Graphs associated to $\text{CAT}(0)$ cube complexes

Mark Hagen

McGill University

Cornell Topology Seminar, 15 November 2011
Outline

- Background on $CAT(0)$ cube complexes
- The contact graph: a combinatorial invariant of cube complexes
- The main geometric property of the contact graph
- Application: weak and strong relative hyperbolicity
- Some further applications
CAT(0) cube complexes

- A $d$-cube is $[-1, 1]^d$ with the Euclidean metric, for $0 \leq d < \infty$.
- A cube complex is a CW-complex whose cells are cubes of various dimensions, glued along their faces by combinatorial isometries, i.e. with "no tricks".
- The cube complex $X$ is nonpositively curved if the link of each 0-cube is a flag complex.
- If $X$ is simply connected and nonpositively curved, it admits a piecewise-Euclidean CAT(0) metric (Gromov, 1987, for finite-dimensional cube complexes; Leary, 2010, for all cube complexes).
- Such an $X$ is a CAT(0) cube complex, but we rarely use the CAT(0) metric here.
- The dimension of the cube complex $X$ is the supremum of the dimensions of its cubes. The degree of $X$ is the supremum of the degrees of its 0-cubes. Note $\dim(X) \leq \deg(X)$. 
The link condition

Figure: Nonpositive curvature: for $n \geq 3$, if you see the corner of an $n$-cube, there is an $n$-cube, i.e. links of 0-cubes are flag complexes.
Hyperplanes

- **A midcube** in $[-1, 1]^d$ is the subspace obtained by restricting exactly one coordinate to 0.
- A **hyperplane** in the $CAT(0)$ cube complex $X$ is a connected union $H$ of midcubes such that for each cube $c$, either $H \cap c = \emptyset$ or $H \cap c$ is a single midcube of $c$.
- The **carrier** $N(H)$ of $H$ is the union of all closed cubes $c$ such that $H \cap c$ is a midcube.

*Figure: A cube complex and some hyperplanes.*
Properties of hyperplanes

Theorem (Sageev, 1995)

Let $X$ be a CAT(0) cube complex and $H$ a hyperplane. Then

1. $H$ is **two-sided**, i.e. $N(H) \cong H \times [-1, 1]$.
2. $H$ and $N(H)$ are convex (in the CAT(0) metric, and in the $\ell^1$ sense defined below).
3. $H$ is a CAT(0) cube complex of dimension at most $\dim(X) - 1$.
4. $H$ separates $X$: the space $X - H$ has exactly two components, $H^+$ and $H^-$, called **halfspaces**.

Subspaces $A, B \subset X$ are **separated by** $H$ if, e.g., $A \subset H^+$ and $B \subset H^-$. The 1-cube $c$ is **dual** to the hyperplane $H$ if $H$ separates the endpoints of $c$. This gives an equivalence relation on the 1-cubes: each 1-cube is dual to a unique hyperplane.
The cubical $\ell^1$ metric

The usual path metric $d$ on $X^1$ is closely related to the hyperplanes:

- $d(x, y)$ counts the number of hyperplanes separating the 0-cubes $x, y$.
- $X^1$ is a median graph and every median graph is the 1-skeleton of a cube complex (Chepoi, 2000).
- $d$ extends to a metric on $X^1$ whose restriction to each cube is the $\ell^1$ metric. Many CAT(0) metric properties of subcomplexes of $X$ -- notably convexity -- hold for this metric as well.
- If $X$ is finite-dimensional, then the inclusion $X^1 \to X$ is a quasi-isometry (with respect to the CAT(0) metric on $X$).

We will focus mostly on the geometry of $X^1$ and of $X$ with the $\ell^1$ metric.
Convexity and isometric embedding

The hyperplane $H$ crosses the subcomplex $A \subset X$ if $H^+ \cap A$ and $H^- \cap A$ are nonempty.

- $A$ is isometrically embedded if and only if $N(H) \cap A$ is connected for all $H$ crossing $A$.
- If $A$ is isometrically embedded, then $A$ is convex if and only if $N(H) \cap N(H') \cap A \neq \emptyset$ for all $H, H'$ crossing $A$ and, if $H, H'$ cross, then $N(H) \cap N(H') \cap A$ contains a 2-cube reflecting this crossing.

For example, hyperplane carriers are convex subcomplexes.
Crossing hyperplanes

Hyperplanes $H, H'$ cross if either of the following equivalent statements holds:

1. There exists a 2-cube $s \subseteq N(H) \cap N(H')$, two of whose 1-cubes are dual to $H$ and the other two dual to $H'$.

2. All four of the quarterspaces $H^\pm \cap (H')^\pm$ are nonempty.

---

Figure: $H$ and $H'$ cross, as seen by examining parts of their carriers.
The crossing graph

The crossing graph of $X$, denoted $\Delta X$, has a vertex for each hyperplane, and two vertices are adjacent when the corresponding hyperplanes cross. Some examples:

- If $X$ is 1-dimensional -- a tree -- then $\Delta X$ is totally disconnected.
- If $X$ is a $d$-cube, then $\Delta X$ is a $d$-clique.
- More examples below.

This is the same as the "transversality graph" introduced by Roller. Intersection graphs of collections of subcomplexes have been studied in many contexts; the crossing graph is the intersection graph of the collection of hyperplanes.
Properties of the crossing graph

- Each $d$-cube in $\mathbf{X}$ is encoded by a $d$-clique in $\Delta \mathbf{X}$. Hence $\dim(\mathbf{X})$ is the supremum of the cardinalities of cliques in $\Delta \mathbf{X}$.
- $\Delta \mathbf{X}$ is disconnected if and only if $\mathbf{X}$ is a wedge sum of nonempty subcomplexes.
- Cubical flat plane theorem (H., 2010): $\mathbf{X}$ is hyperbolic (with either metric) if and only if there exists $M < \infty$ such that, for all $K_{p,q} \subseteq \Delta \mathbf{X}$, $\min(p,q) \leq M$ (``thin bicliques'').
- Disadvantage (Roller, 1998, also H., 2010 and many others): Every simplicial graph is the crossing graph of a $\text{CAT}(0)$ cube complex. So not much can be said, geometrically, about crossing graphs...
Generalizing crossing

For distinct hyperplanes $H, H'$, the following are equivalent:

1. No hyperplane $H''$ separates $H$ and $H'$.
2. $N(H) \cap N(H') \neq \emptyset$.

If $H, H'$ satisfy the above, then they **contact**. Can happen in two ways:

1. $H$ and $H'$ contact if they cross, since $N(H) \cap N(H')$ contains at least one 2-cube.
2. $H$ and $H'$ **osculate** if they contact but do not cross: there exist 1-cubes $c, c'$, dual to $H, H'$ respectively, such that $c$ and $c'$ have a common 0-cube but do not lie in a common 2-cube.

Figure: An osculation.
The contact graph

The contact graph $\Gamma_X$ has a vertex for each hyperplane, with vertices joined by an edge if the corresponding hyperplanes contact. Some examples:

- If $X$ is 1-dimensional -- a tree -- then $\Gamma_X$ is the graph dual to $X$.
- In particular, if $R$ is the cube complex obtained by subdividing $\mathbb{R}$, with 0-cubes at integer points, then $\Gamma_R \cong R$.
- If $X$ and $Y$ are cube complexes, then so is $X \times Y$, and $\Gamma(X \times Y)$ is isomorphic to the join $\Gamma_X \ast \Gamma_Y$.
- For example, $\Gamma R^n$ is the join of $n$ subdivided lines.
A simple contact graph

Figure: A cube complex, its crossing graph, and its contact graph.
Properties of the contact graph

- $\Gamma X$ is simplicial, by definition.
- $\Delta X \subseteq \Gamma X$.
- Each isometrically embedded subcomplex of $X$ projects to a subgraph of $\Gamma X$: the projection of $A \subset X$ contains each vertex corresponding to a hyperplane that crosses $A$, together with each edge that records a hyperplane-contact that occurs in $A$.
- Convex subcomplexes project to induced subgraphs.
- Projections of connected subcomplexes are connected. Hence $\Gamma X$ is connected.
- Sometimes, one can recover a subcomplex of $X$ from a subgraph of $\Gamma X$, but one must be very careful.

The class of contact graphs is much more restricted than the class of crossing graphs, and we can say something about the geometry of the contact graph.
**Theorem (H., 2010)**

Let $X$ be a CAT(0) cube complex. There exist constants $M, C$, independent of $X$, such that $\Gamma X$ is $(M, C)$-quasi-isometric to a tree.

**Remark**

The (tortuous) disc diagram proof gives $M = 4, C = 0$. A simplification gives $M = 5, C = 0$. The quick proof gives $M > 39, C > 30$.

The quick proof is motivated by the observation that, if $H''$ separates $H$ from $H'$, then any path in $\Gamma X$ joining $H$ to $H'$ contains $U$ such that $U = H''$ or $U$ crosses $H''$. 
Quick proof

- Choose any two hyperplanes $H_0, H_n$.
- $H_0, H_n$ are separated by hyperplanes $H_1, \ldots, H_{n-1}$
- Suppose $n$ is even. Any path in $\Gamma X$ joining $H_0$ to $H_n$ contains a hyperplane $H$ with $H = H_{n/2}$ or $H$ crossing $H_{n/2}$, since $H_{n/2}$ separates $H_0, H_n$.
- Let $m$ be the vertex of $\Gamma X$ corresponding to $H_{n/2}$. For $n$ odd, use the midpoint of the edge of $\Gamma X$ joining $H_{(n-1)/2}$ to $H_{(n+1)/2}$.
- Hence there is $m = m(H_0, H_n) \in \Gamma X$ such that $d_{\Gamma X}(P, m) \leq \frac{3}{2}$ for any path $P$ joining $H_0, H_n$. 
Quick proof

Figure: The idea of the quick proof.
Quick proof

This argument verifies Manning’s criterion for $\Gamma X$, for any $\delta > \frac{3}{2}$:

**Theorem (Manning, 2005)**

Let $Y$ be a geodesic space. Suppose there exists $\delta$ such that for all $x, y \in Y$, there exists $m = m(x, y) \in Y$ such that every path $P$ joining $x, y$ satisfies $d(P, m) \leq \delta$. Then $Y$ is $(M, C)$-quasi-isometric to a tree for any $M > 26\delta$, $C > 20\delta$.

The original proof uses disc diagrams in cube complexes, and we sketch the main ideas.
Grading graphs

Let $\Gamma$ be a graph.

- Fix a base vertex $v_0$.
- The **grade** of the vertex $v$ is $g(v) = d_{\Gamma}(v, v_0)$.

For contact graphs, this means we fix a base hyperplane $V_0$ and grade the hyperplane $V$ according to the shortest sequence of hyperplane-contacts needed to travel from $V$ to $V_0$. 
Clusters

If $g(v) = g(v')$, then $v$ and $v'$ are **equivalent** if there is a path $P$ in $\Gamma$ joining $v$ to $v'$ such that, for all vertices $u \in P$, we have $g(u) \geq g(v)$.

Figure: The red vertices are equivalent, being joined by a path containing only vertices of grade at least $n$.

This is an equivalence relation on each grade. The equivalence-classes are called **clusters**.
The cluster tree

Let $\mathcal{T}$ be the graph whose vertices are the clusters of $\Gamma$. For $n \geq 0$, the grade-$n$ cluster $C^n$ is adjacent to the grade-$(n + 1)$ cluster $C^{n+1}$ if there exist $v \in C^n$ and $w \in C^{n+1}$ such that $v, w$ are adjacent in $\Gamma$.

- $\mathcal{T}$ is a tree.
- There is a surjection $\phi : \Gamma \to \mathcal{T}$ that sends each vertex to its cluster.
- $\phi$ sends each edge $(v, w)$ of the form $v \in C^n, w \in C^{n+1}$ to the edge of $\mathcal{T}$ joining $C^n, C^{n+1}$.
- Every other edge of $\Gamma$ joins two vertices in the same cluster, and gets collapsed to a vertex by $\phi$. 
The cluster tree

Figure: The picture on the left shows $\Gamma$ partitioned into grades, and each grade partitioned into clusters. The corresponding cluster tree is shown at right.
Bounding the diameter of a cluster

To prove that $\phi$ is an $(M, 0)$-quasi-isometry amounts to proving that $\text{diam}_{\Gamma}(C) \leq M$ for each cluster $C$. When $\Gamma = \Gamma X$, we do this by setting up a disc diagram:

- Choose grade-$n$ hyperplanes $H_0, H_k$.
- Choose a shortest path $H_0, H_1, \ldots, H_{k-1}, H_k$ in $\Gamma X$ such that each $H_i$ contacts $H_{i+1}$, and each $H_i$ has grade at least $n$ with respect to the base hyperplane $V_0$.
- Choose paths $V_0, V_1, \ldots, V_{n-1}, H_0$ and $V_0, U_1, \ldots, U_{n-1}, H_k$.
- This gives a closed path in $\Gamma X$ containing $V_0, H_0, H_k$. 
The disc diagram

Represent the closed path by a closed path in $X$ that is a concatenation of geodesics in the carriers of $N(V_0), N(U_i), N(V_i), N(H_i)$. This path bounds a **disc diagram** in $X$ that sits on our designated carriers:

**Figure:** The pink carrier is $N(V_0)$. The blue carriers are the $N(H_i)$. Green are the $N(U_i)$ and red are the grade-$n$ hyperplanes of interest.
Quasi-arboreal groups

Definition (Bowditch, 1997)
Let \( G \) be a group and let \( \{P\} \) be a collection of subgroups. Then \( G \) is \textbf{weakly hyperbolic} relative to \( \{P\} \) if \( G \) acts on a graph \( \Gamma \) satisfying:

1. \( \Gamma \) has finitely many \( G \)-orbits of vertices and edges.
2. The stabilizer of each vertex of \( \Gamma \) is conjugate to some element of \( \{P\} \).
3. Each element of \( \{P\} \) is a vertex-stabilizer.
4. \( \Gamma \) is hyperbolic (with the usual graph-metric).

This abstracts the essential features of the original definition, due to Farb. If \( \Gamma \) is quasi-isometric to a tree, then \( G \) is \textit{quasi-arboreal} relative to \( \{P\} \).
Cubical groups are weakly hyperbolic

Note: if $G$ acts on the $\text{CAT}(0)$ cube complex $X$, then $G$ permutes the hyperplanes and preserves contacts between hyperplanes. If the action is proper and cocompact, then there are finitely many orbits of hyperplanes and finitely many orbits of contacts between hyperplanes.

Corollary (H., 2010)

Let $G$ act on the $\text{CAT}(0)$ cube complex $X$. Then $G$ acts on a quasi-tree. Moreover, if the $G$-action on $X$ is proper and cocompact, then $G$ is quasi-arboreal, and thus weakly hyperbolic, relative to the collection of hyperplane-stabilizers.

There are also many examples of quasi-arboreal groups that do not act nicely on cube complexes: Baumslag-Solitar groups, $\text{SL}_n(\mathbb{Z})$ for $n \geq 3$, etc.
Relative hyperbolicity

Definition (Bowditch, 1997)
The graph $\Gamma$ is **fine** if for each edge $e$ and each $n \geq 1$, there are finitely many $n$-cycles containing $e$.
If $\Gamma$ is a $G$-graph showing that $G$ is weakly hyperbolic relative to $\{P\}$, and stabilizers of edges in $\Gamma$ are finite, and $\Gamma$ is fine, then $G$ is **hyperbolic relative to** $\{P\}$.

For a finite-degree $\text{CAT}(0)$ cube complex $X$, when is $\Gamma X$ fine?
When are the stabilizers of edges in $\Gamma X$ finite?
**Fine contact graphs**

**Theorem (H., 2010)**

\[ \Gamma X \text{ is fine if and only if } K_{2,\infty} \notin \Gamma. \]

Idea: first show that \( K_{2,\infty} \subset \Gamma \) if and only if some edge lies in infinitely many 3-cycles or 4-cycles. Then use disc diagrams to show that, if \( \Gamma \) is not fine, some edge lies in infinitely many 3-cycles or 4-cycles.

**Figure**: Above is a \( K_{2,\infty} \) in \( \Gamma X \). Below is the corresponding collection of hyperplanes. The purple edge lies in infinitely many 4-cycles.
G acts on \( \Gamma X \) with finite edge-stabilizers if and only if, for any two hyperplanes \( H, H' \),

\[
\text{stab}(H) \cap \text{stab}(H')^g
\]
is finite whenever these conjugates are distinct.

**Theorem (H., 2010)**

Let \( G \) act properly, cocompactly, and **unambiguously** on \( X \). Then the collection of hyperplane-stabilizers is almost-malnormal only if \( \Gamma X \) does not contain \( K_{2,\infty} \).

“Unambiguity” is a technical condition saying that, if \( H, H' \) are hyperplanes in distinct orbits, then \( \text{stab}(H) \) and \( \text{stab}(H') \) are not conjugate subgroups.
Hyperbolicity relative to ALL hyperplanes

Theorem (H., 2010)
Let $G$ act properly, cocompactly, and unambiguously on $X$. Then $G$ is hyperbolic relative to the collection of hyperplane-stabilizers if and only if the hyperplane-stabilizers form an almost-malnormal collection.

If $K_{2,\infty} \not\subseteq \Gamma X$, then $K_{\infty,\infty} \not\subseteq \Delta X$, so if $G$ is hyperbolic relative to the hyperplane-stabilizers, then $G$ is already word-hyperbolic, by the cubical flat plane theorem.

Questions:

1. What about hyperbolicity relative to some of the hyperplanes?

2. If $G$ acts properly and cocompactly on a cube complex, does it add properly, cocompactly, and unambiguously on a cube complex? The answer is “yes” if we drop “properly”.

Further contact graph applications

1. Bounded contact graphs: which contact graphs are quasi-points?
   The answer relates to:
   
   1.1 Divergence and divergence of geodesics in groups acting on cube complexes (generalizing Behrstock-Charney result on RAAGS).
   1.2 Rank-rigidity for cube complexes (reinterpreting a result of Caprace-Sageev).
   1.3 Leads to a combinatorial analogue of the Tits boundary for cube complexes with the $\ell^1$ metric.

2. Coloring contact graphs (joint work with V. Chepoi):
   
   2.1 Embeddings of cube complexes in finite products of trees.
   2.2 Partial solutions and counterexamples for the nice labeling problem in computer science.