COMMENT ON SECOND-COUNTABILITY AND METRISABILITY OF THE HHS BOUNDARY

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Abstract. Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and let $\partial\mathcal{X}$ be the HHS boundary. We recall the topology, and, for $\mathcal{X}$ proper, we explain why $\mathcal{X} \cup \partial\mathcal{X}$ is second-countable and deduce that $\partial\mathcal{X}$ is metrisable.

Throughout, theorem/section numbers refer to those in [DHS17].

1. Recalling the topology, and first-countability

Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. We refer to Section 2.1 for the definition of $\partial\mathcal{X}$ (as a set), and for the notion of a *support set*. Recall that $\overline{\mathcal{X}}$ denotes $\mathcal{X} \cup \partial\mathcal{X}$.

We now recall the topology. Fix $p = \sum_{U \in supp(p)} a_U^p p_U$. Fix $\epsilon > 0$. For each $U \in supp(p)$, let K_U be a neighbourhood of p_U in $\mathcal{C}U \cup \partial\mathcal{C}U$. Let $\mathcal{N}_{\{K_U\},\epsilon}(p) \subset \mathcal{X}$ be as in Definition 2.11.

The topology on $\overline{\mathcal{X}}$ is defined by analogy to the topology on the Gromov boundary of a hyperbolic space (see e.g. [BH99, p. 429] or [KB02, Definition 2.13]), as follows.

Let $\mathfrak{B}$ be the set of sets of the form $\mathcal{N}_{\{K_U\},\epsilon}(p)$, with $p \in \partial\mathcal{X}$ and K_U as above and $\epsilon > 0$.

Given $p \in \overline{\mathcal{X}}$ and a sequence (x_n) in $\overline{\mathcal{X}}$, we say that $x_n \to x$ as $n \to \infty$ if the following holds. Let $\mathcal{N}$ be either an open ball in $\mathcal{X}$ centred at p, or a set of the form $\mathcal{N}_{\{K_U : U \in supp(p)\},\epsilon}(p)$. Then $x_n \in \mathcal{N}$ for all but finitely many n.

A set $A \subset \overline{\mathcal{X}}$ is closed if for every sequence (a_n) in A that converges to some $p \in \overline{\mathcal{X}}$ in the above sense, $p \in A$. It is readily checked that this defines a topology on $\overline{\mathcal{X}}$.

Remark 1.1. $\partial\mathcal{X}$ is closed in $\overline{\mathcal{X}}$, since each boundary point is disjoint from each ball in $\mathcal{X}$.

Lemma 1.2. *Let $p = \sum_{U \in supp(p)} a_U^p p_U \in \partial\mathcal{X}$, let $supp(p) = \{U_1, \ldots, U_k\}$, for each i let $K_i \subset \mathcal{C}U_i \cup \partial\mathcal{C}U_i$ be a neighbourhood of p_{U_i}, and let $\epsilon > 0$. Then:*

(1) For each i, let $L_i \subset K_i$ be a neighbourhood of p_{U_i}, and let $\delta \in (0, \epsilon)$. Then

$$\mathcal{N}_{\{L_i\},\delta}(p) \subseteq \mathcal{N}_{\{K_i\},\epsilon}(p).$$

(2) For each i, there exists a sequence $\{K_n^i\}_n$ of neighbourhoods of p_{U_i} such that for all neighbourhoods L_i of p_{U_i} and all sufficiently large n, we have

$$\mathcal{N}_{\{K_n^i\}_{i=1}^k, \frac{1}{n}}(p) \subseteq \mathcal{N}_{\{K_i\},\epsilon}(p).$$

If (z_s) is a sequence converging to p, then $z_s \in \mathcal{N}_{\{K_n^i\}_{i=1}^k, \frac{1}{n}}(p)$ for all sufficiently large s.

Proof. Fix $p \in \partial\mathcal{X}$.

First statement: This is immediate from Definitions 2.8, 2.9, and 2.10: any point $q \in \partial\mathcal{X}$ or $x \in \mathcal{X}$ satisfying the conditions for inclusion in $\mathcal{N}_{\{L_i\},\delta}(p)$ satisfies those for inclusion in $\mathcal{N}_{\{K_i\},\epsilon}(p)$ just because each $L_i \subset K_i$ and $\delta < \epsilon$.

Second statement: In view of the first statement, we need to find, for each $U_i \in supp(p)$, a sequence of neighbourhoods K_n^i of p_{U_i} such that any other neighbourhood K of p_{U_i} contains K_n^i for all sufficiently large n. (We then choose n such that $1/n < \epsilon$ and n exceeds the minimal n required for $1 \leq i \leq k$, and apply the first statement.)

Date: December 8, 2020.

Fix a $(1, 10E)$–quasigeodesic ray γ joining $\pi_{U_i}(x_0)$ to p_{U_i} in $\mathcal{C}U_i$, using e.g. [KB02, Remark 2.16]. Let K_n^i be the set of points $z \in \mathcal{C}U_i \cup \partial\mathcal{C}U_i$ such that $(z|\gamma(n))_{\pi_{U_i}(x_0)} > n - 100E$. Note that $p \in K_n^i$ because for $m > n$, we have $(\gamma(m)|\gamma(n))_{\pi_{U_i}(x_0)} \geq n - 15E$.

Let L_i be an arbitrary neighbourhood of p_{U_i} in $\mathcal{C}U_i \cup \partial\mathcal{C}U_i$. Suppose that there are infinitely many values of n such that $K_n^i \not\subset L_i$. For each such n, choose $x_n \in K_n^i - L_i$. Since $(x_n|\gamma(n)) > n - 100E$, we have $x_n \to p_{U_i}$. Thus (x_n) is a sequence of points in $(\mathcal{C}U_i \cup \partial\mathcal{C}U_i) - L_i$ converging to $p_{U_i} \in L_i$, contradicting that L_i is a neighbourhood of p_{U_i}. Hence $K_n^i \subseteq L_i$ for all sufficiently large n, as required.

Finally, let $(u_s)_s$ be a sequence in $\mathcal{C}U_i \cup \partial\mathcal{C}U_i$ such that $u_s \to p_{U_i}$. By the definition of convergence in the Gromov boundary, we have $\liminf_{s,n}(u_s|\gamma(n))_{\pi_{U_i}(x)} = \infty$. By examining a $(1, 10E)$–quasigeodesic triangle with vertices $\pi_{U_i}(x_0), u_s, \gamma(n)$, we find that for all n there exists s_0 such that $u_s \in K_n^i$ for $s \geq s_0$. So K_n^i is a neighbourhood of p_{U_i} in $\mathcal{C}U_i \cup \partial\mathcal{C}U_i$.

Now let (z_s) be a sequence in $\overline{\mathcal{X}}$ such that $z_s \to p$ in $\overline{\mathcal{X}}$. Fix $n \in \mathbb{N}$. Since K_n^i is a neighbourhood of p_{U_i} for each i, convergence demands that $z_s \in \mathcal{N}_{\{K_n^i\},\frac{1}{n}}(p)$ for sufficiently large s. □

Remark 1.3. For each $\mathcal{N} \in \mathfrak{B}$ or $\mathcal{N}$ a ball in $\mathcal{X}$, let $\mathcal{N}'$ be the set of $q \in \mathcal{N}$ such that any $(z_n) \subset \overline{\mathcal{X}}$ with $z_n \to q$ has the property that $z_n \in \mathcal{N}'$ for all but finitely many n. Each $\mathcal{N}'$ is open, since any sequence in its complement converges, if at all, to a point in its complement. Lemma 1.2 implies the following:

- For all $p \in \partial\mathcal{X}$, there exists $\mathcal{N} \in \mathfrak{B}$ with $p \in \mathcal{N}'$. Let $\mathfrak{B}'(p)$ be the set of such $\mathcal{N}'$.
- For all $\mathcal{N}_1', \mathcal{N}_2' \in \mathfrak{B}'(p)$, there exists $\mathcal{N}_3' \in \mathfrak{B}'(p)$ such that $\mathcal{N}_3' \subset \mathcal{N}_1' \cap \mathcal{N}_2'$.
- For all $p, q \in \partial\mathcal{X}$ and all $\mathcal{N}_1' \in \mathfrak{B}'(p)$ and $\mathcal{N}_2' \in \mathfrak{B}'(q)$ and all $z \in \partial\mathcal{X}$ with $z \in \mathcal{N}_1' \cap \mathcal{N}_2' \cap \partial\mathcal{X}$, there exists $\mathcal{N}_3' \in \mathfrak{B}(z)$ such that $z \subset \mathcal{N}_3' \subset \mathcal{N}_1' \cap \mathcal{N}_2'$, and if $z \in \mathcal{N}_1' \cap \mathcal{N}_2' \cap \mathcal{X}$, there exists a ball about z contained in $\mathcal{N}_1' \cap \mathcal{N}_2'$.

So the open balls in $\mathcal{X}$, together with the subsets of the form $\mathcal{N}'$, form a neighbourhood system on $\overline{\mathcal{X}}$. A subset A is open in the resulting topology if and only if for each $a \in A$, there exists $\mathcal{N}'$ in the neighbourhood system such that $a \in \mathcal{N}' \subset A$. A routine check shows that this is the same as the topology defined above: if A is open in the above sense, then it is a union of sets $\mathcal{N}'$, which are open in the original topology. Conversely, suppose that A is open in the original topology, and let $a \in A$. If all of the neighbourhoods $\mathcal{N}_{\{K_n^i\},\frac{1}{n}}(a)$ from Lemma 1.2 contain points not in A, then we have a sequence in $\overline{\mathcal{X}}$ converging to a, contradicting that A is open.

Remark 1.4 (Neighbourhoods and interiors)**.** Fix $p \in \partial\mathcal{X}$, and let $\mathcal{N} = \mathcal{N}_{\{K_i\},\epsilon}(p)$. It follows from Lemma 1.2 that $p \in \mathcal{N}'$, so p is an interior point of $\mathcal{N}$, and $\mathcal{N}$ is really a neighbourhood of p (see also Section 2 of [DHS17]). Moreover, let A be an arbitrary subset of $\overline{\mathcal{X}}$ and let $p \in A$. Suppose that for all sequence (p_n) converging to p, we have $p_n \in A$ for all sufficiently large n. Using the sequence of neighbourhoods in Lemma 1.2, it follows that some $\mathcal{N}' \in \mathfrak{B}'(p)$ is contained in A. Hence p is an interior point of A. We will use this sequential characterisation of interior points throughout.

Proposition 1.5 (Neighbourhood basis, first countable)**.** *For each $p \in \partial\mathcal{X}$, the set of $\mathcal{N}_{\{K_i\},\epsilon}(p)$ (where $\epsilon > 0$ and $\{K_i\}$ varies over the neighbourhoods of p_{U_i} for each i) forms a neighbourhood basis for the topology of $\overline{\mathcal{X}}$ at p. Moreover, $\overline{\mathcal{X}}$ is first-countable.*

Proof. Fix p and let A be an open set in $\overline{\mathcal{X}}$ containing p. As explained above, some neighbourhood $\mathcal{N}_n = \mathcal{N}_{\{K_n^i\},\frac{1}{n}}(p)$ provided by Lemma 1.2 is contained in A. So these neighbourhoods form a neighbourhood basis at p, and there are countably many of them. So $\partial\mathcal{X}$ is first-countable. First countability of the whole of $\overline{\mathcal{X}}$ then follows because the subspace topology on $\mathcal{X}$ is just the metric topology. □

In Section 2, using elements of $\mathfrak{B}$ only (via the above definition of convergence), it is shown that $\overline{\mathcal{X}}$ is Hausdorff, and $\mathcal{X}$ is dense in $\overline{\mathcal{X}}$.

2. Second-countability and metrisability in the proper case

We now assume that $\mathcal{X}$ is a proper metric space. Our goal is to explain the following fact, mentioned in the introduction to the paper:

Proposition 2.1 (Metrisability). *If $(\mathcal{X}, \mathfrak{S})$ is an HHS with $\mathcal{X}$ a proper space, then $\partial\mathcal{X}$ is metrisable.*

Proof. As mentioned above, $\partial\mathcal{X}$ is Hausdorff. We will show below (Lemma 2.5) that $\overline{\mathcal{X}}$ is second-countable (here is one place where properness is needed). In Section 3, it is shown (again using properness of $\mathcal{X}$) that $\overline{\mathcal{X}}$ is compact (what is verified in detail is sequential compactness, which is equivalent to compactness for second-countable spaces, see e.g. [BB13, Proposition 1.6.23]). Since $\partial\mathcal{X}$ is closed, it is also compact, and is second-countable since $\overline{\mathcal{X}}$ is. As a compact Hausdorff space, $\partial\mathcal{X}$ is regular, so the Urysohn metrisation theorem (see e.g. [Mun00, Theorem 34.1]) implies $\partial\mathcal{X}$ is metrisable. □

It remains to show that $\overline{\mathcal{X}}$ is second-countable.

Lemma 2.2. *Let $\mathfrak{S}_1$ be the set of $U \in \mathfrak{F}$ such that $\partial\mathcal{C}U \neq \emptyset$. Then $\mathfrak{S}_1$ is countable. Hence the set $\mathfrak{T}$ of support sets for points in $\partial\mathcal{X}$ is countable.*

Proof. Recall from Section 1.4 that there exists $C < \infty$ such that each map π_U is C–coarsely surjective. Using properness, choose a countable set $\{x_n\}_n \subset X$ such that the 1–balls about the points x_n cover $\mathcal{X}$. After enlarging C uniformly, each map π_U is C–coarsely surjective when restricted to $\{x_n\}$.

Fix a basepoint $x_0 \in \mathcal{X}$ and let $U \in \mathfrak{S}_1$. There exists R, depending on C and the HHS constants only and larger than the distance formula threshold, such that $\mathsf{d}_U(x_0, x_n) > R$ for some x_n. For the given x_n, there are only finitely many such U. So $\mathfrak{S}_1$ is contained in a countable union (indexed by x_n) of finite sets. The final assertion then follows from the bounded cardinality of support sets. □

For each $U \in \mathfrak{S}$, we have a uniformly coarsely dense subset $\{\pi_U(x_n)\}_{n\in\mathbb{N}}$. For simplicity, we can and shall assume E is large enough that this set is E–coarsely dense.

If $q \in \mathcal{C}U \cup \partial\mathcal{C}U$ and $R \geq 0$, let $K(q, R)$ be the set of $z \in \mathcal{C}U \cup \partial\mathcal{C}U$ such that $(z|q)_{\pi_U(x_0)} > R$.

Lemma 2.3. *Let $U \in \mathfrak{S}_1$. There exists a countable set $\mathfrak{B}_0(U)$ of neighbourhoods in $\mathcal{C}U \cup \partial\mathcal{C}U$ such that the following holds. Let $p \in \partial\mathcal{C}U$ and let $\mathcal{N}$ be a neighbourhood in $\mathcal{C}U \cup \partial\mathcal{C}U$ of p. Then there exists $\mathcal{N}_0 \in \mathfrak{B}_0(U)$ such that:*

- *$p \in \mathcal{N}_0$;*
- *$\mathcal{N}_0 \subset \mathcal{N}$;*
- *if $(z_n) \subset \mathcal{C}U \cup \partial\mathcal{C}U$ converges to p, then $z_n \in \mathcal{N}_0$ for all but finitely many n.*

Proof. By shrinking $\mathcal{N}$, we can assume $\mathcal{N} = K(p, R)$ for some $R \geq 0$. Let γ be a $(1, 10E)$–quasigeodesic ray from $\pi_U(x_0)$ to p. As in Lemma 1.2, let $K_n = K(p_n, n - 100E)$, where $p_n \in \pi_U(\mathcal{X})$ is E–close to $\gamma(n)$.

Note that K_n is determined by the natural number n and the point $p(n)$, i.e. it is independent of p and γ. Since $\pi_U(\mathcal{X})$ is countable, K_n is therefore one of countably many subsets of $\mathcal{C}U \cup \partial\mathcal{C}U$.

As in the proof of Lemma 1.2, there exists n_0 such that for all $n \geq n_0$, we have $p \in K_n \subseteq \mathcal{N}$. Similarly, if (z_s) converges to p, then $z_s \in K_n$ for all sufficiently large s. □

We now construct a countable set $\mathfrak{B}_0$ of neighbourhoods in $\mathcal{X} \cup \overline{\mathcal{X}}$.

(1) Let $\mathfrak{T}$ be the countable set of subsets of $\mathfrak{S}_1$ that are support sets for boundary points.

(2) Fix $\{U_i\}_{i=1}^k \in \mathfrak{T}$, let $\mathfrak{B}(U_i)$ be as in Lemma 2.3. For each $K \in \mathfrak{B}(U_i)$, choose $q_i \in K \cap \partial\mathcal{C}U_i$ so that K is a neighbourhood of q_i. (This is possible by the construction in the proof of Lemma 2.3.)
(3) For each i, we thus have a countable collection of points $q_i \in \partial\mathcal{C}U_i$, and a countable collection of neighbourhoods $K(q_i)$. Each q_i has one of the $K(q_i)$ as a neighbourhood.
(4) For the given $\{U_i\}$, we thus have countably many points $\sum_{i=1}^k \alpha_i q_i \in \partial\mathcal{X}$, where $(\alpha_1, \dots, \alpha_k) \in (\mathbb{Q}\cap(0,1])^k$ and $\sum_i \alpha_i = 1$. We also have countably many sets $\{K(q_i)\}_{i=1}^k$ of neighbourhoods, with each $K(q_i) \in \mathfrak{B}(U_i)$. Letting $t > 0$ vary in $\mathbb{Q}$, we thus have countably many neighbourhoods of the form
$$\mathcal{N}_{\{K(q_i)\},t}(\sum_i \alpha_i q_i)$$
in $\overline{\mathcal{X}}$ associated to each support set in $\mathfrak{T}$.
(5) Taking the union over $\mathfrak{T}$ gives a countable set of neighbourhoods of the above form, which we denote by $\mathfrak{B}_0$.

Fix $p \in \partial\mathcal{X}$, and let $supp(p) = \{U_1, \dots, U_k\}$. Write $p = \sum_{i=1}^k a_i^p p_{U_i}$. Fix $\epsilon > 0$ and for each $i \leq k$ let $L_i \subset \mathcal{C}U_i \cup \partial\mathcal{C}U_i$ be a neighbourhood of p_{U_i}. Let $\mathcal{N} = \mathcal{N}_{\{L_i\},\epsilon}(p)$.

Let $\alpha_1, \dots, \alpha_k \in \mathbb{Q} \cap (0,1]$ and $t \in (0,\epsilon) \cap \mathbb{Q}$ be constants to be chosen (in terms of ϵ and the coefficients a_i^p). We will require that $\sum_i \alpha_i = 1$.

For $i \leq k$, choose a neighbourhood $K(q_i) \in \mathfrak{B}_0(U_i)$ in $\mathcal{C}U_i \cup \partial\mathcal{C}U_i$ satisfying the conclusion of Lemma 2.3 with respect to the point p_{U_i} and the neighbourhood L_i. Let $q = \sum_i \alpha_i q_i \in \partial\mathcal{X}$, so $\mathcal{N}_0 = \mathcal{N}_{\{K(q_i)\},t}(q) \in \mathfrak{B}_0$.

Lemma 2.4. *The constants $t, \alpha_1, \dots, \alpha_k$ can be chosen so that:*
(1) $p \in \mathcal{N}_0$.
(2) $\mathcal{N}_0 \subset \mathcal{N}$.
(3) If $(p_n) \subset \overline{\mathcal{X}}$ converges to p, then all but finitely many p_n belong to $\mathcal{N}_0$.

Proof. **Proof of item** (1): By definition, p is a boundary point, and $supp(p) = supp(q) = \{U_i\}$, so p is *non-remote* with respect to q. By construction, for each i, we have $p_{U_i} \in K(q_i)$. So, if
$$|\alpha_i - a_i^p| < t < \epsilon$$
for all i (call this **Constraint I**), then $p \in \mathcal{N}_0$, by Definition 2.9.

Proof of item (2): Let $z \in \mathcal{N}_0$. We need to show that $z \in \mathcal{N}$. Note that if z is a boundary point, it is remote/non-remote for p if and only if it is remote/non-remote for q, because $supp(p) = supp(q)$.

Suppose that z is non-remote for q. Then for each $U_i \in supp(z)$, we have $z_{U_i} \in K(q_i) \subset L_i$. For each such U_i, we also have $|a_{U_i}^z - \alpha_i| < t$, so $|a_{U_i}^z - a_i^p| \leq t + |a_i^p - \alpha_i| < \epsilon$ as long as $|a_i^p - \alpha_i| < \epsilon - t$. Call this **Constraint II**. Finally, we have
$$\sum_{V \in supp(z) - supp(q)} a_V^z < t$$
since $z \in \mathcal{N}_0$. So, since $t < \epsilon$ and $supp(p) = supp(q)$, we have $\sum_V a_V^z < \epsilon$. Hence, by Definition 2.9, $z \in \mathcal{N}$.

Suppose that z is remote for q (and hence for p). Then since $t < \epsilon$, and $K(q_i) \subset L_i$, Definition 2.8 shows that $z \in \mathcal{N}$ provided that, for all $i \neq j$, we have
$$\left| \frac{\alpha_i}{\alpha_j} - \frac{a_i^p}{a_j^p} \right| < \epsilon - t.$$
Call this **Constraint III.**

Finally, suppose that $z \in \mathcal{X}$. Then for all i, $\pi_{U_i}(z) \in K(q_i) \subset L_i$ since $z \in \mathcal{N}_0$. If $T \perp supp(q)$, then $\mathsf{d}_T(x_0, x)/\mathsf{d}_{U_i}(x_0, x) < t < \epsilon$. So by Definition 2.10, $z \in \mathcal{N}$ if **Constraint II** holds.

Choice of t and $\alpha_1, \ldots, \alpha_k$: First, choose $t \in (0, \epsilon/2) \cap \mathbb{Q}$, so any set of α_i satisfying Constraint I will satisfy Constraint II. For later purposes, we can and shall also require that t is chosen so that $a_j^p - t > a_j^p/2$ for $1 \leq j \leq k$, since $a_j^p > 0$ because $U_j \in supp(p)$.

Let $\eta = \max_{1 \leq i \leq k} \frac{1}{a_i^p} \in [1, \infty)$. Let $s \in \left(0, \frac{t}{\eta + 2\eta^2}\right)$. Choose $\alpha_1, \cdots \alpha_k \in (0, 1] \cap \mathbb{Q}$ so that $|\alpha_i - a_i^p| < s$ for each i and $\sum_i \alpha_i = 1$. Since $s < t$, Constraint I, hence Constraint II, holds. The following computation verifies Constraint III for $1 \leq i \leq j \leq j$:

$$\begin{aligned}
\left|\frac{a_i^p}{a_j^p} - \frac{\alpha_i}{\alpha_j}\right| &\leq \left|\frac{a_i^p}{a_j^p} - \frac{\alpha_i}{a_j^p}\right| + \alpha_i \left|\frac{1}{a_j^p} - \frac{1}{\alpha_j}\right| \\
&\leq s\eta + \alpha_i \left|\frac{\alpha_j - a_j^p}{\alpha_j a_j^p}\right| \\
&\leq s\eta + 2s\eta^2 < t < \epsilon.
\end{aligned}$$

So, with the given choices of t and $\alpha_1, \ldots, \alpha_k$, we have $p \in \mathcal{N}_0 \subset \mathcal{N}$.

Proof of item (3): Let $\bar{p}_n^i \in \mathcal{C}U_i \cup \partial\mathcal{C}U_i$ be the projection of p_n to $\mathcal{C}U_i$, so $p_i^n \to p_{U_i}$ for each i. The neighbourhood $K(q_i)$ of p_{U_i} was chosen so that all but finitely many of the p_n^i lie in $K(q_i)$, for each i. So to show that p_n is eventually in $\mathcal{N}_0$, it remains to verify the "quantitative" parts of Definitions 2.8, 2.9, 2.10.

Examining Definition 2.8, 2.9, or 2.10, there are finitely many quantities (involving e.g. the a_i^p and the coefficients of a given p_i^n), each of which can be made arbitrarily small by increasing n since $p_n \to p$. The goal is to make the corresponding quantities, with a_i^p replaced by α_i, smaller than t. We will do this for Definition 2.8; the other arguments are identical.

Fix i, j. For all sufficiently large n (depending on i and j), we have

$$\left|\frac{\mathsf{d}_{U_i}(x_0, \partial\pi_{U_j}(p_n))}{\mathsf{d}_{U_j}(x_0, \partial\pi_{U_j}(p_n))} - \frac{a_i^p}{a_j^p}\right| < t - \left|\frac{\alpha_i}{\alpha_j} - \frac{a_i^p}{a_j^p}\right|,$$

and so

$$\left|\frac{\mathsf{d}_{U_i}(x_0, \partial\pi_{U_j}(p_n))}{\mathsf{d}_{U_j}(x_0, \partial\pi_{U_j}(p_n))} - \frac{\alpha_i}{\alpha_j}\right| < t,$$

as required. So, for sufficiently large n, the latter inequality holds for all $i, j \leq k$. Second, for all sufficiently large n, we have $\sum_{T \perp supp(p)} a_T^{p_n} < t$, since $p_n \to p$, and this is the condition needed for $p_n \in \mathcal{N}_0$ since $supp(q) = supp(p)$. Thus $p_n \in \mathcal{N}_0$ for sufficiently large n, proving (3). □

Lemma 2.5 (Second countable)**.** *$\overline{\mathcal{X}}$ is second-countable.*

Proof. Let $\mathfrak{B}_0'$ be the set interiors of sets in $\mathfrak{B}_0$. Since $\mathfrak{B}_0$ is countable, so is $\mathfrak{B}_0'$. Let $p \in \partial\mathcal{X}$ and let $A \subset \overline{\mathcal{X}}$ be an open set containing p. Then A contains a neighbourhood $\mathcal{N} = \mathcal{N}_{\{L_i\},\epsilon}(p)$. Lemma 2.4 provides $\mathcal{N}_0 \in \mathfrak{B}_0$ such that $p \in \mathcal{N}_0 \subseteq \mathcal{N} \subseteq A$ and the three enumerated conditions hold. The interior of $\mathcal{N}_0$ is contained in A. By item (3) above, any sequence converging to p has all but finitely many terms in $\mathcal{N}_0$, so $p \in Int(\mathcal{N}_0)$. Hence $A \cap \partial\mathcal{X}$ is a union of sets of the form $C \cap \partial\mathcal{X}$, where $C \in \mathfrak{B}_0'$. Since $\mathcal{X}$ is a proper metric space, it is second-countable, so by adding to $\mathfrak{B}_0'$ a countable basis for the topology on $\mathcal{X}$, we get one for $\overline{\mathcal{X}}$. □

References

[BB13] Ronald Beattie and Heinz-Peter Butzmann. *Convergence structures and applications to functional analysis.* Springer Science & Business Media, 2013.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[DHS17] Matthew Durham, Mark Hagen, and Alessandro Sisto. Boundaries and automorphisms of hierarchically hyperbolic spaces. *Geometry & Topology*, 21(6):3659–3758, 2017.

[KB02] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93. Amer. Math. Soc., Providence, RI, 2002.

[Mun00] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128].