ is any relator, then $\text{diam}(\Pi(Y_i)) \leq \epsilon$.

   d) The group $\pi_1 X^*$ acts by isometries on $H$ in such a way that $\Pi$ is $\pi_1 X^*$–equivariant.

   e) Let $H$ be a hyperplane in $\text{Cay}(X^*)$. Then $\text{diam}(\Pi(N(H))) \leq \epsilon$.

Under these conditions, we will prove a lemma — Lemma 3.5 — showing that $\pi_1 X^*$ contains a WPD isometry of $H$ provided it contains a loxodromic one. Later, we choose specific $H$ and $\Pi$.

The proof of Lemma 3.5 will require us to show that certain paths produced by an application of Theorem 2.18 fellow-travel. We need some preliminary lemmas:

Lemma 3.1 (Ladders are thin between cone-cells). Let $L \to \tilde{X}$ be a padded ladder with boundary path $\alpha^{-1}\beta\gamma$, where $\alpha, \beta : [0, \ell] \to \text{Cay}(X^*)$ are geodesics with $\alpha(0) = \beta(0)$ and $\gamma$ is a piece.

Let $\Delta$ be the maximum length of a subpath of $\alpha$ or $\beta$ that lies on a single cone-cell of $L$.

Then there exists $\kappa_0 = \kappa_0(\Delta)$ so that for all $t \leq \ell$, $d(\alpha(t), \beta(t)) \leq \kappa_0$.

Proof. Write $\alpha = \alpha_0\eta_1\alpha_1 \cdots \eta_n\alpha_n$ and $\beta = \beta_0\eta'_1 \cdots \eta'_n\beta_n$, where each $\alpha_i, \beta_i$ lies on the top or bottom boundary path of one of the constituent pseudorectangles of $L$ and each $\eta_i, \eta'_i$ lies on the boundary path of a cone-cell or cut-vertex (and hence has length at most $\Delta$). Denote by $p_i$ the maximal piece in $L$ between the $i^{th}$ cone-cell or cut-vertex and the $i^{th}$ pseudorectangle, so that the $i^{th}$ pseudorectangle is bounded by $\alpha_i p_i^{-1}\beta_i p_i^{-1}$. See the left side of Figure 6.
Remark 3.2 (Rank one elements). As usual (see e.g. [CS11]), $\tilde{\gamma} \in \pi_1 X$ is rank one if it is hyperbolic on $\tilde{X}$ and none of its axes lies in an isometrically embedded Euclidean half-plane. Let $\alpha$ be a combinatorial geodesic axis in $\tilde{X}$ for $g$. Let $W(\alpha)$ be the set of hyperplanes intersecting $\alpha$. Let $C\alpha$ be the graph with vertex set $W(\alpha)$, with vertices $H, V$ adjacent if the corresponding hyperplanes have intersecting carriers.

Let $\tilde{B}$ be the cubical convex hull of $\alpha$, which is a CAT(0) cube complex whose hyperplanes are exactly those in $W(\alpha)$. The graph $C\alpha$ is exactly the contact graph of $\tilde{B}$, i.e. the intersection graph of its set of hyperplane carriers. Considering the action of $\langle g \rangle$ on $\tilde{B}$, we see that there are finitely many $\langle g \rangle$ orbits of hyperplanes in $\tilde{B}$, and each has uniformly bounded coarse intersection with $\alpha$. Hence, by [Hag13] Theorem 2.4, Proposition 2.5, $\langle g \rangle$ has unbounded orbits in $C\alpha$. Hence, since $\langle g \rangle$ acts on $C\alpha$ with finitely many orbits of vertices (each hyperplane of $\tilde{B}$ is dual to one of $\langle g \rangle$–finitely many 1–cubes in $\alpha$), there exists $N$ such that if $H, V$ are hyperplanes intersecting $\alpha$ in 1–cubes lying at distance more than $N$, then $H$ and $V$ cannot cross.

Lemma 3.3. Let $\tilde{\gamma} \in \pi_1 X$ act hyperbolically on $\tilde{X}$, and suppose that $\tilde{\gamma}$ is rank one. Then for each $\tilde{x} \in \tilde{X}^{(0)}$, there exists $\kappa_1$ so that the following holds: if $n \geq 0$ and $P, Q : [0, d_n] \to \tilde{X}$ are combinatorial geodesics joining $\tilde{x}, g^n \tilde{x}$, then $d_{\tilde{X}}(P(t), Q(t)) \leq \kappa_1$ for $0 \leq t \leq d$.

Proof. Let $\alpha \to \tilde{X}$ be a combinatorial geodesic axis for $\tilde{\gamma}$ and let $\tilde{a} \in \alpha$ be a 0–cube. Given $n \geq 0$, let $P, Q : [0, d_n] \to \tilde{X}^{(1)}$ (where $d_n = d_{\tilde{X}}(\tilde{x}, g^n \tilde{x})$) be combinatorial geodesics joining $\tilde{x}, g^n \tilde{x}$ and let $D \to \tilde{X}$ be a minimal-area disc diagram with boundary path $PQ^{-1}$. Note that $D \to \tilde{X}$ is actually an isometric embedding on the 1–skeleton. Indeed, every dual curve in $D$ travels from $P$ to $Q$ since $P, Q$ are geodesics. Hence each dual curve maps to a distinct hyperplane, so that for any vertices $v, v' \in D$, the number of dual curves of $D$ separating $v, v'$ is equal to the number of hyperplanes in $\tilde{X}$ separating their images.

Fix $t \in \{0, 1, \ldots, d_n\}$. The above discussion shows that $d_{\tilde{X}}(P(t), Q(t))$ is bounded by the number of dual curves in $D$ that travel from $P([0, t])$ to $Q([t, d_n])$, plus the number of dual curves from $Q([0, t])$ to $P([t, d_n])$. Each dual curve of the former type crosses each dual curve of the latter type.
Let $\mathcal{H}$ be the set of dual curves of the former type, and let $\mathcal{X}$ be the set of dual curves of the latter type. Let $N_1 = |\mathcal{H}|$ and $N_2 = |\mathcal{X}|$. Note that since all but at most $2d_x(\tilde{a}, \tilde{x})$ hyperplanes that cross $P$ cross $\alpha$, at least $N_1 + N_2 - 2d_x(\tilde{a}, \tilde{x})$ hyperplanes in $\mathcal{X} \cup \mathcal{Y}$ cross $\alpha$.

Let $\mathcal{Y} \subseteq \mathcal{X} \cup \mathcal{Y} \subseteq \mathcal{Y}$ be the subsets consisting of hyperplanes/dual curves that cross $\alpha$. Then by Remark 3.2, we have $\min\{N_1 - 2d_x(\tilde{a}, \tilde{x}), N_2 - 2d_x(\tilde{a}, \tilde{x})\} \leq N$, where $N$ depends only on $\tilde{a}$. But $|\mathcal{X}| = |\mathcal{Y}|$, since for each dual curve crossing $Q(t, 0)$ and $P(t, d_0)$, there must be a dual curve crossing $P(0, t)$ and $Q(t, d_0)$. Hence $\mathcal{Y} \subseteq \mathcal{X}$ has bounded cardinality, and we conclude that $d_x(\tilde{P}(t), Q(t))$ is bounded by some $\kappa_i$ depending only on $\tilde{a}$ and $\tilde{x}$.

**Definition 3.4** $(\Delta$–fast). Let $g \in \pi_1 X^*$ act on $\mathcal{H}$ as a loxodromic isometry and let $\Delta \geq 0$. Then $g$ is $\Delta$–fast if the following holds. Let $\tilde{A}$ be a combinatorial geodesic axis in $\tilde{X}$ for some $\tilde{a} \in \pi_1 X$ mapping to $g$. Let $A$ be the image of $\tilde{A}$ in $\text{Cay}(X^*)$ and let $x \in A$ be a 0–cube. Let $R \geq 0$ and let $\alpha$ be a geodesic in $\text{Cay}(X^*)$ from $x$ to $g^R x$. Then any subpath of $\alpha$ lying in a hyperplane carrier or relator has length at most $\Delta$.

We are now ready for the main lemma:

**Lemma 3.5** (Loxodromic implies WPD). Suppose $g \in \pi_1 X^*$ acts loxodromically on $\mathcal{H}$ and that $g$ is $\Delta$–fast for some $\Delta$. Then for all $\epsilon > 0$, $\tilde{x} \in \mathcal{H}$, there exists $R \in \mathbb{N}$ so that

$$|\{h \in G \mid d_\mathcal{H}(h\tilde{x}, \tilde{x}) \leq \epsilon, d_\mathcal{H}(hg^R \tilde{x}, g^R \tilde{x}) \leq \epsilon\}| < \infty,$$

i.e. $g$ is a WPD element.

**Proof.** Fix $\epsilon > 0$ and let $\tilde{x} \in \mathcal{H}$; since $\Pi$ is coarsely surjective, we can assume $\tilde{x} \in \Pi(x)$ for some vertex $x$ of $\text{Cay}(X^*)$. Let $\tau \geq 1$ be the translation length of $g$ on the graph $\mathcal{H}$. It suffices to prove the claim for a specific $x$, so we can assume that $x$ lies on the image in $\text{Cay}(X^*)$ of the combinatorial geodesic axis of some lift of $g$ to $\pi_1 X$. Hence, since $g$ is $\Delta$–fast, for any $R$ and any geodesic $\alpha$ from $x$ to $g^R x$, any subpath of $\alpha$ lying in a hyperplane carrier or a relator has length at most $\Delta$.

Fix an integer $R$ satisfying $R \geq 10^9(\epsilon + \delta + \epsilon + \Delta)/\tau$.

**Rank one lift:** Let $\tilde{g} \in \pi_1 X$ be any lift of $g$ and let $\tilde{L}$ be a combinatorial geodesic axis for $\tilde{g}$. If $\tilde{g}$ is not rank one, then the image $L$ of $\tilde{L}$ in $\mathcal{H}$ has diameter at most $3\epsilon$, by property [3e] of the map $\Pi$ together with [Hag13] Proposition 5.1. This contradicts that $g$ is loxodromic. Hence $\tilde{g}$ is rank one.

**Setup for verifying WPD condition:** Let $y = g^R x$. Fix a combinatorial geodesic $\alpha$ of $\tilde{X}^*$ from $x$ to $g^R x$. Suppose that $h \in \pi_1 X^*$ satisfies $d_\mathcal{H}(\Pi(x), \Pi(hx)) < \epsilon$ and $d_\mathcal{H}(\Pi(y), \Pi(hy)) < \epsilon$.

**The triangle:** Let $\beta$ be a $\text{Cay}(X^*)$–geodesic from $x$ to $hx$, let $\eta$ be a geodesic from $hy$ to $y$, and let $\gamma$ be a geodesic from $hx$ to $y$, so that we have geodesic triangles $\alpha \beta \gamma$ and $\eta(\alpha)\gamma$ with common side $\gamma$.

**Applying the classification of triangles:** By Theorem 2.18 we have a minimal disc diagram $D = D_1 \cup, D_2 \rightarrow \tilde{X}^*$, with boundary path $\alpha \beta(\alpha h \alpha)^{-1} \eta$, with the following structure:

- The diagram $D_1$ has boundary path $\alpha \beta \gamma$ and $D_2$ has boundary path $\eta(\alpha h \alpha)^{-1} \gamma$.
- For $i \in \{1, 2\}$, the diagram $D_i$ decomposes as $B_1^i \cup B_2^i \cup B_3^i \cup S_i$, where $S_i$ is a standard diagram in the sense of Theorem 2.18 and each $B_j^i$ is a bigon diagram in $X$ (i.e., no cone cells). The boundary path of $S_i$ is a geodesic triangle $A_i B_i C_i$, where $A_i \alpha \alpha^{-1}$, $B_i \beta^{-1}$, $C_i \gamma^{-1}$ are the boundary paths of $B_1^i, B_2^i, B_3^i$ respectively, and $A_2(\alpha h \alpha), B_2(\eta \alpha)$, and $C_2 \gamma$ are the boundary paths of $B_1^i, B_2^i, B_3^i$ respectively.
- The diagram $S_i$ contains a constituent padded ladder $L_i$ whose image in $\tilde{X}^*$ projects under $\Pi$ to a set of diameter at least $R - 2(\epsilon + \delta + \epsilon)$, along with two ladders projecting to sets of diameter $\leq 10(\epsilon + \delta + \epsilon)$. Specifically, the padded ladder $L_1$ is the subdiagram of $D_1$ obtained as follows: either $D_1$ is a ladder, in which case $L_1 = D_1$, or there is a
 cone-cell or tripod with 3 complementary components, all of whose closures are padded ladders; \( L_1 \) is the padded ladder among these that contains \( y \). The padded ladder \( L_2 \) is defined analogously.

The above notation is summarized in Figure 7.

**Figure 7.** The two triangles in the proof of Lemma 3.5. The padded ladder \( L_1 \) is the subdiagram of \( S_1 \) between the red subdiagram (cone-cell or union of 3 square grids) intersecting \( A_1, B_1, C_1 \) and the point \( y \). The ladder \( L_2 \) is the subdiagram of \( S_2 \) between the analogous red subdiagram (on the right) and \( h.x \).

**Bounds on cones and pseudorectangles:** Let \( A'_1 \) be the part of \( A_1 \) on the boundary path of the ladder \( L_1 \). Then there is a decomposition \( A'_1 = \rho_0 \sigma_1 \rho_1 \cdots \sigma_s \rho_s \), where each \( \rho_i \) lies on a pseudorectangle and each \( \sigma_i \) lies on the boundary path of a cone-cell. Our choice of \( \Delta \) ensures that \( |\sigma_i|, |\rho_i| \leq \Delta \), with the following exception: we may have \( |\rho_i| > \Delta \) if the pseudorectangle carrying \( \rho_i \) is horizontally degenerate.

Similarly, the maximal subpath \( A'_2 \) of \( A_2 \) lying on the ladder \( L_2 \) decomposes as \( \rho_0 \sigma_1 \cdots \sigma_t \rho_t \), where each \( \rho_i \) lies on a pseudorectangle, each \( \sigma_i \) lies on a cone-cell, and each \( |\sigma_i|, |\rho_i| \leq \Delta \), except that we may have \( |\rho_i| > \Delta \) if \( \rho_i \) is carried on a horizontally degenerate pseudorectangle. See Figure 8.

**The paths** \( C'_1, C'_2 \): For each \( i \), let \( R_i \) be the pseudorectangle carrying \( \rho_i \) and let \( \rho'_i \) be the part of the boundary path of \( R_i \) parallel to (i.e. crossing the same dual curves as) \( \rho_i \). Let \( K_i \) be the cone-cell carrying \( \sigma_i \) and let \( \sigma'_i \) be the part of \( \partial K_i \) between \( \rho'_{i-1} \) and \( \rho'_i \), as shown in Figure 8. Let \( C'_1 = \rho'_0 \sigma'_1 \cdots \sigma'_t \rho'_s \) be the part of \( C_1 \) formed by concatenating these paths. Define \( \varsigma_i, \eta_i \), and the resulting subpath \( C'_2 \) of \( C_2 \) analogously.

**Fellow-traveling of** \( \alpha, A'_1 \): There exist \( s, \kappa_1 \geq 0 \), depending only on \( g, x \), such that \( d(\alpha(t + s), A'_1(t)) \leq \kappa_1 \) for all \( t \). To see this, recall that \( \hat{g} \) is rank one. Now, \( \alpha^{-1} A_1 \) lifts to a geodesic bigon in \( \tilde{X} \), and Lemma 3.3 shows that \( \alpha \) and \( A_1 \) lie at Hausdorff distance \( \kappa'_1 \) bounded in terms of \( g \) and \( x \). Then choose \( s \) so that \( d(\alpha(s), A'_1(0)) \leq \kappa'_1 \). A computation supplies \( \kappa_1 \) in terms of \( \kappa'_1 \).

Parametrising \( h \alpha \) so that \( (h \alpha)(t) = h \cdot \alpha(t) \), and parametrising \( A_2 \) so that it starts at \( h.x \), we have \( d(h \alpha(t), A'_2(t)) \leq \kappa_1 \), by the same argument. (In particular, we choose a lift \( \tilde{h}g\tilde{h}^{-1} \) of \( hgh^{-1} \) and note that the convex hull of its axis is a translate of that of \( \tilde{g} \), which is why applying Lemma 3.3 for \( hgh^{-1} \) yields the same constant that it did for \( g \) above.)

**Fellow-traveling of** \( C'_1, C'_2 \): Next, consider the subdiagram \( E = B^1 \cup \gamma B^2 \) of \( D \) bounded by \( C_1 \) and \( C_2 \). Since \( C_1, C_2 \) are geodesics, and \( E \) is a square diagram, every dual curve starting on \( C_1 \) ends on \( C_2 \) and every dual curve starting on \( C_2 \) ends on \( C_1 \).
Let \( K, K' \) be dual curves that emanate from \( C'_1 \) and cross each other. Let \( C''_1 \) be the subpath of \( C'_1 \) between and including the 1–cubes dual to \( K \) and \( K' \). These 1–cubes \( e, f \) respectively lie on common cone-cells or pseudorectangles with points \( a_e, a_f \) on \( A'_1 \).

We can choose these so that \( d(e, a_e), d(f, a_f) \leq 2\Delta \).

Now, \( a_e, a_f \) respectively lie at distance at most \( \kappa_1 \) from points on \( b_e, b_f \in \alpha \).

Hence \( d(e, b_e) \leq 2\Delta + \kappa_1 \) and \( d(f, b_f) \leq 2\Delta + \kappa_1 \). Thus \( d(b_e, b_f) \geq |C''_1| - 4\Delta - 2\kappa_1 \).

On the other hand, \( d_H(\Pi(b_e), \Pi(b_f)) \leq 2\varepsilon + 4\Delta + 2\kappa_1 \), by property (3e) of \( \Pi \) and the fact that \( K, K' \) cross. (Indeed, there is a path in \( \mathcal{N}(K) \cup \mathcal{N}(K') \) from \( e \) to \( f \), so \( d_H(\Pi(e), \Pi(f)) \leq 2\varepsilon \), and \( \mathcal{N}(K), \mathcal{N}(K') \) map to hyperplane carriers in \( \text{Cay}(X^*) \), whose images in \( H \) have diameter at most \( \varepsilon \).

Let \( \eta_\alpha = \max \frac{d(p, q)}{d_H(\Pi(p), \Pi(q))} \), where \( p, q \) vary over vertices of \( \alpha \) with distinct images in \( H \), and let \( \eta \) be the maximal \( \eta_\alpha \) over the (finitely many) choices of \( \alpha \) with the given endpoints. Let \( \zeta_\alpha \) be the maximum of \( d(p, q) \) as \( p, q \) vary over vertices of \( \alpha \) with \( \Pi(p) = \Pi(q) \), and let \( \zeta \) be the maximum of the \( \zeta_\alpha \) over all choices of \( \alpha \). Note that \( \eta, \zeta \) depend on \( g, x \) and \( R \), but not \( h \).

So, \(|C''_1| \leq (2\varepsilon + 10\Delta + 2\kappa_1)\eta + \zeta + 10\Delta + 2\kappa_1 \). In other words, there exists \( N \) depending only on \( g, x, R \) such that any two dual curves that emanate from \( C'_1 \) at distance more than \( N \) cannot cross.

Parametrisate \( C_1, C_2 \) so that \( C_1(0) = C_2(0) = hx \). Our choice of \( R \) ensures that there exist \( t_0, t'_0 \), depending only on \( R, \varepsilon, \Delta, \kappa_1, \kappa_2 \), and \( \tau \), so that \( C_1(t), C_2(t) \) lie on \( C'_1, C'_2 \) respectively when \( 0 \leq t \leq t'_0 \).

Let \( t = \frac{t_0 - t'_0}{2} \) and let \( z = C_1(t) \). Let \( z' = C_2(t) \). Now, as in the proof of Lemma 3.3, \( d(z, z') \) is bounded by the number of dual curves \( K \) in the square diagram \( E \) that cross \( C_1 \) before \( z \) and \( C_2 \) after \( z' \), or vice versa. Our choice of \( R \) ensures that any such \( K \) cannot cross \( C_1 - C'_1 \) or \( C_2 - C'_2 \), since property (3e) would then provide a shortcut in \( H \) from \( \Pi(hx) \) to \( \Pi(gx) \). Thus, as in the proof of Lemma 3.3, \( d(z, z') \) is bounded in terms of \( N \), say by some \( \kappa_2 \).

**Fellow-traveling of \( A'_1, C'_1 \) and \( A'_2, C'_2 \):** By Lemma 3.1, there exists \( \kappa_3 \), depending on \( \Delta \) and the small-cancellation assumption, so that \( d(A'_1(t), C'_1(t)) \leq \kappa_3 \). The same is true for \( A'_2, C'_2 \).

**Conclusion:** Let \( z, z' \) be as above. Then \( z \) is uniformly close to \( \alpha(t) \) (the distance is bounded by \( \kappa_1 + \kappa_2 \), and the same is true for \( ha(t), z' \). Hence \( d(\alpha(t), ha(t)) \leq 2(\kappa_1 + \kappa_2) + \kappa_3 \), which does not depend on \( h \). Since \( \text{Cay}(X^*) \) is locally finite and \( \pi_1 X^* \) acts freely on \( \text{Cay}(X^*) \), the
action of \( \pi_1 X^* \) on \( \text{Cay}(X^*) \) is metrically proper and hence there are only finitely many such \( h \), as claimed. \( \square \)

4. The hyperbolic space \( \mathcal{H} \) and the projection \( \Pi : \text{Cay}(X^*) \to \mathcal{H} \)

Let \( \langle X \mid \{Y_i\} \rangle \) be a cubical \( C'(\frac{1}{14}) \) presentation and define a space \( \mathcal{H} \) as follows. First, let \( \mathcal{H}' \) be the 1–skeleton of \( \tilde{X}^* \). This consists of the 1–skeleton of \( \text{Cay}(X^*) \), together with a combinatorial cone on each lift of each \( Y_i \). We form \( \mathcal{H} \) from \( \mathcal{H}' \) by adding a combinatorial cone on the carrier of each hyperplane.

We also have a projection \( \Pi : \text{Cay}(X^*) \to \mathcal{H} \), defined as follows. On the 1–skeleton of \( \text{Cay}(X^*) \), we declare \( \Pi \) to be the inclusion. If \( c \) is a cube of \( \text{Cay}(X^*) \) with \( \dim(c) \geq 2 \), we send \( c \) arbitrarily to a point in the image of its 1–skeleton. However, we require this choice to be made \( \pi_1 X^* \)–equivariantly, so that \( \Pi \) is \( \pi_1 X^* \)–equivariant. Obviously \( \Pi \) is coarsely surjective and 1–Lipschitz on the 1–skeleton of \( \text{Cay}(X^*) \). By construction, \( \Pi \) sends each cone to a set of diameter \( \leq 2 \), while hyperplane carrier in \( \text{Cay}(X^*) \) are sent to subsets of \( \mathcal{H} \) with diameter at most 2. Hence, to see that \( \mathcal{H} \) and \( \Pi \) satisfy the conditions required in Section 3, we need only to prove that \( \mathcal{H} \) is hyperbolic.

**Lemma 4.1** (Square bigons have thin projection). Let \( \alpha, \beta \to \text{Cay}(X^*) \) be geodesics with common endpoints, and suppose that \( \alpha \beta \) bounds a disc diagram \( D \to \tilde{X}^* \) that does not contain any cone-cells. Then \( \Pi(\alpha), \Pi(\beta) \) lie at uniformly bounded Hausdorff distance in \( \mathcal{H} \).

**Proof.** Let \( e \) be a 1–cube of \( \alpha \) and let \( K \) be the dual curve in \( D \) emanating from \( e \) and mapping to a hyperplane \( H \) of \( \text{Cay}(X^*) \). Since \( \alpha \) is a geodesic, \( K \) terminates at a 1–cube \( f \) of \( \beta \), whence \( d_{\mathcal{H}}(\Pi(e), \Pi(f)) \leq 2 \). Hence \( \Pi(\alpha) \subseteq N_2(\Pi(\beta)) \) and the proof is complete by symmetry. \( \square \)

**Proposition 4.2.** The graph \( \mathcal{H} \) is hyperbolic.

**Proof.** It suffices to prove that the 0–skeleton of \( \text{Cay}(X^*) \), with the subspace metric inherited from \( \mathcal{H} \), is hyperbolic. First, suppose that \( \alpha \beta \gamma \) is a geodesic triangle in \( \text{Cay}(X^*) \). Then Lemma 4.1 combines with Theorem 2.18 and the fact that \( \Pi \) sends cones to uniformly bounded sets to show that each of \( \Pi(\alpha), \Pi(\beta), \Pi(\gamma) \) is contained in the \( \delta' \)–neighborhood in \( \mathcal{H} \) of the union of the other two, for some uniform \( \delta' \). The Guessing Geodesics Lemma (see e.g. [Ham07, Proposition 3.5]) now implies that \( \mathcal{H} \) is hyperbolic. \( \square \)

**Theorem 4.3.** Let \( \langle X \mid \{Y_i\}_{i \in \mathbb{Z}} \rangle \) be a \( C'(\frac{1}{14}) \) presentation with \( X \) locally finite, and let \( G = \pi_1 X^* \). Then any \( g \in G \) acting on the space \( \mathcal{H} \) constructed above as a fast loxodromic element acts on \( \mathcal{H} \) as a WPD element, whence either \( G \) is virtually cyclic or acylindrically hyperbolic.

**Proof.** The assertion that \( g \) is a WPD element follows from Lemma 3.5. Hyperbolicity of \( \mathcal{H} \) comes from Proposition 4.2. Applying [Osi16, Theorem 1.2, (AH3 \( \Rightarrow \) AH2)] completes the proof. \( \square \)

5. Proof of Theorem 3

5.1. Preservation of loxodromics. We now study the question of when \( \pi_1 X^* \) contains a loxodromic isometry of \( \mathcal{H} \), using knowledge of which elements of \( \pi_1 X \) act loxodromically on the contact graph \( C \tilde{X} \) of \( \tilde{X} \), which is the intersection graph of the set of hyperplane carriers in \( \tilde{X} \).

Let \( p : \tilde{X} \to \text{Cay}(X^*) \) be the universal covering map (regarding \( \text{Cay}(X^*) \) as the cover of \( X \) corresponding to the subgroup \( K = \langle \{\pi_1 Y_i\}_{i \in \mathbb{Z}} \rangle \) of \( \pi_1 X \)). Let \( \mathcal{H} \) be the graph obtained from \( \tilde{X}(1) \) by coning off the 1–skeleton of each hyperplane carrier.

Form a new graph \( \hat{\mathcal{H}} \) from \( \mathcal{H} \) by coning off every subgraph of \( \tilde{X}(1) \subset \hat{\mathcal{H}} \) which is the 1–skeleton of an elevation \( \tilde{Y}_i \to \tilde{X} \) of some \( Y_i \to X \). Observe that \( p \) induces a quotient map \( \hat{p} : \hat{\mathcal{H}} \to \mathcal{H} \), which restricts to \( p \) on \( \tilde{X}(1) \) and which sends the cone-point \( v_H \) over the hyperplane carrier.
Lemma 5.2. Let $\Gamma$ be the graph with a vertex for each hyperplane carrier in $X$, and a vertex for each elevation $Y_i \hookrightarrow X$ of each $Y_i \rightarrow X$, with adjacency corresponding to intersection. Then $\Gamma$ is $\pi_1 X$–equivariantly quasi-isometric to $H$ and $H$ is $\pi_1 X$–equivariantly quasi-isometric to $C \bar{X}$.

Proof. The map $f : \Gamma(0) \rightarrow \bar{H}$ that sends each vertex (corresponding to a hyperplane carrier or an elevation $Y_i$) to the corresponding cone-point. Since each point of $X$ lies in a hyperplane carrier, the map $f$ is quasi-surjective. If $v, w$ are vertices of $\Gamma$, corresponding to subcomplexes $\bar{C}_v, \bar{C}_w$, and $v, w$ are adjacent, then $\bar{C}_v \cap \bar{C}_w \neq \emptyset$, so $d_{\bar{H}}(f(v), f(w)) \leq 2$. Hence $f$ is coarsely Lipschitz.

By sending each cone-point $v$ in $\bar{H}$ to the vertex of $\Gamma$ corresponding to the subcomplex over which $v$ is the cone, we obtain a coarsely Lipschitz quasi-inverse for $f$, so $f$ is a quasi-isometry.

That $f$ is $\pi_1 X$–equivariant follows immediately from the definition of $f$. This shows that $\Gamma$ is $\pi_1 X$–equivariantly quasi-isometric to $H$. The other assertion is proved in a similar manner in [Hag14, Section 5].

Since our ultimate goal is to understand when $\pi_1 X^*$ contains a loxodromic isometry of $H$, and existing tools (mainly from [CS11, Hag13]) tell us when elements of $\pi_1 X$ act loxodromically on $C \bar{X}$, we need to relate these phenomena.

Lemma 5.3 (Loxodromics persist). Let $\tilde{g} \in \pi_1 X$ act loxodromically on $\bar{H}$. Then either $\tilde{g}$ acts loxodromically on $\bar{H}$ or $\tilde{g}$ stabilizes some elevation $\tilde{Y}_i \subseteq \tilde{X}$ of some $Y_i \rightarrow X$.

Proof. By Lemma 5.2 $\bar{H}$ is quasi-isometric to the intersection graph $\Gamma$ of the set of hyperplane carriers and various elevations $\bar{Y}_i$ in $\bar{X}$. This graph is connected. Hence it suffices to show that $\langle \tilde{g} \rangle$ acts loxodromically on $\Gamma$ provided that $\tilde{g}$ doesn’t stabilize any $\bar{Y}_i$.

Let $\tilde{A} \subseteq \tilde{X}$ be a combinatorial geodesic axis for $\tilde{g}$ (by replacing $\bar{X}$ by its first cubical subdivision, we may assume that such an axis exists [Hag]). Fix a 0–cube $\tilde{a} \in \tilde{A}$ and fix $n > 0$. Let $P$ be the subpath of $\tilde{A}$ joining $\tilde{a}$ to $g^n \tilde{a}$.

Let $\tilde{Q}$ be a geodesic of $\Gamma$ joining vertices corresponding to subcomplexes containing $\tilde{a}$ and $g^n \tilde{a}$. Let the vertex-sequence of $\tilde{Q}$ be $C_0, \ldots, C_N$, where each $C_i$ is either a hyperplane carrier in $\bar{X}$ or an elevation $\bar{Y}_i \subseteq \bar{X}$. Then we have a combinatorial path $Q = \alpha_0 \cdots \alpha_N$ joining $\tilde{a}$ to $g^n \tilde{a}$, where each $\alpha_i$ is a geodesic in $C_i$. The closed path $QP^{-1}$ bounds a disc diagram $D \rightarrow \bar{X}$.

Suppose that the choices of $\tilde{Q}$, $Q$, and $D$ have been made so that the area of $D$ is minimal among all possible such choices. Let $K$ be a dual curve in $D$ emanating from $P$. Then $K$ ends on $Q$, since $P$ is a geodesic of $\bar{X}$. If $K$ is a dual curve emanating from $Q$, then it emanates from some $\alpha_i$. Suppose that $K$ ends on $\alpha_j$. Then $j \neq i$ since $\alpha_i$ is a geodesic. If $|i - j| > 2$, then since the hyperplane to which $K$ maps has carrier $\Gamma$–adjacent to $C_i, C_j$, we have contradicted that $\tilde{Q}$ is a $\Gamma$–geodesic. If $j = i \pm 2$, then we can replace $C_{i \pm 1}$ with the carrier of the hyperplane.
to which $K$ maps, providing a new choice of $\tilde{Q}$ leading to a lower-area choice of $D$. Finally, if $j = i \pm 1$, then we can apply hexagon moves to show that $\alpha_i$ has a terminal segment coinciding with an initial segment of $\alpha_{i+1}$ (say); we can remove the resulting spur. We conclude that $D$ can be chosen so that all dual curves travel from $Q$ to $P$. In other words, $\tilde{Q}$ is a geodesic of $\tilde{X}$. (The preceding argument is almost exactly the proof of Proposition 3.1 of [BHS17].)

Since $\tilde{g}$ acts loxodromically on $\tilde{H}$, Lemma 5.2 shows that there is a constant $\kappa_0 \geq 1$ so that for all hyperplanes $H$ of $\tilde{X}$, there are at most $\kappa_0$ hyperplanes that intersect $\tilde{A}$ and also intersect $H$ (i.e. $\tilde{A}$ has uniformly bounded projection to each hyperplane). Hence, for each $i$ for which $C_i$ is a hyperplane carrier, at most $\kappa_0$ dual curves in $D$ travel from $P$ to $\alpha_i$.

We now claim that there exists $\kappa_1$ so that, for all $i \in I$ and all elevations $\tilde{Y}_i \subseteq \tilde{X}$ of $Y_i \to X$, there are at most $\kappa_1$ hyperplanes intersecting $\tilde{Y}_i$ and $\tilde{A}$. First, since $g$ is loxodromic on $\tilde{H}$, Lemma 5.2 and Theorem 2.3 of [Hag13] imply that there exists $\kappa'$ such that if $d_X(H \cap \tilde{A}, H' \cap \tilde{A}) > \kappa'$, for hyperplanes $H, H'$, then $H \cap H' = \emptyset$. It follows that the cubical convex hull $\tilde{B}$ of $\tilde{A}$ lies at finite Hausdorff distance from $\tilde{A}$. Suppose that no $\kappa_1$ with the desired property exists. Then for any $L > 0$, there exists $\tilde{Y}_i$ so that $diam(\tilde{g}_B(\tilde{Y}_i)) > L$, where $\tilde{g}_B : \tilde{X} \to B$ is the combinatorial closest-point projection, or gate map.

The gate map is discussed in [BHS17 Section 2.1]. For our purposes, we just need the following properties: if $B$ is a convex subcomplex of $\tilde{X}$, then for any 0–cube $b \in \tilde{X}$, the image $\tilde{g}_B(b)$ is the unique closest 0–cube of $\tilde{B}$ to $b$, and if $B'$ is some other subcomplex, then the set of hyperplanes crossing $\tilde{g}_B(B')$ is precisely the set of hyperplanes crossing $\tilde{B}$ and $\tilde{B}'$. In particular, if there is no $\kappa_1$ with the claimed property, then for any $L$, we can choose $\tilde{Y}_i$ so that at least $L$ hyperplanes cross both $\tilde{Y}_i$ and $\tilde{A}$; this gives the lower bound on $diam(\tilde{g}_B(\tilde{Y}_i))$.

Taking $L$ much larger than the translation length of $g$, we see that there exists $\tilde{Y}_i$ such that either $\tilde{g}Y_i = \tilde{Y}_i$ or the following holds: for some $r$ bounded below by a linear function of $L$, the geodesic $\tilde{A}$ has a subpath $\tilde{A}_r$ that joins $\tilde{g}\tilde{a}$ to $\tilde{g}^n\tilde{a}$ and is a piece between $\tilde{Y}_i$ and $\tilde{g}\tilde{Y}_i$. This is a contradiction for sufficiently large $L$ because of the bound on the lengths of geodesic pieces. Hence $\tilde{g} \in \text{Stab}(\tilde{Y}_i)$.

We have shown that if $\tilde{g}$ does not stabilize some elevation $\tilde{Y}_i$ of some $Y_i$, then $d_X(\tilde{a}, g^n\tilde{a}) \leq (\kappa_0 + \kappa_1)N \leq (\kappa_0 + \kappa_1)(\lambda d_{\tilde{X}}(\tilde{a}, g^n\tilde{a}) + \mu)$, where $\lambda, \mu$ are constants for the quasi-isometry $\Gamma \to \tilde{H}$. Since the left-hand side grows linearly in $n$, it follows that $\tilde{g}$ is loxodromic on $\tilde{H}$.

We now consider two properties of lifts of an element of $\pi_1 X^*$ to $\pi_1 X$. Embeddability of a lift guarantees that its image has infinite order. Asystolicity is stronger and more concrete.

**Definition 5.4 (Embeddable, asystolic).** Fix $g \in \pi_1 X^*$. Any $\tilde{g} \in \pi_1 X$ mapping to $g$ is a lift of $g$. Let $\tilde{A}$ be a combinatorial geodesic axis for $\tilde{g}$, which exists provided $\tilde{g} \neq 1$, since $\pi_1 X$ is torsion-free and isometries of $\tilde{X}$ are combinatorially semisimple [Hag].

Recall that $p : \tilde{X} \to Cay(X^*)$ denotes the universal covering map. Letting $A = p(\tilde{A})$, we say that $\tilde{g}$ is an embeddable lift of $g$ if $p$ restricts on $A$ to a cubical isomorphism $\tilde{A} \to A$, i.e. $A$ is an embedded combinatorial line in Cay($X^*$), for some choice of $\tilde{g}$–axis $\tilde{A}$. If $\tilde{g}$ is an embeddable lift, then $\tilde{g}$ has infinite order, and $\tilde{g}^n$ is an embeddable lift of $g^n$ for all $n > 0$.

Given $\lambda \in [0, 1)$, the lift $\tilde{g}$ is $\lambda$–asystolic if, for each axis $A$ of $\tilde{g}$, and each subpath $P$ of $\tilde{A}$ such that $P \subset \tilde{Y}_i$, where $\tilde{Y}_i$ is an elevation of a relator $Y_i$, we have $|P| < \lambda|Y_i|$. Note that if $\tilde{g}$ is an $\lambda$–asystolic lift of $g$, then $\tilde{g}^n$ is an $\lambda$–asystolic lift of $g^n$ for all $n > 0$.

**Lemma 5.5 (Embeddability from $\frac{35}{72}$–asystolicity).** Let $\tilde{g} \in \pi_1 X$ and let $\tilde{A}$ be a combinatorial geodesic axis for $\tilde{g}$. Let $g$ be the image of $\tilde{g}$ in $\pi_1 X^*$. Suppose that for any subpath $P$ of $\tilde{A}$ lying
in an elevation \( \tilde{Y}_i \) of a relator \( Y_i \), we have \( |\tilde{P}| < \frac{35}{72} \| Y_i \| \). Then \( \tilde{g} \) is an embeddable lift of \( g^n \), for all \( n \in \mathbb{Z} \). In particular, \( g \) has infinite order.

**Proof.** It suffices to prove the claim for \( n = 1 \).

Suppose that \( \tilde{g} \) is not an embeddable lift of \( g \). Then \( p : \tilde{A} \to A \) is not injective, so there exist distinct 0–cubes \( \tilde{y}, \tilde{y}' \in \tilde{A} \) such that \( p(\tilde{y}) = p(\tilde{y}') \). In other words, letting \( \tilde{P} \) be the subpath of \( \tilde{A} \) joining \( \tilde{y} \) to \( \tilde{y}' \), the path \( P = p \circ \tilde{P} \) in \( \text{Cay}(X^*) \) is a nontrivial closed path. Let \( D \to \tilde{X}^* \) be a minimal-complexity disc diagram with boundary path \( P \).

We claim that \( P \) has no spurs. Indeed, if \( P \) has a spur \( ee^{-1} \), then \( \tilde{P} \) contains a subpath \( \tilde{e}_1 \tilde{e}_2 \), where \( \tilde{e}_1, \tilde{e}_2 \) are distinct 1–cubes such that \( \tilde{e}_1 \cap \tilde{e}_2 \) is a 0–cube. By choosing \( \tilde{y}, \tilde{y}' \) as close as possible, we can assume that the endpoints of \( \tilde{e}_1 \tilde{e}_2 \) are \( \tilde{y}, \tilde{y}' \), and \( \tilde{e}_1, \tilde{e}_2 \) are lifts of the 1–cube \( e \). Hence \( \tilde{y} = h \tilde{y}' \) for some nontrivial \( h \in \ker(\pi_1 X \to \pi_1 X^*) \), so \( h \) fixes the 0–cube \( \tilde{e}_1 \cap \tilde{e}_2 \). This contradicts that \( \pi_1 X \) acts on \( \tilde{X} \) freely.

Hence Theorem 2.16 implies that \( D \) is one of the following:

- A single vertex. This is impossible since \( P \) is nontrivial.
- A single cone-cell. In this case, \( P \) is an essential path in a relator \( Y_i \), by minimality of the complexity. Hence \( |\tilde{P}| \gg |Y_i| \), contradicting asystolicity.
- A ladder, or a diagram with at least three features of positive curvature (shells or generalised corners).

In either of the latter two cases, there are at least two features of positive curvature, and neither is a spur. Suppose that \( C \) is a shell in \( D \) with boundary path \( OI \), with \( O \) the inner path and \( I \) the outer path. Let \( Y_i \) be the relator to which the path \( \partial_p C \) maps. By minimality of complexity, \( \partial_p C \) is an essential path in \( Y_i \), for otherwise we could replace \( C \) by a square diagram, reducing the complexity of \( D \). Thus \( |\partial_p C| \gg |Y_i| \). But by Theorem 2.17 we thus have \( |O| > |Y_i|/2 \). Now, \( O \) lifts to a subpath of \( \tilde{P} \) that lies in some elevation \( \tilde{Y}_i \) of \( Y_i \) and has length more than \( |Y_i|/2 \), contradicting asystolicity.

Hence every feature of positive curvature along \( \partial_p D \) is a generalised corner of a square. By shuffling (without changing the boundary path), we can assume that these are exposed squares, i.e. each generalised corner of a square along \( P \) is actually a length–2 subpath of the boundary path of a square.

Since there are at least two of these squares, at least one, denoted \( s \), satisfies the following: \( \partial_p s = Ief \), where \( ef \) is a subpath of \( \partial_p D \) and the vertex in which \( e, f \) intersect is not \( p(\tilde{y}) \).

Hence we can perform a square homotopy, removing \( s \) from \( D \), to obtain a new diagram \( D' \) in which \( ef \) is replaced by \( I \) in the boundary path. Note that \( |\partial_p D'| = |P| \), and \( p(\tilde{y}) \in \partial_p D' \).

Thus we can replace \( \tilde{A} \) by a \( \langle \tilde{g} \rangle \)–invariant geodesic \( \tilde{A}' \) as follows: lift \( ef \) to a path \( \tilde{e} \tilde{f} \) in \( \tilde{P} \), lift \( s \) to a square \( \tilde{s} \) meeting \( \tilde{A} \) in the subpath \( \tilde{e} \tilde{f} \), and homotop \( \tilde{A} \) across \( \tilde{s} \). Do the same at each \( \langle \tilde{g} \rangle \)–translate of \( \tilde{s} \). Let \( \tilde{P}' \) be the subpath of \( \tilde{A}' \) from \( \tilde{y} \) to \( \tilde{y}' \).

Continuing in this way, we eventually find that \( \tilde{A} \) is square-homotopic in \( \tilde{X} \) to a \( \langle \tilde{g} \rangle \)–invariant combinatorial geodesic \( \tilde{B} \) such that \( \tilde{B} \) contains a path \( \tilde{O} \) such that \( \tilde{O} \) lies in some \( \tilde{Y}_i \) and satisfies \( |\tilde{O}| > |Y_i|/2 \). Moreover, \( \tilde{y}, \tilde{y}' \in \tilde{B}, \) and \( \tilde{O} \) is a subpath of the subpath \( \tilde{Q} \) of \( \tilde{B} \) joining \( \tilde{y} \) to \( \tilde{y}' \).

Hence there exists a subpath \( \tilde{Q}_1 \) of \( \tilde{B} \) with the following properties:

- the path \( \tilde{Q}_1 \) contains a subpath \( \tilde{O}_1 \) that lies in \( \tilde{Y}_i \) and is maximal with that property;
- we have \( |\tilde{O}_1| > |Y_i|/2 \);
- either \( \tilde{O}_1 \) is unbounded, or \( \tilde{Q}_1 \) starts and ends on \( \tilde{A} \).

First suppose that \( \tilde{O}_1 \) is bounded and let \( \tilde{P}_1 \) be the subpath of \( \tilde{A} \) subtended by the endpoints of \( \tilde{Q}_1 \).

Consider the geodesic bigon \( \tilde{Q}_1 \tilde{P}_1^{-1} \) in \( \tilde{X} \). Let \( E \to \tilde{X} \) be a minimal area disc diagram with \( \partial_p E = \tilde{Q}_1 \tilde{P}_1^{-1} \). Moreover, since \( \tilde{Y}_i \) is convex, we make our choice allowing the geodesic \( \tilde{O}_1 \) to
vary, fixing the endpoints; any such geodesic lies in \( \overline{Y}_i \). In particular, if \( E \) is chosen to be of minimal area among all disc diagrams with the given boundary path (with \( \overline{O}_1 \) allowed to vary as above), then no two dual curves emanating from \( \overline{O}_1 \) can cross.

If \( \overline{Q}_1 = \overline{O}_1 \), then \( \overline{P}_1 \) lies in \( \overline{Y}_i \), and, since \( |\overline{P}_1| = |\overline{Q}_1| > \|Y_i\|/2 \), this contradicts our hypotheses.

Now write \( \overline{Q}_1 = U \overline{O}_1 V \), with at least one of \( U, V \) a nontrivial path. We now allow \( U, V \) to vary, fixing their endpoints, and assume that \( E \) had minimal area over all of these choices. Hence no two dual curves emanating from \( U \) can cross, and the same is true of \( V \).

Hence consider 1–cubes \( r, s \) immediately preceding and succeeding \( \overline{O}_1 \) in \( \overline{Q}_1 \). At least one of \( r \) or \( s \) exists. If the hyperplane \( H_r \) dual to \( r \) crosses \( \overline{Y}_i \), then convexity of \( \overline{Y}_i \) implies \( r \subset \overline{Y}_i \), contradicting maximality of \( \overline{O}_1 \). Hence, \( H_r \) does not cross \( \overline{Y}_i \).

Thus every dual curve emanating from \( U \) maps to a hyperplane disjoint from \( \overline{Y}_i \), and the same is true of dual curves emanating from \( V \). Now, the uniform \( C'(\frac{1}{144}) \) condition provides a uniform constant \( M \) such that all cone-pieces and wall-pieces have diameter at most \( M \). Hence \( K_r \) crosses at most \( M \) of the dual curves emanating from \( \overline{O}_1 \). Indeed, since all the dual curves in \( E \) map to distinct hyperplanes, if \( K_r \) crosses \( n \) of these dual curves, then \( \overline{Y}_i \) contains a piece of \( \gamma(H_r) \) of length \( n \).

We conclude that at most 2\( M \) of the dual curves \( K \) emanating from \( \overline{O}_1 \) have positive length. Hence \( \overline{O}_1, \overline{P}_1 \) have a common subpath of length greater than \( \|Y_i\|/2 - 2M \). Since \( M < \|Y_i\|/144 \), we conclude that \( \overline{P}_1 \), and hence \( \overline{A} \), has a subpath that lies in \( \overline{Y}_i \) and has length more than 35\( \|Y_i\|/72 \), a contradiction.

The remaining case is where \( \overline{O}_1 \) is unbounded. In other words, \( \overline{Y}_i \) contains a sub-ray of the axis \( B \) of \( \langle \hat{g} \rangle \). Since \( \overline{Y}_i \) is compact, this implies that some power of \( \hat{g} \) stabilises \( \overline{Y}_i \), so \( B \subset \overline{Y}_i \).

Let \( \overline{A}_n \) be the subpath of \( \overline{A} \) between \( \hat{y} \) and \( \hat{g}^n \hat{y} \), let \( U, U' \) be geodesics joining \( \hat{y}, \hat{g}^n \hat{y} \) to closest 0–cubes of \( \overline{Y}_i \), and let \( V \) be a geodesic of \( \overline{Y}_i \) joining the terminal points of \( U, U' \). Let \( F \to X \) be a minimal-area disc diagram bounded by the paths \( \overline{A}_n, U, U', V \). Then, by allowing \( U, U', V \) to vary, fixing their endpoints, and assuming that \( D \) is minimal over all such choices, we have that no two dual curves emanating from \( V \) can cross. Now, the number of dual curves intersecting \( U, U' \) is bounded independently of \( n \), since \( \overline{A} \) lies in a uniform neighbourhood of \( \overline{Y}_i \). Hence, when \( n \) is sufficiently large, we see that either some dual curve travelling from \( \overline{A}_n \) to \( V \) has length 0, or some hyperplane \( H \) crosses \( U \) and \( U' \).

In the former case, \( \overline{A} \) contains a point of \( \overline{Y}_i \). It follows by convexity of \( \overline{Y}_i \) that \( \overline{A} \subset \overline{Y}_i \), contradicting asystolicity. If the former case does not hold for any \( n \), then there is a hyperplane \( H \) separating \( \overline{A} \) from \( \overline{Y}_i \). The hyperplane \( H \) does not cross \( \overline{Y}_i \), but every hyperplane crossing \( \overline{A} \) crosses \( \overline{Y}_i \) and \( H \). Hence \( \overline{Y}_i \) contains arbitrarily large wall-pieces, a contradiction.

**Conclusion:** We have shown that, if \( \overline{A} \) contains no subpath of any \( \overline{Y}_i \) of length more than 35\( \|Y_i\|/72 \), then \( \hat{g} \) is an embeddable lift of \( g \). In particular, \( g \) has infinite order. \( \square \)

We are now ready for our main technical lemma, which explains how to identify when an element of \( \pi_1 X \) that is loxodromic on the contact graph survives in \( \pi_1 X^* \) as an element that is loxodromic on \( \mathcal{H} \).

**Lemma 5.6 (Asystolic loxodromics persist).** Let \( g \in \pi_1 X^* \). Suppose that \( \hat{g} \) is a \( \frac{35}{72} \)-asystolic lift of \( g \). Suppose that \( \hat{g} \) is loxodromic on \( \mathcal{H} \). Then \( g \) is loxodromic on \( \mathcal{H} \).

Suppose, moreover, that \( \hat{g} \) is \( \frac{17}{36} \)-asystolic. Then \( g \) acts on \( \mathcal{H} \) as a loxodromic WPD element.

**Proof of Lemma 5.6** The proof has several parts. Since \( \frac{17}{36} \)-asystolicity implies \( \frac{35}{72} \)-asystolicity, we will assume \( \frac{35}{72} \)-asystolicity for the purpose of showing that \( g \) is loxodromic, and \( \frac{17}{36} \)-asystolic only for the purpose of showing that \( g \) is fast (recall Definition 3.4).
Embeddability and bounded coarse intersections with hyperplanes: Since \( g \) is 35/72–asystolic, Lemma 5.3 implies that \( \tilde{g} \) is embeddable and hence \( g \) has infinite order. Hence no lift of \( g \) stabilises an elevation of a relator. Thus, if \( \tilde{g} \) is loxodromic on \( \hat{H} \), then Lemma 5.3 ensures that \( \tilde{g} \) (which is necessarily rank one and has no power stabilising a hyperplane) is loxodromic on \( \hat{H} \). Moreover, since \( \tilde{g} \) is loxodromic on \( \hat{H} \), there exists \( \mathbf{p}_g < \infty \) such that for all hyperplanes \( \hat{H} \), we have \( \text{diam}(\mathbf{g}_X(\hat{H}))(A) \leq \mathbf{p}_g \), where \( A \) is a combinatorial axis for \( \tilde{g} \) in \( X \). In other words, at most \( \mathbf{p}_g \) of the hyperplanes crossing \( \tilde{A} \) can cross \( \hat{H} \).

Bounded coarse intersections with elevations of relators: Let \( M \) be the upper bound on diameters of pieces, i.e. \( M = \frac{1}{17} \inf_i \|Y_i\| \).

We claim that: there exists \( \mathbf{q}_g < \infty \) such that, for all subcomplexes \( \tilde{Y}_i \) that are lifts of relators to \( \tilde{X} \),

\[
\text{diam}(\mathbf{g}_{\tilde{Y}_i}(\tilde{A})) < \mathbf{q}_g,
\]

and

\[
\text{diam}(\mathbf{g}_{\tilde{Y}_i}(\tilde{A})) < \frac{1}{2} \|Y_i\|.
\]

Moreover, under the additional \( \frac{17}{39} \)-asystolicity assumption, we will actually get \( \text{diam}(\mathbf{g}_{\tilde{Y}_i}(\tilde{A})) < \frac{35}{72} \|Y_i\| \).

First, we will show that there exists \( \mathbf{q}_g' \) such that \( |\tilde{P}| \leq \mathbf{q}_g' \) whenever \( \tilde{P} \) is a subpath of \( \tilde{A} \) that lies in some \( \tilde{Y}_i \). Indeed, if not, then for all \( i \in \mathbb{N} \), there exists \( \tilde{Y}^N \) (an elevation of a relator \( Y_{iN} \)) and a subpath \( \tilde{P}_N \) of \( \tilde{A} \) such that \( \tilde{P}_N \) lies in \( \tilde{Y}^N \) and has length more than \( N \). By applying powers of \( \tilde{g} \), we can assume that \( \tilde{P}_N \) joins \( \tilde{a} \) to \( \tilde{g}^{kN} \tilde{a} \), where \( \tilde{a} \in \tilde{A}(0) \) and \( k_N \geq N/\tau_g - 1 \), where \( \tau_g \) is the combinatorial translation length of \( \tilde{g} \).

Hence the subpath \( \tilde{Q}_N \) of \( \tilde{P}_N \) joining \( \tilde{a} \tilde{g}^{-1} \tilde{a} \) lies in \( \tilde{Y}^{-N} \cap \tilde{g}\tilde{Y}^N \). Thus either \( \tilde{Y}^N = \tilde{g}\tilde{Y}^{-N} \), or \( \tilde{Q}_N \) is a piece. Since \( \tilde{g} \) is embeddable, the former option is impossible, so \( \tilde{Q}_N \) is a piece of length at least \( N - 2\tau \). For \( N > M + 2\tau \), this is a contradiction. We conclude that there must exist \( \mathbf{q}_g' \) with the claimed property.

Meanwhile, just by asystolicity, if \( \tilde{P} \) is a subpath of \( \tilde{A} \) lying in some \( \tilde{Y}_i \), we have \( |\tilde{P}| < \frac{35}{72} \|Y_i\| \).

(Or \( |\tilde{P}| < \frac{17}{39} \|Y_i\| \) under the stronger of the two asystolicity assumptions.)

Now fix \( \tilde{Y}_i \). Let \( R \) be a geodesic in \( \mathbf{g}_{\tilde{Y}_i}(\tilde{A}) \); so, the hyperplanes intersecting \( R \) all intersect \( \tilde{A} \) and \( \tilde{Y}_i \). Under the gate map, \( R \) is the image of some geodesic \( R' \) in \( \tilde{A} \) joining two 0–cubes \( y, y' \).

(So, \( R \) joins \( \mathbf{g}_{\tilde{Y}_i}(y), \mathbf{g}_{\tilde{Y}_i}(y') \).)

Next, let \( \beta, \beta' \) be geodesics joining \( y, \mathbf{g}_{\tilde{Y}_i}(y) \) and \( y', \mathbf{g}_{\tilde{Y}_i}(y') \) respectively. Let \( D \to \tilde{X} \) be a minimal-area disc diagram bounded by \( \beta, \beta', R, R' \). We allow \( \beta, \beta', R \) to vary among geodesics with the given endpoints; varying such a geodesic does not change the hyperplanes it crosses. We make all such choices so that, among them, the resulting \( D \) has minimal area. (We emphasise that we do not allow \( R' \) to vary, since we need it to be a subpath of \( \tilde{A} \).) See Figure 9.

Let \( K, K' \) be dual curves in \( D \) emanating from \( \beta \). Then minimality of area ensures that \( K, K' \) do not cross. The same holds with \( \beta' \) replaced by \( \beta \) or \( R \). Next, let \( K \) be a dual curve emanating from \( R \). The hyperplane to which \( K \) maps must not separate \( y, \mathbf{g}_{\tilde{Y}_i}(y) \), since it crosses \( \tilde{Y}_i \), so \( K \) cannot cross \( \beta \). Similarly, \( K \) cannot cross \( \beta' \). So, every dual curve emanating from \( R \) terminates on \( R' \).

Hence every dual curve \( K \) travels from \( R \) to \( R' \) or from \( R' \) to \( \beta \) or \( \beta' \), or from \( \beta \) to \( \beta' \). Now, if \( K \) travels from \( \beta \) to \( R' \), then \( K \) maps to a hyperplane \( H \) not crossing \( \tilde{Y}_i \) (since \( H \) separates \( \mathbf{g}_{\tilde{Y}_i}(y) \) from \( y \)), so its yields a wall-piece in \( \tilde{Y}_i \) and hence crosses at most \( M \) of the dual curves traveling from \( R \) to \( R' \). The same holds for dual curves traveling from \( \beta' \) to \( R' \). Hence there are at least \( |R'| - 2M \) dual curves that travel from \( R \) to \( R' \) and do not cross any other dual
curves. These dual curves thus have length 0, so \( R, R' \) have a common subpath of length at least \(|R'| - 2M\). The preceding discussion shows that \(|R'| - 2M < q'_g\), so \(|R'| < q'_g + 2M\). Taking \( q_g = q'_g + 2M \) proves the claim.

Moreover, the preceding discussion also shows that \(|R'| < \frac{35}{72}\|Y_i\| + 2M\) (under \( \frac{35}{72}\)–asystolicity) or \(|R'| < \frac{17}{36}\|Y_i\| + 2M\) (under \( \frac{17}{36}\)–asystolicity). Hence, under the weaker assumption, we get \(|R'| < \|Y_i\|/2\), since \( 144M < \inf_j \|Y_j\| \). Under the stronger assumption, we get \(|R'| < \frac{35}{72}\|Y_i\|\).

(Since there is no uniform bound on the systoles of the relators, this is insufficient to give the bound \( q_g \) above, because \( \tilde{A} \) can pass through infinitely many orbits of elevations of relators. We use the bound \( \|Y_i\|/2 \) below to prove that \( g \) is loxodromic, and the bound \( \frac{35}{72}\|Y_i\| \) for proving that \( g \) is fast.)

The path \( A \to \text{Cay}(X^*) \): Let \( \tilde{a} \in \tilde{A} \) be a 0–cube. Let \( a = p(\tilde{a}) \), so that \( g^n a = p(\tilde{g}^n \tilde{a}) \) for each \( n > 0 \). Let \( A_0 = p \circ \tilde{A} \). Since \( \tilde{g} \) is an embeddable lift of \( g \), the path \( A_0 \) is an embedded bi-infinite combinatorial path in \( \text{Cay}(X^*) \).

Paths in \( \mathcal{H} \): Fix \( n > 0 \). Choose a sequence \( C_0, \ldots, C_N \) satisfying:

- for each \( j \leq N \), either \( C_j \) is the carrier of a hyperplane in \( \tilde{X}^* \) or \( C_j \to \tilde{X}^* \) is a lift of some \( Y_i \to X \) (abusing notation, we use the same name for \( C_j \) as for its image);
- we have \( a \in C_0 \) and \( g^n a \in C_N \);
- we have \( C_j \cap C_{j+1} \neq \emptyset \) for all \( j \leq N \);
- the number \( N \) is minimal with the above properties.

For each \( j \), let \( \alpha_j \) be a combinatorial geodesic of \( C_j \), chosen so that \( Q = \alpha_0 \alpha_2 \cdots \alpha_N \) is an embedded piecewise-geodesic from \( a \) to \( g^n a \).

Constructing a disc diagram: Let \( A_n \) be the subpath of \( A \) joining \( a \) to \( g^n a \). Let \( D \to \tilde{X}^* \) be a minimal disc diagram with boundary path \( A_n Q^{-1} \). Moreover, choose the \( C_j \) and \( Q \), subject to the above constraints, so that \( D \) has minimal complexity among all diagrams constructed in the preceding manner. See Figure 10.

The first square homotopy: If \( D \) contains an exposed square (or generalised corner of a square) corresponding to a length–2 subpath \( e f \) (with \( e, f \) single 1–cubes) that lies on \( A_n \), then we can perform a square homotopy (shuffling first if necessary) to replace \( A_n \) by a square-homotopic path with the same length and endpoints. Hence \( D = D' \cup_P D'' \), where \( D' \) is a square diagram bounded by the paths \( P, A_n \) and \( D'' \) is a diagram bounded by \( P \) and \( Q \), and there are no exposed squares/generalsised corners along \( P \).

Moreover, since \( D' \) is a square diagram, it lifts to a square diagram \( D' \to \tilde{X} \) bounded by the subpath of \( \tilde{A} \) joining \( \tilde{a} \) to \( \tilde{g} \tilde{a} \) (since \( \tilde{A} \) is a lift of \( A \)) and a lift \( \tilde{P} \) of the embedded path \( P \).

Figure 9. The diagram in \( \tilde{X} \) bounded by \( R, R', \beta, \beta' \), along with some allowable dual curves. At least \(|R'| - 2M\) of the vertical dual curves do not cross any other dual curve, and thus have length 0. Hence, for each \( Y_i \), at most \( \|Y_i\|/2 \) hyperplanes cross both \( \tilde{A} \) and \( \tilde{Y}_i \) in \( \tilde{X} \).
The path $\tilde{A}$ is a geodesic, so every dual curve in $D'$ starting on $A_n$ ends on $P$. Now, since $|P| = |A_n|$, we also have that $\tilde{P}$ is a geodesic of $\tilde{X}$, so dual curves in $D'$ starting on $P$ end on $A_n$. Hence, if $P_1$ is a subpath of $P$ lying in a cone-cell $Y$, we have $|P_1| < \|Y_i\|/2$, where $Y_i$ is the relator to which $Y$ maps. If $P_2$ is a subpath of $P$ lying in a hyperplane carrier, $|P_2| \leq p_q$.

**Shortcuts and complexity reductions:** Now, let $K$ be a dual curve or cone-cell in $D''$. Suppose that $K$ intersects $\alpha_i, \alpha_j$. Then minimality of $N$ implies that $|i - j| \leq 2$, for otherwise the hyperplane or relator to which $K$ maps would provide a shortcut between $C_i$ and $C_j$, contradicting minimality of $N$. Moreover, if $j - i = 2$, then we can replace $C_{i+1}$ by the hyperplane or relator to which $K$ maps, resulting in a new disc diagram (constructed as above) that is a proper subdiagram of $D''$ and hence has lower complexity. Hence $|i - j| \leq 1$.

**Ruling out shells:** Suppose that $K$ is a positively-curved shell with boundary path $OI$, where the outer path $O$ is a subpath of $P$. Then by Theorem 2.17 $|O| > |\partial_p K|/2$. Now, $\partial_p K$ is an essential path in some relator $\|Y_i\|$, so $|\partial_p K| \geq \|Y_i\|$. On the other hand, $|O| < \|Y_i\|/2$, by the discussion above. This is a contradiction. Hence there are no shells or generalised corners of squares along $P$. Since $P$ is embedded, there are also no spurs. Hence $D''$ contains no positively-curved cell whose outer path is a subpath of $P$.

Likewise, suppose that $K$ is a positively-curved shell with boundary path $OI$, where the outer path $O$ is a subpath of $Q$. Then the relator $Y$ to which $K$ maps must intersect $C_i, C_j$ for some $i, j$. We saw above that this can only happen if $|i - j| \leq 1$. Hence $O$ is the concatenation of at most 2 cone-pieces or wall-pieces, so $|O| \leq 2M$. But Theorem 2.17 requires that $|O| > 72M$, a contradiction. Hence there is no positively-curved shell whose outer path is a subpath of $Q$. Since $Q$ is an embedded path, by construction, it also contains no spur.

**The second square homotopy:** We are working toward an application of the ladder theorem — in view of the diagram trichotomy, we now just need to remove generalised corners along $Q$ using square homotopies, as follows.

Let $K$ be a dual curve with one end on $Q$, i.e. one end on some $\alpha_i$. We have seen already that $K$ cannot have its other end on $\alpha_j$ for $|i - j| \geq 2$. Moreover, since $\alpha_i$ is a geodesic, $K$
cannot have its other end on \( \alpha_i \). Finally, \( K \) cannot end on \( \alpha_{i \pm 1} \). Indeed, otherwise \( D \) would have a subdiagram \( E \) bounded by the subpath of \( \alpha_i \alpha_{i+1} \) (say) subtended by the 1–cubes dual to \( K \), along with a path on \( \mathcal{N}(K) \). The small-cancellation conditions and Theorem \ref{thm:2.16} imply that \( E \) is a square diagram. Choosing \( K \) to be innermost, every dual curve in \( E \) travels from \( \mathcal{N}(K) \) to \( \alpha_i \alpha_{i+1} \). Now, no two dual curves emanating from \( \alpha_i \) (or \( \alpha_{i+1} \)) can cross, because an innermost such pair would give an exposed square in \( D \) along \( \alpha_i \); convexity of \( C_i \) would then yield a contradiction with minimal complexity. Hence \( E \) has no squares, so the two 1–cubes of \( Q \) dual to \( K \) coincide, contradicting that \( \alpha_i, \alpha_{i+1} \) do not share a 1–cube (since \( Q \) is an embedded path). See Figure \ref{fig:1.11}

![Figure 11](image)

**Figure 11.** If a dual curve \( K \) in \( D \) intersects \( \alpha_i, \alpha_{i+1} \), then \( D \) has a subdiagram \( E \) as shown. (Also shown are \( C_i, C_{i+1} \)). This must be a square diagram, and convexity of \( C_i, C_{i+1} \) then allow us to conclude, from minimality of complexity, that the 1–cubes dual to the first and last points of \( K \) coincide. This contradicts that \( Q \) was an embedded path. This is the last step in showing that no dual curve starts and ends on \( Q \).

Let \( Q_1 \) be a path in \( D'' \) such that:

- \( Q_1 \) and \( Q \) have the same endpoints;
- \( |Q_1| \leq |Q| \);
- \( Q_1 \) has no spurs;
- the subdiagram \( F \) of \( D'' \) bounded by \( Q \) and \( Q_1 \) is a square diagram, and has as many squares as possible subject to the above constraints.

Note that every dual curve in \( F \) with one end on \( Q_1 \) has one end on \( Q \), since \( F \) is a square diagram and no dual curve in \( D'' \) starts and ends on \( Q \), by the preceding discussion. Since \( |Q_1| \leq |Q| \), it follows that every dual curve in \( F \) with one end on \( Q_1 \) has an end on \( Q \).

Let \( D''_1 \) be the subdiagram of \( D'' \) bounded by \( Q_1 \) and \( P \). Suppose that \( ef \) is a length–2 subpath of \( Q_1 \) corresponding to a generalised corner of a square in \( D''_1 \). By shuffling, we can assume that \( ef \) is a subpath of the boundary path of a square \( s \) in \( D''_1 \). Then \( \partial_p s = efe'f' \), and we can replace \( ef \) by \( e'f' \) (and remove spurs if necessary) in \( Q_1 \) to obtain a new path \( Q'_1 \), of length at most \( |Q_1| \), such that the subdiagram bounded by \( Q \) and \( Q'_1 \) has more squares than \( F \), a contradiction. Hence there are no generalised corners of squares, or spurs, of \( D''_1 \) lying along \( Q_1 \). See Figure \ref{fig:1.12}

**Extracting a ladder:** Suppose that \( K \) is a shell in \( D''_1 \) with boundary path \( OI \), where the outer path \( O \) is a subpath of \( Q_1 \). Then \( O \) is also a subpath of \( \partial_p F \), and every dual curve in \( F \) emanating from \( O \) ends on some \( \alpha_i \). Let \( i_0, i_1 \) be the minimal and maximal values of \( i \) such that \( F \) contains a dual curve \( K' \) that travels from \( \alpha_i \) to \( O \). Let \( Y \) be the relator to which \( K \) maps. Then there is a sequence \( C_{i_0}, H, Y, H', C_{i_1} \), where \( H, H' \) are hyperplanes and consecutive terms intersect. Hence, by minimality of \( N \), we have \( i_1 - i_0 \leq 4 \). See Figure \ref{fig:1.13}

Thus, since every dual curve emanating from \( O \) ends on \( \alpha_i \) for some \( i \in \{ i_0, \ldots, i_1 \} \), we have that \( O \) is square-homotopic (fixing endpoints) to a concatenation of at most 5 cone-pieces or
Figure 12. The diagram $D''_1$ has no generalised corner of a square along $Q_1$, for otherwise we could enlarge the subdiagram $F$ with a square homotopy.

Figure 13. The diagram $D$ is the union of square diagrams $D', F$ and a ladder $D''_1$. Dual curves in $F$ travel from $Q$ to $Q_1$, and dual curves in $D'$ travel from $P$ to $A_n$. The cone-cells in $D''_1$ have small projection to $A_n$ because of asystolicity of $g$. They have bounded projection to $Q$ because of minimality of $N$: any dual curve in $F$ emanating from a common cone-cell of $D''_1$ must end on $\alpha_i$ for at most one of 5 values of $i$. This is ultimately used to show that $D''_1$, and hence all of $D$, is a square diagram.
function of $N$, by Lemma 5.2 If $N$ grows sublinearly in $n$, this means that $\tilde{g}$ is not loxodromic on $\tilde{H}$. But recall (from the very beginning of the proof) that $\tilde{g}$ is loxodromic on $\tilde{H}$ because it is embeddable and loxodromic on $H$. Hence $N$ grows linearly in $n$, so $g$ is loxodromic on $H$.

**Fast:** Let $n \geq 0$. Let $S$ be a geodesic in Cay($X^*$) from $a$ to $g^na$. Write $S = S_1 \cdots S_N$, with each $S_i$ lying in a relator or hyperplane carrier. Let $D \to \tilde{X}^*$ be a minimal-complexity diagram bounded by $A_n$ and $S$. Then, as above, $D$ contains no positively curved shell with outer path along $A_n$, and $D$ has no positively curved shell with outer path along $S$, by Theorem 2.17 since $S$ is a geodesic. Hence, arguing as above (“Extracting a ladder”), we see first that $D$ is square-homotopic to a ladder $D'$ with boundary path $A_0'T^{-1}$, where $T$ is a geodesic square-homotopic to $S$ and $A_0'$ is square-homotopic to, and has the same length as, $A_n$.

Now, for each cone-cell $K$ of $D'$, let $\gamma$ be the part of $\partial_pK$ lying on $T$ and let $\sigma$ be the part of $\partial_pK$ lying on $A_0'$. Since all dual curves in $D$ starting on $A_n$ end on $A_0'$ (because $A_0'$ was obtained from $A_n$ by square homotopies across generalised corners), $\frac{17}{36}$–asystolicity implies that $|\sigma| < \frac{25}{36}||Y||$, where $Y$ is the relator to which $K$ maps. Now, since $T$ is a geodesic, shortcuts of the type shown on the right in Figure 6 are impossible, so $|\gamma| \leq |\sigma| + 2M$. Hence $|\partial_pK| < ||Y|| \left(\frac{17}{36} + \frac{25}{36}\right) < ||Y||$. Thus $\partial_pK$ bounds a square diagram in $X$, contradicting minimal complexity. Hence $D$ contains no cone-cells.

Hence $D$ is a square diagram, and dual curves starting on $S$ end on $A_n$. We thus have $|S_i| \leq \nu_q$ for all $i$, so $g$ is $\Delta$–fast for some $\Delta$. Since $g$ is a fast loxodromic on $H$, Lemma 3.5 shows that $g$ is a WPD element.

### 5.2. Building fast loxodromics in the essential case

In this subsection, we additionally assume that $\pi_1X$ acts essentially on $\tilde{X}$, in the sense of [CS11]. In particular, there is no proper $\pi_1X$–invariant convex subcomplex. Since $X$ is compact, the action of $\pi_1X$ on $\tilde{H}$ is cobounded.

By [Hag13, Theorem 5.4] (which in turn relies on results in [CS11]), and Lemma 5.2 one of the following holds:

1. $\tilde{X} \cong A \times B$, where $A, B$ are unbounded CAT(0) cube complexes. In this case, either $\pi_1X^*$ is finite or $\pi_1X^* = \pi_1X$. Indeed, if $\tilde{Y}_i = \tilde{X}$, then $\tilde{Y}_i \to X$ is a covering map. Since $\tilde{Y}_i$ is compact, this implies that $Y_i$ is a finite cover, whence $\pi_1^i$ is finite. Otherwise, since $\tilde{Y}_i \subseteq \tilde{X}$ is convex, we have $\tilde{Y}_i = A' \times B'$, where $A' \subset A$ and $B' \subset B$ are convex subcomplexes, and one of the two containments is proper. Hence there is a hyperplane $H$ such that $\tilde{Y}_i$ is contained in the carrier of $H$, but $H$ does not cross $\tilde{Y}_i$. This contradicts any metric small-cancellation condition, because it implies that there is no bound on wall-pieces, unless $Y_i$ is contractible. In the latter case, we can remove $Y_i$ from the set of relators without changing $\pi_1X^*$. Thus we can assume $I = \emptyset$, so $\pi_1X^* = \pi_1X$.

2. There exists $\tilde{g} \in \pi_1X$ acting loxodromically on $\tilde{H}$.

We now restrict to case [2]. Given $\tilde{g} \in \pi_1X$, let $\tau_{\tilde{g}} \geq 1$ be the combinatorial translation length of $\tilde{g}$ on $\tilde{X}$. Let $\mathcal{L}$ be the (nonempty) set of $\tilde{g} \in \pi_1X$ such that $\tilde{g}$ is loxodromic on $\tilde{H}$. Let $L = L(X)$ be $\min_{\tilde{g} \in \mathcal{L}} \tau_{\tilde{g}}$.

Now let $\alpha_0 = \alpha_0(X) \leq \frac{1}{114}$ be a constant to be determined. Throughout the rest of this section, we assume that $\langle X | \{Y_{i1} | i \in I\} \rangle$ is a $C^{\alpha}(\alpha)$ cubical presentation, where $\alpha \leq \alpha_0$.

**Lemma 5.7.** Suppose that $\tilde{X}$ contains a (nontrivial) piece. There exists $\tilde{g} \in \pi_1X$ such that $\tilde{g}$ is loxodromic on $\tilde{H}$ and $\frac{17}{36}$–asystolic.

**Proof.** Suppose that $\tilde{g} \in \mathcal{L}$ satisfies $\tau_{\tilde{g}} = L = L(X)$. Suppose that $\tilde{g}$ is not $\frac{17}{36}$–asystolic. Then, by definition, for some combinatorial geodesic axis $\tilde{A}$ of $\tilde{g}$, there is a subpath $\tilde{P}$ of $\tilde{A}$ such that $\tilde{P}$ lies in some $\tilde{Y}_i$ and $|\tilde{P}| \geq 17||Y_i||/36$. 

Let \( t \geq 1 \) be a constant to be determined, and suppose \( \alpha_0 \in [0, \frac{1}{17}] \). Then \( \min_j \| Y_j \| > \alpha_0^{-1} > tL \), since there is a piece and our presentation satisfies \( C'(\alpha_0) \). Hence \( |\tilde{P}| > \frac{17L}{36} \).

Let \( k = \left\lfloor \frac{17}{36} \right\rfloor - 1 \). Then we have a 0–cube \( \tilde{a} \in \tilde{P} \) such that \( \tilde{a}, \tilde{g}_1\tilde{a}, \ldots, \tilde{g}_k\tilde{a} \) lie in \( \tilde{P} \), and hence in \( \tilde{Y}_i \). Thus \( \tilde{Q} = \tilde{P} \cap \tilde{P}^{\ast} \) is a geodesic that lies in \( \tilde{Y}_i \cap \tilde{g}\tilde{Y}_i \). Hence either \( \tilde{g}\tilde{Y}_i = \tilde{Y}_i \) or \( \tilde{Q} \) is a piece of \( \tilde{Y}_i \) in \( \tilde{g}\tilde{Y}_i \). If \( \tilde{g}\tilde{Y}_i = \tilde{Y}_i \), then \( \tilde{g} \) is conjugate into \( \tilde{Y}_i \), and hence has translation length at least \( \alpha_0^{-1} > tL \), which is impossible since \( \gamma_\tau = L \).

So, \( \tilde{Q} \) is a piece, whence \( |\tilde{Q}| < \frac{1}{17} \| Y_i \| \). Now, \( |\tilde{Q}| = |\tilde{P}| - 2L \), so \( |\tilde{Q}| > \frac{17}{36} \| Y_i \| - 2L \). From these two estimates, it follows that \( \| Y_i \| < \frac{288}{67} L \). This is impossible if \( t \geq \frac{288}{67} \). To enable the argument above, we also need \( k \geq 2 \), so \( t \geq 7 \) suffices. \( \square \)

To summarise, we have:

**Proposition 5.8** (Acylindrically hyperbolicity when \( \pi_1 X \) acts essentially). Let \( X \) be a compact nonpositively curved cube complex such that \( \pi_1 X \) acts essentially on \( \tilde{X} \). Let \( \mathcal{L} \) be the (possibly empty) set of \( \tilde{g} \in \pi_1 X \) such that \( \tilde{g} \) acts loxodromically on \( C\tilde{X} \), and let \( L = L(X) \) be the minimal combinatorial translation length of \( \tilde{g} \) on \( \tilde{X} \), for \( \tilde{g} \in \mathcal{L} \). Let \( \alpha_0 = \min \{ \frac{1}{17}, \frac{1}{36} \} \) and let \( \alpha \in [0, \alpha_0) \).

Let \( \{ Y_i \to X \} \) be a (possibly infinite) set of local isometries of nonpositively curved cube complexes with each \( Y_i \) compact. Suppose that \( \langle X \mid \{ Y_i \} \rangle \) is a \( C'(\alpha) \) cubical presentation, and let \( X^* \) be the presentation complex. Then one of the following holds:

1. \( \pi_1 X^* \) is finite or two-ended.
2. Each \( Y_i \) is contractible, and \( \tilde{X} \cong A \times B \), where \( A, B \) are unbounded CAT(0) cube complexes, and \( \pi_1 X^* \cong \pi_1 X \).
3. \( \pi_1 X^* \) is acylindrically hyperbolic.

**Proof.** We can assume that each \( Y_i \) is non-contractible, by removing contractible relators. Indeed, doing so does not affect the small-cancellation hypothesis or the group \( \pi_1 X^* \). Next, our preceding discussion has handled the case where \( \mathcal{L} = \emptyset \) (this is the case where \( \tilde{X} \) is a product, which we saw yields either the first or second conclusion of the proposition). We have also already dealt with the case where \( \tilde{H} \) has two ends (i.e. \( C\tilde{X} \) has two ends): we saw above that this leads to the first conclusion of the proposition.

In the remaining case, provided \( \tilde{X} \) contains a (nontrivial) piece, Lemma 5.7 produces \( \tilde{g} \in \pi_1 \tilde{X} \) that is loxodromic on \( C\tilde{X} \) (hence on \( \tilde{H} \)) and \( \frac{17}{36} \)–asystolic. Lemma 5.6 implies that \( \tilde{g} \) is a fast loxodromic on \( \tilde{H} \). Lemma 5.5 and Theorem 4.3 now imply that \( \pi_1 X^* \) is acylindrically hyperbolic.

So, it remains to consider the case where there are no pieces. If \( \mathcal{I} = \emptyset \), then we are done, so let \( \tilde{Y}_i \) be an elevation of a relator. Since \( \tilde{Y}_i \) is not a single 0–cube (otherwise we could discard it), some hyperplane \( H \) crosses \( \tilde{Y}_i \). Now, any hyperplane \( V \) crossing \( H \) must also cross \( \tilde{Y}_i \), for otherwise there would be a nontrivial wall-piece of \( \mathcal{N}(V) \) in \( \tilde{Y}_i \). Hence each hyperplane crossing \( \tilde{Y}_i \) is contained in \( \tilde{Y}_i \). Moreover, no hyperplane can cross \( \tilde{Y}_i \) and also cross some \( \tilde{Y}_j \neq \tilde{Y}_i \), for otherwise there would be a cone-piece. We conclude that \( \tilde{X} \) decomposes as a tree of spaces such that each edge space is a 0–cube and each elevation of each relator is a vertex space. Hence \( \pi_1 X \cong \pi_1 Z \ast (\ast_{i \in \mathcal{I}} \pi_1 \tilde{Y}_i) \), where \( Z \) is a compact nonpositively curved cube complex and each \( \tilde{Y}_i \) is a nonpositively curved cube complex having \( Y_i \) as a finite cover. Thus either \( \pi_1 X^* \) is a nontrivial free product, or \( \pi_1 X^* \cong \pi_1 Z \). In either case, Proposition 5.8 is verified. \( \square \)

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3** Let \( X, L = L(X) \), and \( \alpha_0 \) be as above. Let \( \langle X \mid \{ Y_i \} \rangle \) be a \( C'(\alpha_0) \) cubical presentation with each \( Y_i \) compact.

By [CS11] Proposition 3.5, there is a convex \( \pi_1 X \)–invariant subcomplex \( \tilde{Z} \subset \tilde{X} \) on which \( \pi_1 X \) acts cocompactly. (Although it is not made explicit in [CS11], \( \tilde{Z} \) is in general only a
subcomplex of the first cubical subdivision of $\tilde{X}$, because the action may have inversions across hyperplanes. But, we can pass to the cubical subdivision and regard edges as having length $\frac{1}{2}$ without affecting the argument, and thus assume that $\tilde{Z}$ is a subcomplex.)

Let $Z = \pi_1 X \setminus \tilde{Z}$, which is a compact nonpositively curved cube complex. Let $\tilde{g} \in L$. Then the translation length of $\tilde{g}$ on $\tilde{Z}$ is at least $\tau_\tilde{g}$, since $\tilde{Z} \subset \tilde{X}$. On the other hand, the gate map $g_\tilde{Z} : \tilde{X} \to \tilde{Z}$ is $\pi_1 X$–equivariant and 1–Lipschitz, so the translation length of $\tilde{g}$ on $\tilde{Z}$ is at most $\tau_\tilde{g}$. Hence $L(Z) = L$.

The inclusion $\tilde{Z} \to \tilde{X}$ descends to a local isometry $Z \to X$. We form a new cubical presentation for $\pi_1 X^*$ by attaching to $Z$ all of the components of each fiber product $Y_i \otimes_X Z \to Z$. Pieces in the new cubical presentation arise by intersecting pieces in $\tilde{X}$ with $\tilde{Z}$, and the systoles of the relators have not decreased, so the $C'(a_0)$ condition persists. The theorem now follows from Proposition 5.8.

5.3. Computing $L(X)$ for the main examples. We now compute $L(X)$ for some standard examples:

- Suppose that $X$ is a wedge of circles, i.e. a 1–dimensional compact nonpositively curved cube complex with a single 0–cube and at least one 1–cube. Then $\tilde{X}$ is a tree, and $C\tilde{X}$ is quasi-isometric to $\tilde{X}$. So, each nontrivial $\tilde{g} \in \pi_1 X$ is loxodromic on $C\tilde{X}$, and so $L(X)$ is just the girth of $X$, i.e. 1. So, Theorem [B] holds under the $C'(\frac{1}{14})$ condition in this case (and in fact, this is not optimal since there are alternative proofs available when $\dim X = 1$).

- Suppose that $X$ is a graph. Then, exactly as above, $L(X)$ is the girth of $X$.

- Suppose that $X$ is the Salvetti complex of a right-angled Artin group $A(\Gamma)$ with (finite) presentation graph $\Gamma$. Let $a_1, \ldots, a_n$ be the generators of $A(\Gamma)$, with one for each vertex of $\Gamma$. Then $L \neq \emptyset$ if and only if $\Gamma$ does not decompose as the join of two proper subgraphs. In fact, if $\tilde{g} \in A(\Gamma)$ is represented by a reduced, cyclically reduced word $a_1^{e_1} \cdots a_k^{e_k}$, then $\tilde{g}$ is loxodromic on $C\tilde{X}$ if and only if the vertices $a_{i_1}, \ldots, a_{i_k}$ are not all contained in a common subgraph of $\Gamma$ that decomposes as a nontrivial join (this follows by combining [KMTL7] Theorem 2.2 with the discussion in the last paragraph of p. 22 in [KKL4]). Letting $a, b$ be distinct generators not contained in a common join, we thus have that $\tilde{g} = ab$ is loxodromic on $C\tilde{X}$, so $L(X) \leq 2$.

References


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