UNIFORM UNDISTORTION FROM BARYCENTRES, AND APPLICATIONS TO HIERARCHICALLY HYPERBOLIC GROUPS

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Abstract. We show that infinite cyclic subgroups of groups acting uniformly metrically properly on injective metric spaces are uniformly undistorted. In the special case of hierarchically hyperbolic groups, we use this to study translation lengths for actions on the associated hyperbolic spaces. Then we use quasimorphisms to produce examples where these latter results are sharp.

Contents

1. Introduction

An important descriptor of an isometric action of a group G on a metric space (X, d) is the *(stable) translation length* $\tau_X : G \to [0, \infty)$, given by

$$\tau_X(g) = \lim_{n\to\infty} \frac{\mathsf{d}(x, g^n x)}{n},$$

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which is well defined and independent of the choice of $x \in X$ by the triangle inequality. Translation lengths were first considered for X a Riemannian manifold, but the notion arises frequently when considering actions on more general spaces. The many well-known results where translation lengths play an important role include the work of Gromoll–Wolf on actions on nonpositively-curved manifolds [GW71] and work of Gersten–Short on biautomaticity [GS91].

While the precise values of the function τ_X depend on the choices of both X and the action, many properties of the image of τ_X are not. For instance, if G acts isometrically on X and Y and they admit a G–equivariant (K, K)–quasi-isometry, then $\frac{1}{K}\tau_X(g) \leqslant \tau_Y(g) \leqslant K\tau_X(g)$ for any $g \in G$. One consequence is that the set of loxodromic isometries (those $g \in G$ for which $\tau_X(g) > 0$) is invariant depends only on the G–equivariant quasi-isometry type of X.

A G–action on X is *translation discrete* if there is a constant $\tau_0 > 0$ such that, for each $g \in G$, either $\langle g\rangle$ has bounded orbits in X (in particular, $\tau_X(g) = 0$), or $\tau_X(g) \geqslant \tau_0$. Translation discreteness is also invariant under G–equivariant quasi-isometries. In this paper, we consider translation discreteness in two main settings: for X a proper Cayley graph; and for (improper) actions on hyperbolic spaces arising in the context of hierarchically hyperbolic groups.

1.1. Translation discreteness in Cayley graphs

The first context in which we study translation discreteness is when G is a finitely generated group, and $X = G$ is equipped with a word-metric coming from a finite generating set. In this setting, $\tau_G(g) > 0$ means that $\langle g\rangle$ is an *undistorted* infinite cyclic subgroup of G, i.e., the map $\mathbb{Z} \to G$ given by $n \mapsto g^n$ is a quasi-isometric embedding. Translation discreteness of the action of G on itself therefore equates to infinite cyclic subgroups being *uniformly* undistorted.

Well-known examples of finitely generated groups that contain *distorted* infinite cyclic subgroups include the Baumslag–Solitar groups $BS(p, q)$ with $|p| \neq |q|$ and virtually nilpotent groups that are not virtually abelian [DK18, Lem. 14.15], among many others. In the other direction, all infinite cyclic subgroups are undistorted when G is hyperbolic [Gro87, Cor. 8.1.D], CAT(0) [BH99, Prop. 6.10], or, more generally, semihyperbolic [AB95, Thm 7.1]. Hence this property is often considered a form of "coarse non-positive curvature".

In many examples of interest, the stronger property of having uniformly undistorted infinite cyclic subgroups holds. This is the case, for example, in hyperbolic groups [Gro87], CAT(0) groups [Con00b], Garside groups [HO21b], Helly groups [HO21a], groups satisfying various (graphical) small-cancellation conditions [ACGH19], mapping class groups [Bow08], and $\mathrm{Out}(F_n)$ [Ali02]. This property is often established directly by constructing uniform-quality quasi-axes in some space on which G acts. A beautiful example is Haglund's construction of combinatorial axes for loxodromic isometries of CAT(0) cube complexes [Hag07], which generalises the classical case of trees [Ser03].

Uniform undistortion of infinite cyclic subgroups has stronger consequences than mere undistortion does. For example, if G has uniformly undistorted infinite cyclic subgroups, then solvable subgroups of G with finite virtual cohomological dimension must be abelian [Con00a], and finitely generated abelian subgroups of G are undistorted (though not necessarily uniformly) [But19].

Remark 1.1 (Non-examples and blended behaviour)**.** Many of the best-known examples of groups G with *distorted* cyclic subgroups exhibit a sort of blended behaviour: there exists $\tau_0 > 0$ such that for all $g \in G$, either $\tau_G(g) = 0$ or $\tau_G(g) \geqslant \tau_0$. For example, if G is the integer Heisenberg group $\langle x, y, z \mid [x, z] = [y, z] = 1, [x, y] = z\rangle$, then $\tau_G(z^n) = 0$ for all $n \in \mathbb{Z}$, but every other element has nontrivial image under the 1–Lipschitz epimorphism $G \to \mathbb{Z}^2$ given by $x \mapsto (1, 0), y \mapsto (0, 1), z \mapsto (0, 0)$, whence $\tau_G(g) \geqslant 1$ for all $g \in G - \langle z\rangle$. Similar behaviour is exhibited by the Baumslag-Solitar group $G = \langle a \rangle *_{a^b = a^2}$: the function τ_G vanishes on $\langle a \rangle$, but by considering axes in the Bass–Serre tree of the given splitting, one sees that τ_G is bounded away from 0 on $G - \langle a \rangle$.

Nevertheless, there do exist groups G where $\tau_G(g) > 0$ for all $g \in G - \{1\}$, but τ_G takes arbitrarily small values. This was investigated in detail by Conner [Con00a], who analysed translation discreteness in solvable groups. For example, Conner gives an explicit linear map $M \in GL_4(\mathbb{Z})$ such that all cyclic subgroups of the group $G = \mathbb{Z}^4 \rtimes_M \mathbb{Z}$ are undistorted, but not uniformly [Con00a, Eg. 7.1].

Examining the list of translation discrete examples, one finds a large overlap with the class of semihyperbolic groups, which have undistorted cyclic subgroups as noted above. However, it appears to be an open question whether all semihyperbolic groups are translation discrete. Our first theorem is a positive result in this direction. See Section 2 for the definition of an injective metric space.

Theorem 1.2. *Let G be a group acting metrically properly and coboundedly on an injective metric space X. There exists $\tau_0 > 0$ such that $\tau_G(g) \geqslant \tau_0$ for all infinite-order $g \in G$, i.e., G is translation discrete. Moreover, there exists $q > 0$ such that each infinite-order $g \in G$ admits a q–quasi-axis in G.*

This theorem is established as follows. As explained in Section 2, the injective space X *has barycentres* in the sense of Definition 2.1. Translation discreteness is thus a consequence of Proposition 2.8, and the statement about quasi-axes follows from Proposition 2.11.

The results in Section 2 are more general than the above statement, in terms of the action and in terms of the space X; the reader is referred there for the precise statements. In particular, they engage more directly with the above question about semihyperbolic groups by covering groups with *reversible, conical* bicombings, a refinement of semihyperbolicity.

We emphasise that Theorem 1.2 does not require the space X to be proper. If X is proper, then the statement about quasi-axes can be strengthened: each infinite-order $g \in G$ is semisimple by [BH99, Thm II.6.10], hence has positive translation length by [Lan13, Prop. 1.2] and so has a geodesic axis. In particular, translation discreteness was already known for groups acting geometrically on proper injective spaces. This includes Helly groups [CCG$^+$20, Thm 6.3] and cocompactly cubulated groups, using that finite-dimensional CAT(0) cube complexes are equivariantly bilipschitz-equivalent to injective spaces [Bow20].

Theorem 1.2 applies to the class of *hierarchically hyperbolic groups (HHGs)*, which were introduced in [BHS17, BHS19] as a common generalisation of mapping class groups and fundamental groups of compact special cube complexes, and which act properly and coboundedly on injective spaces [HHP20]. We give background on hierarchical hyperbolicity in Section 3, and more discussion appears in the next subsection. In Proposition 3.10, we establish:

Corollary 1.3 (HHGs are translation discrete)**.** *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. There exists $\tau_0 > 0$ such that $\tau_G(g) \geqslant \tau_0$ for all infinite-order $g \in G$.*

It was shown in [DHS17, DHS20] that HHGs have undistorted infinite cyclic subgroups; Corollary 1.3 strengthens this. From Corollary 1.3 and [But19, Thm 4.2], one recovers the fact that hierarchically hyperbolic groups have undistorted abelian subgroups, previously established in [HHP20, Cor. H] and [HRSS22, Prop. 2.17].

1.2. Non-proper actions arising from hierarchical hyperbolicity

Many well-known examples illustrate the importance of actions on hyperbolic spaces under hypotheses weaker than properness. Questions about translation lengths in this context have a long history, beginning with translation discreteness of hyperbolic groups on their Cayley graphs [Gro87, Swe95, Del96]. Actions of mapping class groups on curve graphs are important guiding examples, and Bowditch established translation discreteness for pseudo-Anosovs, as a consequence of acylindricity of the action [Bow08]. There has been considerable subsequent study of the translation length spectrum for mapping class groups and analogous examples, e.g., [BSS23, AT17, BW16, Man13, Gen22].

After mapping class groups, the most relevant examples for our present purposes are quasi-trees associated to right-angled Artin groups (the *extension graph* [KK13, KK14]) and CAT(0) cube complexes more generally (the *contact graph* [Hag14]). The action of a cubulated group G on the contact graph $\mathcal{C}X$ of the cube complex X is WPD [BHS17] but need not be acylindrical [She22], and the set of positive translation lengths need not be bounded away from zero [Gen22]. However, when X admits a *factor system* as in [BHS17], which happens under a variety of hypotheses [HS20], including when the action of G is cospecial [HW08], the action on $\mathcal{C}X$ is acylindrical, and nonzero translation lengths are therefore bounded below. These examples are discussed in more detail in Section 5.

Mapping class groups and compact special groups (including right-angled Artin groups) are early examples of hierarchically hyperbolic groups, the class of which has now been considerably enlarged; see, e.g., [BHS19, BR20a, RS20, BR20b, Che22, BHMS20, HMS21, HRSS22]. A hierarchically hyperbolic group is, by definition, equipped with a cofinite G–set $\mathfrak{S}$ in which each element U is associated to a hyperbolic space $\mathcal{C}U$ in such a way that G acts by isometries on the extended metric space $\prod_{U\in\mathfrak{S}}\mathcal{C}U$, permuting factors. In particular, $\mathrm{Stab}_G(U)$ acts by isometries on $\mathcal{C}U$, and one can ask about the translation length spectra for these actions.

In [DHS17], it is shown that each $g \in G$ has an associated uniformly finite subset $\mathrm{Big}(g) \subset \mathfrak{S}$ such that, after replacing g by a uniform power, each $U \in \mathrm{Big}(g)$ is invariant under the action of $\langle g\rangle$, which moreover has unbounded orbits in $\mathcal{C}U$. In [DHS20], it is shown that $\tau_{\mathcal{C}U}(g) > 0$ for each $U \in \mathrm{Big}(g)$. It follows from this that $\tau_G(g) > 0$. However, the argument in [DHS20] cannot be adapted to yield a *uniform* lower bound on $\tau_{\mathcal{C}U}(g)$ for all $U \in \mathrm{Big}(g)$. In this paper, the logic is reversed: to study $\tau_{\mathcal{C}U}(g)$, we *start* with the lower bound on $\tau_G(g)$ provided by Corollary 1.3, and use this to analyse $\tau_{\mathcal{C}U}(g)$ for $U \in \mathrm{Big}(g)$.

One cannot hope for a global statement to the effect that $\mathrm{Stab}_G(U)$ has discrete translation length spectrum on $\mathcal{C}U$. This is illustrated by the following theorem, proved in Section 5.

Theorem 1.4. *Let G be an infinite hierarchically hyperbolic group that is elementary or acylindrically hyperbolic. Let $[\alpha] \in H^2(G,\mathbb{Z})$ be representable by a bounded cocycle, and let E_α be the corresponding $\mathbb{Z}$–central extension of G.*

The extension E_α admits a hierarchically hyperbolic structure $(E_\alpha, \mathfrak{S})$ such that the following holds for some E_α-invariant $A \in \mathfrak{S}$. For all $\epsilon > 0$, there exists $g \in E_\alpha$ such that $A \in \mathrm{Big}(g)$ and $\tau_{\mathcal{C}A}(g) \in (0, \epsilon)$.

The theorem applies, for example, to any central extension of any infinite hyperbolic group G, since $H^2_b(G,\mathbb{Z}) \to H^2(G,\mathbb{Z})$ is onto in that case [Min02]. Even $\mathbb{Z}^2$ admits hierarchically hyperbolic structures involving arbitrarily small translation lengths; see Example 5.2. Such examples establish sharpness of the following result, which covers all HHG structures on a given group. To state it, we recall that there is a constant n such that $g^n \in \mathrm{Stab}_G(U)$ for any $g \in G$ and any $U \in \mathrm{Big}(g)$.

Theorem 1.5. *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. There exists $\tau_0 > 0$, depending only on G and $\mathfrak{S}$, such that for any infinite-order $g \in G$, there exists $U \in \mathrm{Big}(g)$ such that $\tau_{\mathcal{C}U}(g^n) \geqslant \tau_0$.*

Theorem 1.5 does not assert that the action of any particular $\mathrm{Stab}_G(U)$ on $\mathcal{C}U$ is translation discrete, and Theorem 1.4 rules out the possibility of replacing "there exists $U \in \mathrm{Big}(g)$" by "for all $U \in \mathrm{Big}(g)$" in Theorem 1.5.

On the other hand, Theorem 1.5 does give enough control of translation lengths to prove strong results. For example, Theorem 1.5 plays a role in recent work of Abbott–Ng–Spriano and Gupta–Petyt on uniform exponential growth in HHGs and many cubulated groups [ANS+19]. It is also used in forthcoming work by Zalloum [Zal23] where the author shows that in any irreducible virtually torsion-free HHG, one can find a Morse element (and more generally, free stable subgroups) uniformly quickly. It is also related to a subtle point about *product regions* and their coarse factors, see [DHS20, §2] and [CHK22, §15].

1.3. Questions

To find a uniform quasi-axis for an element g of an HHG $(G, \mathfrak{S})$, we used the G–action on an injective space. This is similar to how bicombings on injective spaces are used in [HHP20] to produce equivariant bicombings on G. As noted in that paper, it is unknown whether those bicombing quasigeodesics are *hierarchy paths* for the given HHG structure, and we can ask the same about the quasi-axes:

Question 1.6. *Fix a hierarchically hyperbolic group $(G, \mathfrak{S})$ and a word metric on G. Does there exist a constant D such that for all infinite order $g \in G$, there is a $\langle g \rangle$–invariant D–quasi-axis projecting to an unparametrised (D, D)–quasigeodesic in $\mathcal{C}U$ for all $U \in \mathfrak{S}$?*

The content of the above question is whether D can be chosen independently of g. One could also ask a more general version of the question, in which $\mathbb{Z}$ subgroups are replaced by $\mathbb{Z}^n$ subgroups. Again, it is known that any such subgroup stabilises a *hierarchically quasiconvex* subspace F that is quasi-isometric to $\mathbb{R}^m$ for some $m \geqslant n$ [HRSS22, Prop. 2.17], but the hierarchical quasiconvexity parameters may depend on the choice of subgroup. Since the active ingredient in the proof of Corollary 1.3 is the action on a space with barycentres, a starting point could be the following.

Question 1.7. *Let G act properly and coboundedly on a space X with barycentres. Are the $\mathbb{Z}^n$ subgroups of G uniformly undistorted for each $n \geqslant 2$?*

Button showed in [But19] that *uniform* undistortion of $\mathbb{Z}$ subgroups implies (not *a priori* uniform) undistortion of $\mathbb{Z}^n$ subgroups; the question asks whether the implicit constants can be made uniform, at least for each fixed choice of n. Similarly to the cyclic case, proving uniform undistortion of abelian subgroups generally requires establishing some form of a *flat-torus* theorem [GW71, LY72, BH99], as suggested by Question 2.12 and by a very general result of this type due to Descombes–Lang [DL16]. We discuss the latter result, and prove some statements effectivising Button's result, in Section 2. However, it seems plausible that the answer to Question 1.7 and hence to Question 2.12 is negative in the given generality, and counterexamples would be very interesting.

Next, we mentioned earlier the results of Genevois and Shepherd about the spectrum of translation-lengths for the action of a cocompactly cubulated group on the contact graph $\mathcal{C}X$ of the CAT(0) cube complex X. A natural question is:

Question 1.8. *Let G act properly and cocompactly on the CAT(0) cube complex X, and suppose that $\tau_{\mathcal{C}X}$ is bounded away from 0 on loxodromic elements of G. Is the induced action on $\mathcal{C}X$ acylindrical?*

In other words, is inapplicability of Bowditch's translation-length result [Bow08] the only obstacle to acylindricity? One can ask analogous questions in the more general context of *quasimedian graphs*, studied in [Gen22], or for the *curtain models* studied in [PSZ22].

We mention a few additional avenues for research suggested by these results. First, one could try to articulate conditions on a group G with a proper cobounded action on an injective space X such that the spectrum of stable translation lengths on X determines X up to G–equivariant isometry, possibly among actions on injective spaces in some restricted class. One could also attempt to characterise HHG structures up to some natural equivalence by translation length spectra. The aim would be a useful notion of the "space of injective/HHG structures" for a given G. Some motivation for this idea comes from the marked ℓ^1–length spectrum rigidity result for certain classes of actions on cube complexes [BF21].

For many finitely generated groups G, more delicate results regarding the spectrum of translation lengths are known. For instance, when G is hyperbolic, there is an integer N that depends only on the hyperbolicity constant of a given Cayley graph where all infinite order elements g satisfy $\tau_G(g) \in \frac{1}{N}\mathbb{Z}$. More generally, the same statement holds for *M-Morse* elements of any *Morse local-to-global* group G (see [RST22] for the definition). Namely, for each Morse gauge M there exists $N \in \mathbb{Z}$ such that if $g \in G$ is M–Morse, then $\tau_G(g) \in \frac{1}{N}\mathbb{Z}$ provided that G is Morse local-to-global [RST22]. Thus, given a group G which is translation discrete, one could investigate whether the translation lengths $\tau_G(g)$ are rational for all $g \in G$. For example, it is not known whether the translation lengths (on the Cayley graph) of all elements in the mapping class group are rational.

Finally, we refer the reader to Section 5.5 for a few additional, more technical, questions building on the central extension constructions in Section 5.

1.4. Outline of the paper

Section 2 discusses translation lengths for actions on spaces with barycentres, which extract a key property of injective spaces. Section 3 covers background on hierarchical hyperbolicity and applies Proposition 2.8 to prove uniform undistortion for HHGs. This is the starting point for the proof of Theorem 1.5, which occupies Section 4. Finally, Section 5 discusses the sharpness of Theorem 1.5 and analogues for well-known groups.

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2. Barycentres and translation lengths

Definition 2.1. Let us say that a metric space (X, d) *has barycentres* if for each n there is a map $b_n\colon X^n \to X$ such that b_n is:

- idempotent: for every $x \in X$ we have $b_n(x^n) = x$, where x^n denotes the tuple $(x, \ldots, x) \in X^n$;
- symmetric: b_n is invariant under permutation of co-ordinates;
- $\operatorname{Isom} X$–invariant: for every $g \in \operatorname{Isom} X$ we have $gb_n(x_1, \ldots, x_n) = b_n(gx_1, \ldots, gx_n)$;

- $\frac{1}{n}$–Lipschitz: $\mathsf{d}\left(b_n(x_1,\ldots,x_n), b_n(y_1,\ldots,y_n)\right) \leqslant \frac{1}{n}\sum_{i=1}^n \mathsf{d}(x_i,y_i)$.

In fact, it would be natural to assume something *a priori* rather stronger than idempotence, namely that the barycentre of a repeated tuple agrees with the barycentre of the tuple, in the sense of Remark 2.5. However, we do not explicitly need this in our arguments, and when X is complete it is actually a consequence of the above definition, as explained in the aforementioned remark.

Building on work of Es-Sahib–Heinich and Navas [EH99, Nav13], Descombes showed that every complete metric space with a *reversible, conical* bicombing (and every proper space with a conical bicombing) that is Isom–invariant has barycentres [Des16, Thm 2.1]. This includes many examples of interest, including all *CUB spaces* [Hae22], examples of which are produced in [HHP23]. Most importantly for the purposes of this paper, it includes all *injective metric spaces* by work of Lang [Lan13, Prop. 3.8], as mentioned in the introduction. However, there is a more direct way to see that injective spaces have barycentres [Pet22, §7.2, §7.4], as we now describe.

Definition 2.2. A metric space X is *injective* if for every metric space B and every subset $A \subset B$, if $f\colon A \to X$ is 1–Lipschitz then there exists a 1–Lipschitz map $\hat{f}\colon B \to X$ with $\hat{f}|_A = f$.

For example, given a metric space Y, let $\mathbb{R}^Y = \{Y \to \mathbb{R}\}$, and for $f, g \in \mathbb{R}^Y$ let $\mathsf{d}_\infty(f,g) = \sup_{y\in Y}|f(y)-g(y)|$. Every component of the extended metric space $(\mathbb{R}^Y, \mathsf{d}_\infty)$ is injective.

Lemma 2.3. *Injective metric spaces have barycentres.*

Proof. Let (X,d) be an injective space. The map $x \mapsto \mathsf{d}(x,\cdot)$ defines an isometric embedding of X into $\mathbb{R}^X$, so we can view X as a subset. We now recall a construction from [Dre84, §1]. First, let $P_X \subset \mathbb{R}^X$ be the set of functions f such that $f(x)+f(y) \geqslant \mathsf{d}(x,y)$ for all $x, y \in X$. Observe that the map $(\mathbb{R}^X)^n \to \mathbb{R}^X$ defined by $(f_1,\ldots,f_n) \mapsto \frac{1}{n}\sum_{i=1}^n f_i$ sends tuples of functions in P_X to functions in P_X. Let $T_X \subset P_X$ be the set of f such that

$$f(x) = \sup_{y\in X}\{\mathsf{d}(x,y) - f(y)\}$$

for all $x \in X$. Observe that X, regarded as above as a subset of $\mathbb{R}^X$, is contained in T_X. On the other hand, as noted in [Dre84], injectivity of X implies that $X \to T_X$ is surjective, and we can identify X with T_X.

Each isometry $\Psi\colon X \to X$ extends to a linear isomorphism $f \mapsto f\Psi^{-1}$ which is an isometry on each component of $\mathbb{R}^X$ and which preserves P_X and T_X. Dress defines a 1–Lipschitz retraction $p\colon P_X \to T_X = X$ in [Dre84, §1.9] which, by construction, is $\mathrm{Isom}(X)$–equivariant (see also [Lan13, Prop. 3.7.(2)]). (From Definition 2.2, one could construct a 1–Lipschitz retraction $\mathbb{R}^X \to X$ directly, but we use the map p to ensure equivariance.)

Given points $x_1,\ldots,x_n$ in X, consider their affine barycentre $\frac{1}{n}\sum_{i=1}^n \mathsf{d}(x_i,\cdot) \in P_X$. The maps defined by $b_n\colon (x_1,\ldots,x_n) \mapsto p\left(\frac{1}{n}\sum_{i=1}^n \mathsf{d}(x_i,\cdot)\right)$ satisfy the requirements of Definition 2.1, and hence provide barycentres for X. □

Lemma 2.4. *If X has barycentres, then every pair of points is joined by an isometric image of a dense subset of an interval. If X is complete, then it has an* $\mathrm{Isom}\,X$*–invariant, reversible, conical geodesic bicombing; and every space with an* Isom*–invariant conical geodesic bicombing has barycentres.*

In our applications, we only need the first assertion in the above lemma, and have included the statements about bicombings only since they may be of independent interest.

Proof of Lemma 2.4. Given $x, y \in X$, the point $b_2(x,y)$ has $\mathsf{d}(x, b_2(x,y)) = \mathsf{d}(y, b_2(x,y)) = \frac{1}{2}\mathsf{d}(x,y)$. Iterating, we get an isometrically embedded dyadic interval from x to y, and if X is complete then we get a geodesic by taking limits. By the properties of barycentres, these geodesics form an $\operatorname{Isom} X$–invariant conical bicombing. The fact that every space with an Isom–invariant conical bicombing has barycentres is [Des16, Thm 2.1]. □

Remark 2.5. If X is complete, then Lemma 2.4 implies that it has a reversible conical bicombing. According to [Des16, Prop. 2.4], the barycentres can then be perturbed so that they additionally satisfy $b_{nm}(x_1^m, \dots, x_n^m) = b_n(x_1, \dots, x_n)$ for every $n, m, x_1, \dots, x_n$, where z^m denotes the tuple $(z, \dots, z) \in X^m$. We could therefore have assumed this stronger property to begin with in most situations. Moreover, observe that the barycentres on X naturally extend to its metric completion.

Recall that for a group G acting on a metric space (X, d) and an element $g \in G$, the stable translation length is denoted $\tau_X(g) = \lim_{n\to\infty} \frac{1}{n}\mathsf{d}(x, g^n x)$, which is independent of x. We also write $|g| = \inf\{\mathsf{d}(x, gx) : x \in X\}$. Observe that, by repeatedly applying the triangle inequality, we always have $\tau_X(g) \leqslant |g|$. The following was noted for injective spaces in [Pet22, Rem. 7.25]. We provide a proof for completeness.

Lemma 2.6. *If G acts on a metric space X with barycentres, then $|g| = \tau_X(g)$ for all $g \in G$.*

Proof. Fix $x \in X$, and let $x_n = b_n(x, gx, \dots, g^{n-1}x)$. We compute

$$\begin{aligned}\mathsf{d}(x_n, gx_n) &= \mathsf{d}\Big(b_n(x, gx, \dots, g^{n-1}x),\, b_n(gx, g^2x, \dots, g^n x)\Big)\\ &= \mathsf{d}\Big(b_n(x, gx, \dots, g^{n-1}x),\, b_n(g^n x, gx, \dots, g^{n-1}x)\Big) \leqslant \frac{1}{n}\mathsf{d}(x, g^n x).\end{aligned}$$

Hence $\tau_X(g) \leqslant |g| \leqslant \mathsf{d}(x_n, gx_n) \to \tau_X(g)$. □

Definition 2.7. Let G be a group acting on a metric space X. The action is said to be *acylindrical* if for every $\varepsilon > 0$ there exist R, N such that if $\mathsf{d}(x,y) > R$, then

$$|\{g \in G : \mathsf{d}(x, gx), \mathsf{d}(y, gy) \leqslant \varepsilon\}| \leqslant N.$$

The action is *uniformly proper* if for every $\varepsilon > 0$ there exists N such that

$$|\{g \in G : \mathsf{d}(x, gx) \leqslant \varepsilon\}| \leqslant N$$

for all $x \in X$.

Uniform properness implies acylindricity, and proper cobounded actions are uniformly proper. The proof of the first part of the following proposition is slightly simpler than Bowditch's proof for acylindrical actions on hyperbolic graphs [Bow08], and also recovers it because hyperbolic graphs are coarsely dense in their injective hulls [Lan13].

Proposition 2.8. *Let G act on a metric space X with barycentres. If the action is:*

- acylindrical*, then there exists $\delta > 0$ such that $\tau_X(g) > \delta$ for every $g \in G$ whose action is not elliptic;*
- uniformly proper*, then $\tau_X(g) > \delta$ for every infinite-order $g \in G$;*
- proper and cobounded*, then G has finitely many conjugacy classes of finite subgroups.*

Proof. Supposing that the action is acylindrical, let R and N be such that if $\mathsf{d}(x,y) > R$ then $|\{g \in G : \mathsf{d}(x, gx), \mathsf{d}(y, gy) \leqslant 1\}| \leqslant N$. Suppose that $g \in G$ is not elliptic. By Lemma 2.6

there is some $x \in X$ such that $\mathsf{d}(x, gx) \leqslant \tau_X(g) + \frac{1}{2N}$. As g is not elliptic, there is some n such that $\mathsf{d}(x, g^n x) > R$. If $\tau_X(g) \leqslant \frac{1}{2N}$, then

$$\mathsf{d}(g^n x, g^{n+i}x) \;=\; \mathsf{d}(x, g^i x) \;\leqslant\; i\left(\tau_X(g) + \frac{1}{2N}\right) \;\leqslant\; \frac{i}{N}$$

for all $i \geqslant 0$. Considering $i \in \{0, \dots, N\}$ gives a contradiction, proving the first statement.

If the action is proper, then no infinite-order element is elliptic, and if the action is uniformly proper then it is acylindrical.

Finally, suppose the action is proper and cobounded. If $F = \{1, f_2, \dots, f_n\}$ is a finite subgroup of G, then $b_n(1, \dots, f_n)$ is fixed by F. Using that finite subgroups have fixed points, a standard argument shows that G has finitely many conjugacy classes of finite subgroups. □

For a constant $\varepsilon \geqslant 0$ and an element g of a group acting on a metric space X with barycentres, let $M_\varepsilon(g) = \{x \in X : \mathsf{d}(x, gx) \leqslant \tau_X(g) + \varepsilon\}$. If $\varepsilon > 0$, then this set is nonempty by Lemma 2.6. Given a proper cocompact action, $M_0(g) \neq \varnothing$ as well, see [DL16, Lem. 4.3].

Lemma 2.9. *$M_\varepsilon(g)$ is closed under taking barycentres and is stabilised by the centraliser of g.*

Proof. If $\{x_1, \dots, x_n\} \subset M_\varepsilon(g)$, then

$$\begin{aligned}\mathsf{d}\left(b_n(x_1, \dots, x_n), g b_n(x_1, \dots, x_n)\right) \;&=\; \mathsf{d}\left(b_n(x_1, \dots, x_n), b_n(gx_1, \dots, gx_n)\right)\\ &\leqslant\; \frac{1}{n}\sum_{i=1}^{n} \mathsf{d}(x_i, gx_i) \;\leqslant\; \tau_X(g) + \varepsilon.\end{aligned}$$

If h commutes with g and $x \in M_\varepsilon(g)$, then $\mathsf{d}(hx, ghx) = \mathsf{d}(hx, hgx) \leqslant \tau_X(g) + \varepsilon$. □

Definition 2.10. For $q \geqslant 0$, a *q–quasiaxis* of an isometry g of a metric space X is a $\langle g \rangle$–invariant subset $A_g \subset X$ admitting a q–coarsely surjective $(1+q, q)$–quasiisometry $\mathbb{R} \to A_g$.

Proposition 2.11. *Let G act on a metric space X with barycentres. For every $\varepsilon > 0$, every $g \in G$ with $\tau_X(g) > 0$ has a ε–quasiaxis.*

Hence, if G acts on X acylindrically, then for all $\epsilon > 0$, every non-elliptic $g \in G$ has an ϵ–quasiaxis. If the action is uniformly proper, then this holds for all $g \in G$ of infinite order.

Proof. As noted in Remark 2.5, the barycentre maps on X naturally extend to its completion, so there is no loss in assuming that X is complete. Fix $\varepsilon > 0$. Let $g \in G$ satisfy $\tau := \tau_X(g) > 0$ and $x \in M_\varepsilon(g)$. Let $I : [0, \mathsf{d}(x, gx)] \to X$ be the unit-speed geodesic from x to gx provided by the proof of Lemma 2.4. Consider the subset $A_g = \bigcup_{n \in \mathbb{Z}} g^n I$, which is stabilised by g. We have $A_g \subset M_\varepsilon(g)$ by Lemma 2.9. It remains to show that A_g is an ε–quasiline.

Given $t \in \mathbb{R}$, write $t = n\tau + r$, where $n \in \mathbb{Z}$ and $r \in [0, \tau)$, and let $f(t) = g^n I(r)$, which is well-defined since $\tau \leqslant |I|$. This defines a map $f : \mathbb{R} \to A_g$ that is ε–coarsely onto. Let $t_1, t_2 \in \mathbb{R}$, and write $t_i = n_i \tau + r_i$ for $i \in \{1, 2\}$. If $n_1 = n_2$, then by definition we have $\mathsf{d}(f(t_1), f(t_2)) = |t_1 - t_2|$. Otherwise, we may assume that $n_1 < n_2$, and we compute:

$$\begin{aligned}\mathsf{d}(f(t_1), f(t_2)) \;&\leqslant\; \mathsf{d}(f(t_1), g^{n_1+1}x) + \mathsf{d}(g^{n_1+1}x, g^{n_2}x) + \mathsf{d}(g^{n_2}x, f(t_2))\\ &\leqslant\; (\tau + \varepsilon - r_1) + (n_2 - n_1 - 1)(\tau + \varepsilon) + r_2\\ &=\; r_2 - r_1 + \tau(n_2 - n_1) + \varepsilon(n_2 - n_1) \;\leqslant\; |t_2 - t_1| + \varepsilon|t_2 - t_1| + \varepsilon.\end{aligned}$$

We similarly obtain a lower bound as follows:

$$\begin{aligned}\mathsf{d}(f(t_1), f(t_2)) &\geqslant \mathsf{d}(g^{n_1}x, g^{n_2+1}x) - \mathsf{d}(g^{n_1}x, f(t_1)) - \mathsf{d}(g^{n_2+1}x, f(t_2)) \\ &\geqslant (n_2 - n_1 + 1)\tau - r_1 - (\tau + \varepsilon - r_2) \\ &= \tau(n_2 - n_1) + r_2 - r_1 - \varepsilon = |t_2 - t_1| - \varepsilon.\end{aligned}$$

Combining these estimates, we see that f is a $(1+\varepsilon, \varepsilon)$–quasi-isometry. The statement about acylindrical and uniformly proper actions now follows using Proposition 2.8. □

The same proof shows that if X is complete and g is non-elliptic with $M_0(g) \neq \varnothing$, then g has a geodesic axis. However, in our applications we will not be able to arrange for M_0 to be nonempty, because X can fail to be proper.

It is natural to ask for a higher-dimensional version of the above result. More precisely:

Question 2.12. *Let G act uniformly properly on a metric space X with barycentres and let $n > 1$. Does there exist λ such that for all subgroups $H \leqslant G$ with $H \cong \mathbb{Z}^n$, there is an H–invariant subspace $F \subset X$ that is (λ, λ)–quasi-isometric to $\mathbb{R}^n$?*

We finish this section by partially addressing this question.

Proposition 2.13. *Let G act properly coboundedly on a metric space X with barycentres, and let $\varepsilon > 0$. If $H = \langle g_1, \dots, g_n\rangle \cong \mathbb{Z}^n$ is a free abelian subgroup, then there is a $(1+\varepsilon, \varepsilon)$–coarsely Lipschitz map $(\mathbb{R}^n, \ell^1) \to X$ whose image is H–invariant.*

For $n = 1$, the above proposition gives a $\langle g_1\rangle$–equivariant uniformly coarsely Lipschitz axis in X, but does not recover the full statement of Proposition 2.11 because it gives only one of the bounds needed for a quasi-isometric embedding.

Proof of Prop. 2.13. Let $\delta \in (0, 1]$ be given by Proposition 2.8, and fix $\varepsilon > 0$.

We first show that $\bigcap_{i=1}^n M_\varepsilon(g_i) \neq \varnothing$, arguing by induction on n. By Lemma 2.6, $M_\varepsilon(g_1) \neq \varnothing$. Fix $j \in \{2, \dots, n\}$, and assume by induction that there exists $x \in \bigcap_{i=1}^{j-1} M_\varepsilon(g_i)$. By Lemma 2.9, for every m, the point $y_m = b_m(x, g_jx, \dots, g_j^{m-1}x)$ lies in $\bigcap_{i=1}^{j-1} M_\varepsilon(g_i)$, using that $[g_j, g_i] = 1$ for all i. Moreover,

$$\mathsf{d}(y_m, g_jy_m) = \mathsf{d}\left(b_m(x, g_jx, \dots, g_j^{m-1}x), b_m(g_j^mx, g_jx, \dots, g_j^{m-1}x)\right) \leqslant \frac{1}{m}\mathsf{d}(x, g_j^mx).$$

Thus $y_m \in \bigcap_{i=1}^{j} M_\varepsilon(g_i)$ for sufficiently large m. So $\bigcap_{i=1}^n M_\varepsilon(g_i) \neq \varnothing$. Fix $x \in \bigcap_{i=1}^n M_\varepsilon(g_i)$.

Next, let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. For brevity, let $\tau_i = \tau_X(g_i)$ for $1 \leqslant i \leqslant n$. Let $D^n \subset \mathbb{R}^n$ be the set of vectors of the form $\sum_{i=1}^n r_i\tau_ie_i$, where r_i is a dyadic rational. Define a map $f\colon D^n \to X$ as follows.

Set $f(0) = x$. For $i \geqslant 1$, suppose that f has been defined on $\left(D^{i-1} \times \{0\}^{n-i+1}\right) \cap \prod_{i=1}^n[0, \tau_i)$. Given an element p thereof, set $f(p + \frac{1}{2}\tau_ie_i) = b_2(f(p), g_if(p))$. We can define $f(q)$ for every $q \in D^i \times \{0\}^{n-i} \cap \prod_{i=1}^n[0, \tau_i)$ by repeatedly taking barycentres in this way. Inductively, this defines f on $D^n \cap \prod_{i=1}^n[0, \tau_i)$. Specifically, for any $p \in D^{n-1} \times \{0\}$, the restriction of f to the set of dyadic rationals in $\{p\} \times [0, \tau_n)$ has image an isometrically embedded (dense subset of an) interval of length $\mathsf{d}(f(p), g_nf(p))$ from $f(p)$ to $g_nf(p)$.

Given $p \in D^n$, we can write

$$p = (p_i\tau_i + a_i\tau_i)_{i=1}^n,$$

where each $p_i \in [0, 1)$ is a dyadic rational and each $a_i \in \mathbb{Z}$. We define $[p] = (p_i\tau_i)_{i=1}^n$. Then we let $f(p) = g_1^{a_1} \cdots g_n^{a_n} f([p])$. Observe that f is well-defined and the image of f is H–invariant by construction. We now check that f is coarsely Lipschitz.

First, for convenience, let $C = D^n \cap \prod_{i=1}^n [0, \tau_i)$ and let $\bar{C}$ be its closure in D^n. Letting H act on D^n by declaring g_i to be a unit translation by $\tau_i e_i$, we see that C contains exactly one point in each H–orbit (and f is an H–equivariant map defined as above on C using barycentres and then extending equivariantly).

Now let $p, q \in D^n$. Let γ be an ℓ^1–metric geodesic in $\mathbb{R}^n$ from p to q such that $\gamma \cap D$ is dense in γ. Then $\gamma = \gamma_1 \cdots \gamma_m$, where each γ_j is an ℓ^1–metric geodesic whose intersection with D^n lies in some H–translate of $\bar{C}$, and these translates are distinct for distinct j. We will argue that f is $(1+\varepsilon)$–Lipschitz on C, so by equivariance, f is $(1+\varepsilon)$–Lipschitz on each H–translate of C. Hence, letting $p = p_0, \dots, p_{m-1}$ be the initial points of $\gamma_1, \dots, \gamma_m$ and $p_m = q$ the terminal point of γ_m, we have $\mathsf{d}(f(p_i), f(p_{i+1})) \leqslant (1+\varepsilon)\|p_i - p_{i+1}\|_1$ for each i. We conclude that

$$\mathsf{d}(f(p), f(q)) \leqslant \sum_{i=0}^{m-1} \mathsf{d}(f(p_i), f(p_{i+1})) \leqslant (1+\varepsilon)\sum_i |\gamma_i| = (1+\varepsilon)\|p-q\|_1,$$

as required. So it remains to verify that f is $(1+\varepsilon)$–Lipschitz on C.

If $p, q \in C$, we can write $p = \sum_{i=1}^n p_i \tau_i e_i$ and $q = \sum_{i=1}^n q_i \tau_i e_i$. By construction,

$$\begin{aligned}\mathsf{d}(f(p), f(q)) &\leqslant \sum_{i=1}^n (\tau_i + \varepsilon)|p_i - q_i| = \sum_{i=1}^n (1 + \frac{\varepsilon}{\tau_i})\tau_i |p_i - q_i| \\ &\leqslant (1+\frac{\varepsilon}{\delta})\sum_{i=1}^n \tau_i |p_i - q_i| = (1+\frac{\varepsilon}{\delta})\|p-q\|_1.\end{aligned}$$

Here we used that $\tau_i \geqslant \delta > 0$ for all i. Since the above works for all ε, we are done. □

One problem with Proposition 2.13 is that the map f could, *a priori*, fail to be colipschitz. This is addressed by the following statement, which is similar to [But19, Thm 4.2].

Lemma 2.14. *Let $H = \langle g_1, \dots, g_n \rangle \cong \mathbb{Z}^n$ be a free abelian group acting on a metric space X. For every $T, \delta > 0$ there exists $\delta' = \delta'(n, \delta, T) > 0$ such that the following holds. If every $h \in H$ has $\tau_X(h) > \delta$ and $\max\{\tau_X(g_i)\} \leqslant T$, then in fact every $h \in H$ has $\tau_X(h) \geqslant \delta' \,\mathsf{d}_H(1, h)$.*

Proof. In the terminology of [But19, Thm 4.2], τ_X defines a $\mathbb{Z}$–norm on H. Consider the group embedding $H \to \mathbb{R}^n$ given by $g_i \mapsto e_i$, where the e_i are the standard basis vectors. The $\mathbb{Z}$–norm τ_X extends to a norm N on $\mathbb{R}^n$. By linearity, $N(x) \geqslant r\|x\|_1$ for all $x \in \mathbb{R}^n$, where $r = \inf\{N(z) : \|z\|_1 = 1\}$. Let us find a lower bound for r.

Let $x \in \mathbb{R}^n$ have $\|x\|_1 = 1$. By an application of the pigeonhole principle, there is a constant $M = M(n, \delta, T)$ and a natural number $q \leqslant M$ such that there exist integers $p_1, \dots, p_n$ such that $|qx_i - p_i| \leqslant \frac{1}{2n}\frac{\delta}{T}$ (see, e.g., [HW75, Thm 201]). Following [Ste85], let $p = (p_1, \dots, p_n)$, so that $\|qx - p\|_1 \leqslant \frac{1}{2}\frac{\delta}{T}$. Because N is a norm, every point z with $\|z\|_1 = 1$ has $N(z) \leqslant \max\{N(e_i)\} \leqslant T$, which shows that $N(qx - p) \leqslant \frac{\delta}{2}$. As $\|qx\|_1 \geqslant \|x\|_1 = 1$, the vector p must be nonzero, hence $N(p) > \delta$, and therefore $N(qx) > \frac{\delta}{2}$. We have shown that $r > \frac{\delta}{2M}$.

If $h \in H$, then $\|h\|_1 = \mathsf{d}_H(1, h)$, so we can compute

$$\tau_X(h) \;=\; N(h) \;\geqslant\; \frac{Tr}{\delta}\|h\|_1 \;\geqslant\; \frac{\delta}{2M(n)}\,\mathsf{d}_H(1,h). \qquad \square$$

Proposition 2.13 generalises a result of Descombes–Lang for proper spaces with *convex, consistent* bicombings [DL16, Thm 1.2], which includes proper injective spaces of finite dimension by [DL15]. They prove that if G acts properly cocompactly on such a space X in such a way that the bicombing is G–invariant, then every free-abelian subgroup $A < G$ of rank n acts by translations on some subset $Y \subset X$ isometric to $(\mathbb{R}^n, N)$, where N is some norm. Although Y is very well controlled, it does not seem clear whether this implies that abelian subgroups of G are uniformly undistorted, because the norm N depends on the choice of A.

3. Background on hierarchical hyperbolicity

A *hierarchically hyperbolic structure* on a space (X, d) is a package of associated data, which is usually abbreviated to $(X, \mathfrak{S})$. Despite the compact notation, this package holds a large amount of information, much of which is not directly relevant to our purposes here (though it is all *indirectly* relevant, via the "distance formula" below). We therefore summarise the main components of the definition and some basic results needed here. For the detailed definition, see [BHS19, §1]; for a mostly self-contained exposition of the theory, see [CHK22, Part 2].

Firstly, $\mathfrak{S}$ denotes the *index set*, whose elements are called *domains*. In some of the following statements, we refer to a constant $E \geqslant 1$, which is part of the data of a hierarchically hyperbolic structure and which is fixed in advance. In particular, in any property of individual domains $V \in \mathfrak{S}$, the constant E is independent of V.

(1) For each domain $W \in \mathfrak{S}$ there is an associated E-hyperbolic geodesic space $\mathcal{C}W$ and an E–coarsely surjective (E, E)-coarsely Lipschitz map $\pi_W : X \to \mathcal{C}W$.
(2) $\mathfrak{S}$ has mutually exclusive relations $\sqsubseteq$, $\perp$, and $\pitchfork$ satisfying the following.
  - $\sqsubseteq$ is a partial order called *nesting*. If $\mathfrak{S} \neq \varnothing$, then $\mathfrak{S}$ contains a unique $\sqsubseteq$-maximal element S.
  - $\perp$ is a symmetric and anti-reflexive relation called *orthogonality*. If $U \sqsubseteq V$ and $V \perp W$, then $U \perp W$.
  - There exists an integer c called the *complexity* of X such that every $\sqsubsetneq$–chain has length at most c, and every pairwise orthogonal set has cardinality at most c.
  - $\pitchfork$, called *transversality*, is the complement of $\perp$ and $\sqsubseteq$.
(3) If $U \sqsubsetneq V$ or $U \pitchfork V$, then there is an associated set $\rho^U_V \subset \mathcal{C}V$ of diameter at most E. If $U \sqsubsetneq V \sqsubsetneq W$, then $\mathsf{d}_{\mathcal{C}W}(\rho^U_W, \rho^V_W) \leqslant E$.
(4) If $U \pitchfork V$ and $x \in G$ satisfies $\mathsf{d}_{\mathcal{C}U}(\pi_U(x), \rho^V_U) > E$, then $\mathsf{d}_{\mathcal{C}V}(\pi_V(x), \rho^U_V) \leqslant E$.

We emphasise that the above list is just a subset of the full definition of a hierarchically hyperbolic structure.

Definition 3.1 (Hierarchically hyperbolic group)**.** A finitely generated group G with word metric $\mathsf{d} = \mathsf{d}_G$ is a *hierarchically hyperbolic group* (or *HHG*) if it has a hierarchically hyperbolic structure $(G, \mathfrak{S})$ such that the following additional equivariance conditions hold.

- G acts on $\mathfrak{S}$. The action is cofinite and preserves the three relations $\sqsubseteq$, $\perp$, and $\pitchfork$.
- For each $g \in G$ and each $U \in \mathfrak{S}$, there is an isometry $g : \mathcal{C}U \to \mathcal{C}gU$. These isometries satisfy $g \circ h = gh$.
- For all $x, g \in G$ and $U \in \mathfrak{S}$, we have $g\pi_U(x) = \pi_{gU}(gx)$. Moreover, if $V \in \mathfrak{S}$ and either $U \pitchfork V$ or $V \sqsubsetneq U$, then $g\rho^V_U = \rho^{gV}_{gU}$.

We are following the definition given in [PS20] as it appears to be the most compact, but the notion was originally introduced in [BHS17, BHS19]. The original definition was shown to be equivalent to the present, simpler, one in [DHS20, §2].

The final part of a hierarchically hyperbolic structure mentioned above is called a *consistency* condition. A related part of the definition is a "bounded geodesic image" axiom, and though we do not use it directly, it combines with consistency to provide the following statement, which will be important for us. It is part of [BHS19, Prop. 1.11]. For two points

$x, y \in X$, it is standard to simplify notation by using $\mathsf{d}_U(x, y)$ to denote $\mathsf{d}_{\mathcal{C}U}(\pi_U(x), \pi_U(y))$, and similarly for subsets of G.

Lemma 3.2 (Bounded geodesic image)**.** *Let $x, y \in G$, and suppose that $U, V \in \mathfrak{S}$ satisfy $V \subsetneq U$. If there exists a geodesic $\gamma \subset \mathcal{C}U$ from $\pi_U(x)$ to $\pi_U(y)$ such that $\mathsf{d}_U(\rho_U^V, \gamma) > E$, then $\mathsf{d}_V(x, y) \leqslant E$.*

Definition 3.3 (Relevant domains)**.** Let $D \geqslant 0$ and let $x, y \in G$. Then $\mathrm{Rel}_D(x, y)$ denotes the collection of all $U \in \mathfrak{S}$ with $\mathsf{d}_U(x, y) \geqslant D$.

Another axiom from [BHS19, Def. 1.1] is the "large link" axiom, which we also will not use directly, but instead use via the following consequence:

Lemma 3.4 (Passing-up Lemma, [BHS19, Lem. 2.5])**.** *For every $C > 0$ there is an integer $P(C)$ such that the following holds. Let $U \in \mathfrak{S}$ and let $x, y \in G$. If there is a set $\{V_1, \ldots, V_{P(C)}\}$ with $V_i \subsetneq U$ and $\mathsf{d}_{V_i}(x, y) > E$ for all i, then there exists some domain $W \sqsubseteq U$ such that $V_i \subsetneq W$ for some i and $\mathsf{d}_W(x, y) > C$.*

One of the most important features of a hierarchically hyperbolic structure is that one has a "distance formula" [BHS19, Thm 4.5], which allows one to approximate distances in G using projections to the domains.

Theorem 3.5 (Distance formula)**.** *Let $(G, \mathfrak{S})$ be an HHG. There exists D_0, depending only on the HHG structure, such that the following holds. For every $D \geqslant D_0$ there exists A_D such that for all $x, y \in G$ we have*

$$\frac{1}{A_D}\,\mathsf{d}_G(x, y) - A_D \;\leqslant \sum_{U \in \mathrm{Rel}_D(x,y)} \mathsf{d}_U(x, y) \;\leqslant\; A_D\,\mathsf{d}_G(x, y) + A_D.$$

Moreover, the dependence of A_D on D is entirely determined by the HHG structure.

The axioms in [BHS19, Def. 1.1] were chosen largely to enable one to prove Theorem 3.5.

We are interested in infinite cyclic subgroups of the HHG $(G, \mathfrak{S})$ and how they act on the HHG structure. Accordingly, we recall the following definition from [DHS17, §6.1].

Definition 3.6 (Bigsets)**.** Let $(G, \mathfrak{S})$ be an HHG. For each $g \in G$, let $\mathrm{Big}(g)$ be the set of domains $U \in \mathfrak{S}$ such that $\operatorname{diam} \pi_U(\langle g \rangle) = \infty$.

For an element g of an HHG $(G, \mathfrak{S})$, the set $\mathrm{Big}(g)$ is empty if and only if g has finite order [DHS17, Prop. 6.4]. The following properties are established in [DHS17, §6].

Lemma 3.7. *Let $(G, \mathfrak{S})$ be an HHG. Given $g \in G$, write $\mathrm{Big}(g) = \{U_i\}_{i \in I}$.*

1. *$gU_i \in \mathrm{Big}(g)$ for all i.*
2. *$U_i \perp U_j$ for all $i \neq j$. In particular, $|I| \leqslant c$, where c is the complexity of $\mathfrak{S}$.*
3. *For all $i \in I$, we have $g^{c!}U_i = U_i$, and so $\langle g^{c!} \rangle$ acts on each $\mathcal{C}U_i$ by isometries.*
4. *There exists $D = D(g, \mathfrak{S})$ such that $\operatorname{diam} \pi_V(\langle g \rangle) \leqslant D$ for all $V \notin \mathrm{Big}(g)$.*

Remark 3.8. Many of the statements in Lemma 3.7 hold when $\langle g \rangle$ is replaced by more complicated subgroups of G — see [DHS17, §9] and [PS20] — but we will not use this here.

We will use the following proposition, which is [DHS20, Thm 3.1]:

Proposition 3.9. *If g is an infinite-order element of an HHG $(G, \mathfrak{S})$, of complexity c, then $g^{c!}$ acts loxodromically on $\mathcal{C}U$ for all $U \in \mathrm{Big}(g)$. In particular, $\tau_G(g) > 0$.*

The assertion about $\tau_G(g)$ follows since π_U is coarsely Lipschitz and $\langle g^{c!}\rangle$–equivariant.

The proof of Proposition 3.9 given in [DHS20] relies in an essential way on the constants $D(g,\mathfrak{S})$ from Lemma 3.7.(4) and cannot be adapted to give a lower bound on either $\tau_U(g^{c!})$ or $\tau_G(g)$ that is independent of g. Indeed, we shall see in Section 5 that there need not be a uniform lower bound on $\tau_U(g^{c!})$ that holds for *all* $U \in \mathrm{Big}(g)$. On the other hand, the following proposition states that $\tau_G(g)$ *can* be uniformly lower-bounded. This fact, which relies on the results of Section 2, is an important ingredient in establishing Theorem 1.5.

Proposition 3.10 (Uniform undistortion in HHGs)**.** *Let $(G,\mathfrak{S})$ be an HHG. There exists $\tau_0 > 0$ such that $\tau_G(g) \geqslant \tau_0$ for every infinite-order $g \in G$. Hence there exists $K = K(G,\mathfrak{S})$ such that for all infinite-order $g \in G$ and all $x \in G$, we have $\mathsf{d}_G(x, g^n x) > Kn$ for all $n \geqslant 0$.*

Proof. By [HHP20, Cor. 3.8, Lem. 3.10], there is a metrically proper, cobounded action of G on an injective metric space X. Fix a basepoint $x_0 \in X$ and a constant $\mu \geqslant 1$ such that the orbit map $G \to X$ given by $h \mapsto hx_0$ is a (μ,μ)–quasi-isometry.

By Lemma 2.3, X has barycentres. Since the action of G on X is proper and cobounded, it is uniformly proper. Hence Proposition 2.8 provides a constant $\delta > 0$ such that $\tau_X(g) \geqslant \delta$ for all infinite-order $g \in G$. A computation shows $\tau_G(g) \geqslant \frac{\delta}{\mu}$. Recalling that $\tau_G(g^n) \leqslant \mathsf{d}(x, g^n x)$ for all $n \geqslant 0$ and $x \in G$, and that $\tau_G(g^n) = n\tau_G(g)$, we have $\mathsf{d}_G(x, g^n x) \geqslant n\delta\mu$. Taking $K = \frac{\delta}{2\mu}$ completes the proof. □

4. Proof of Theorem 1.5

Let $(G,\mathfrak{S})$ be a hierarchically hyperbolic group and $g \in G$ an infinite order element, with $\mathrm{Big}(g) = \{U_1, \ldots, U_m\}$. By Lemma 3.7, replacing g by $g^{c!}$, we can and shall assume that $gU_i = U_i$ for all i. Independently of g, we bound $\tau_{U_i}(g)$ below for some i.

Our strategy is as follows. First, we carefully construct a uniform quality quasi-axis for g in each U_i and a point $x \in G$ whose projection to each $\mathcal{C}U_i$ lies on this quasi-axis. We next show that the terms in the distance formula for $d(x, g^n x)$ can be divided into two sets: the domains that are orthogonal to all U_i and the domains that nest into some U_i. The first technical step is to give an upper bound to the contribution to $d(x, g^n x)$ from domains that are orthogonal to all U_i. This gives a lower bound on the contribution from domains that nest into some U_i. The second technical step in the proof uses the passing-up lemma and a counting argument to show that, in fact, some U_i itself must have a uniformly large contribution to the distance formula. This will then give a uniform lower bound on the translation length $\tau_{U_i}(g)$. Because the dependence of the constants at each step is crucial to our arguments, we describe every step in detail to make this explicit.

Step 1: Quasi-axes

For each $i \leqslant m$, Proposition 3.9 says that g acts on $\mathcal{C}U_i$ as a loxodromic isometry. A standard fact of hyperbolic spaces is that every loxodromic isometry has a quasiaxis. We make this precise with the following.

Lemma 4.1. *There is a constant R such that the following hold. Let $k \geqslant 1$ be such that $\mathcal{C}U$ is k–hyperbolic for all $U \in \mathfrak{S}$. There exists $\alpha_i \subset \mathcal{C}U_i$ such that:*

- *α_i, with the subspace metric inherited from $\mathcal{C}U_i$, is (Rk, Rk)–quasi-isometric to $\mathbb{R}$;*
- *α_i is Rk–quasiconvex; and*
- *α_i is $\langle g\rangle$–invariant.*

Proof. Since g is loxodromic, g has exactly two fixed points in the Gromov boundary $\partial\mathcal{C}U_i$ [Gro87], and these two points can be joined by a bi-infinite $(1, 20k)$–quasigeodesic γ (see [KB02, Remark 2.16], for instance). Any two such quasigeodsics lie at Hausdorff-distance at most $50k$. In particular, $\alpha' = \bigcup_{n\in\mathbb{Z}} g^n\gamma$ is $\langle g\rangle$–invariant and $50k$–Hausdorff-close to γ. Let α be the union of all geodesic segments in $\mathcal{C}U_i$ whose endpoints lie in α'. Note that $g\alpha = \alpha$ and α is $2k$–quasiconvex.

Fix $p \in \gamma$. Since $\tau_{U_i}(g) > 0$, there exists $\ell = \ell(g, p)$ such that $\mathsf{d}_{U_i}(p, g^{\ell n}p) \geqslant 100k|n|$ for all $n \in \mathbb{Z}$. Let I be a geodesic segment joining p to $g^{\ell}p$, and let $\beta = \bigcup_{n\in\mathbb{Z}} g^{\ell n}I \subset \alpha$.

For each $n \in \mathbb{Z}$, there exists $p'_n \in \gamma$ with $\mathsf{d}_{U_i}(p'_n, g^{\ell n}p) \leqslant 50k$. Since $\mathsf{d}_{U_i}(p'_n, p) \geqslant 100k|n| - 50k$, any point in γ lies between p'_n and p'_{n+1} for some n, so a thin quadrilateral argument shows $\mathsf{d}_{Haus}(\beta, \gamma) \leqslant 50k$. Thus β is a quasiline with constants depending only on k, and $\mathsf{d}_{Haus}(\beta, \alpha') \leqslant 100k$. Since β is uniformly quasiconvex, it then follows that α is uniformly Hausdorff-close to β. Hence α is a uniformly quasiconvex $\langle g\rangle$–invariant quasiline. Setting $\alpha_i = \alpha$ completes the proof. □

Remark 4.2. Since R is a universal constant, there is no harm in increasing E to assume that $E \geqslant Rk$. Thus, when we later refer to Lemma 4.1, we shall take the constants in its conclusion to all be E. Actually, we shall later make one final increase of E by an amount dependent only on the partial realisation axiom; see Section 4.

Corollary 4.3. *Let $x \in \alpha_i$. If $n > 0$ is such that $\mathsf{d}_{U_i}(x, g^n x) \geqslant 14E$, then $\tau_{U_i}(g) \geqslant \frac{E}{n}$.*

Proof. Since α_i is E–quasiconvex and (E, E)–quasi-isometric to $\mathbb{R}$, considering the $\langle g\rangle$–equivariant coarse closest point projection $\mathcal{C}U_i \to \alpha_i$ shows that

$$\mathsf{d}_{U_i}(x, g^n x) \leqslant \inf_{y\in\mathcal{C}U_i} \mathsf{d}_{U_i}(y, g^n y) + 10E.$$

According to [Lan13, Prop. 1.3], $\mathcal{C}U_i$ is E–coarsely dense in its injective hull H, which is E–hyperbolic. Lemma 2.6 shows that there is some $y' \in H$ such that $\mathsf{d}_H(y', g^n y') \leqslant \tau_{U_i}(g^n) + E$. Choosing $y \in \mathcal{C}U_i$ so that $\mathsf{d}_H(y, y') \leqslant E$, we see that $\mathsf{d}_{U_i}(x, g^n x) \leqslant n\tau_{U_i}(g) + 13E$. In particular, if $\mathsf{d}_{U_i}(x, g^n x) \geqslant 14E$, then $\tau_{U_i}(g) \geqslant \frac{E}{n}$. □

In view of this corollary, our task is to produce a uniform constant J, independent of g, such that $\mathsf{d}_{U_i}(x, g^J x) > 14E$ for some i.

Step 2: Choosing which point to move

We fix, for the remainder of the proof, a point $x \in G$ as follows. For each $i \in \mathrm{Big}(g)$, fix some $x_i \in \alpha_i$. Since the elements of $\mathrm{Big}(g)$ are pairwise orthogonal by Lemma 3.7, the partial realisation axiom [BHS19, Def. 1.1.(8)] provides a point $x \in G$ such that

- $\mathsf{d}_{U_i}(x, x_i) \leqslant E$ for all i, and
- $\mathsf{d}_V(x, \rho_V^{U_i}) \leqslant E$ for all pairs (i, V) where either $U_i \sqsubsetneq V$ or $U_i \pitchfork V$.

With one final uniform enlargement of E, for convenience only, we replace each α_i by its E–neighbourhood in $\mathcal{C}U_i$, so that, for this fixed $x \in G$, we have $\pi_{U_i}(x) \in \alpha_i$ for all i.

Step 3: Organising distance formula terms

Let $E' = \max\{5E, D_0\}$, where D_0 is the constant from the distance formula, Theorem 3.5. First, we partition the E'–relevant domains as follows. Fix $n \geqslant 0$, and let

$$\mathcal{W}^n = \{W \in \mathrm{Rel}_{E'}(x, g^n x) \,:\, W \perp U_i \text{ for all } i\}$$

and

$$\mathcal{V}_i^n = \{V \in \mathrm{Rel}_{E'}(x, g^n x) \,:\, V \sqsubseteq U_i\},$$

where $i \in \{1, \ldots, m\}$. We denote the union of the $\mathcal{V}_i^n$ by $\mathcal{V}^n$.

Because U_i nests in itself, we have $U_i \in \mathcal{V}_i^n$ for each i, and $\mathcal{V}_i^n \cap \mathcal{V}_j^n = \varnothing$ for $i \neq j$, as $U_i \perp U_j$. Similarly, $\mathcal{W}^n \cap \mathcal{V}^n = \varnothing$. The sets $\mathcal{V}^n$ and $\mathcal{W}^n$ fit into the following distance estimate.

Lemma 4.4. *For all $n \in \mathbb{Z}$, if $V \in \mathrm{Rel}_{5E}(x, g^n x)$, then either $V \perp U_i$ for all i, or $V \sqsubseteq U_i$ for some i. Consequently, there exists a constant A independent of n such that*

$$\frac{1}{A}\,\mathsf{d}_G(x, g^n x) - A \;\leqslant\; \sum_{V \in \mathcal{V}^n} \mathsf{d}_V(x, g^n x) \;+ \sum_{W \in \mathcal{W}^n} \mathsf{d}_W(x, g^n x) \;\leqslant\; A\,\mathsf{d}_G(x, g^n x) + A.$$

Proof. Fix $n \in \mathbb{Z}$. If $V \in \mathfrak{S}$ satisfies $U_i \sqsubsetneq V$ or $U_i \pitchfork V$ for some i, then $\mathsf{d}_V(x, g^s x) \leqslant 3E$ for all $s \in \mathbb{N}$. To see this, note that $\rho^{g^s U_i}_{g^s V} = \rho^{U_i}_{g^s V}$, since $gU_i = U_i$, and, by definition of x, we have $\mathsf{d}_{g^s V}(x, \rho^{U_i}_{g^s V}) \leqslant E$. We also have $\mathsf{d}_{g^s V}(g^s x, \rho^{U_i}_{g^s V}) = \mathsf{d}_V(x, \rho^{U_i}_V) \leqslant E$. Hence, it follows from the triangle inequality that $\mathsf{d}_{g^s V}(g^s x, x) \leqslant 3E$, and translating by g^{-s} gives the desired result. (The extra E comes from the fact that $\rho^{\bullet}_{\bullet}$ are sets of diameter at most E.) Thus every $V \in \mathrm{Rel}_{5E}(x, g^n x)$ must be either nested in some U_i, or orthogonal to all U_i. In particular, $\mathrm{Rel}_{E'}(x, g^n x) = \mathcal{V}^n \cup \mathcal{W}^n$. The second statement is given by the distance formula, Theorem 3.5, with threshold $E' \geqslant D_0$. □

Step 4: Controlling orthogonal terms

Next, we give a lower bound on the contribution to $\mathsf{d}_G(x, g^n x)$ coming from elements of $\mathcal{V}^n$ by finding an upper bound on the contribution to $\mathsf{d}_G(x, g^n x)$ coming from elements of $\mathcal{W}^n$.

Lemma 4.5. *Given $E' \geqslant \max\{5E, D_0\}$, there exist $\epsilon = \epsilon(\mathfrak{S}, E') > 0$ and $N = N(g, x, E')$ as follows. For all $n \geqslant N$, there exists $U_k \in \mathrm{Big}(g)$ satisfying*

$$\sum_{V \in \mathcal{V}_k^n} \mathsf{d}_V(x, g^n x) \geqslant \epsilon n.$$

Proof. By Lemma 3.7(4), there is a constant $D = D(\mathfrak{S}, g, x)$ such that $\mathrm{diam}(\pi_V(\langle g \rangle \cdot x)) < D$ for all $V \notin \mathrm{Big}(g)$. Lemma 3.7 is stated for $x = 1$, but the bound for $x = 1$ yields a bound for arbitrary x in terms of $\mathsf{d}_G(1, x)$ and E, since the maps π_V are all (E, E)–coarsely Lipschitz.

<u>Claim:</u> There is a constant P such that $\sum_{W \in \mathcal{W}^n} d_W(x, g^n x) \leqslant PD$ for all $n \in \mathbb{Z}$.

<u>Proof:</u> Let $C = \max\{5E, 2D\}$. Let $P = P(C)$ be the constant from the passing-up lemma, Lemma 3.4. Fix $n \geqslant 0$. By definition, $\mathcal{W}^n$ is disjoint from $\mathrm{Big}(g)$, so $\mathsf{d}_W(x, g^n x) \leqslant D$ for all $W \in \mathcal{W}^n$. Thus, if the claim did not hold then we would have $|\mathcal{W}^n| > P$. Also by definition, $\mathsf{d}_W(x, g^n x) > E$ for all $W \in \mathcal{W}^n$. By the passing-up lemma, this would imply the existence of some $V \in \mathfrak{S}$ such that $V \sqsupsetneq W$ for some $W \in \mathcal{W}^n$, and with $\mathsf{d}_V(x, g^n x) > C \geqslant D$. The latter property forces V to lie in $\mathrm{Big}(g)$, but then $W \sqsubsetneq V$ and $W \perp V$, which is a contradiction. ◇

Proposition 3.10 provides a positive $K = K(G, \mathfrak{S})$ such that $\mathsf{d}_G(x, g^n x) > Kn$ for all $n \geqslant 0$. For such n we have

$$\frac{Kn}{A} - A \leqslant \sum_{V \in \mathcal{V}^n} d_V(x, g^n x) + \sum_{W \in \mathcal{W}^n} d_V(x, g^n x),$$

where A, provided by Lemma 4.4, is independent of n. By the claim, the latter term is bounded above by PD, which is independent of n. Let $N = \frac{2A}{K}(A + PD)$. We have shown that if $n \geqslant N$, then

$$\sum_{V \in \mathcal{V}^n} d_V(x, g^n x) \;\geqslant\; \frac{Kn}{A} - A - PD \;\geqslant\; \frac{Kn}{2A}.$$

Since the $\mathcal{V}_k^n$ are disjoint for fixed n, the conclusion holds with $\epsilon = \frac{K}{2Am}$, where $m = |\operatorname{Big}(g)|$ is bounded by the complexity of $\mathfrak{S}$ and K and A are independent of both n and g. □

For the remainder of the proof of Theorem 1.5, fix a domain $U = U_k$ such that the conclusion of Lemma 4.5 holds for arbitrarily large n. Let $\mathbb{N}_\epsilon$ be the set of such n, and let $\alpha = \alpha_k$.

Step 5: Accumulating distance in nested domains

There are now two cases to consider, depending on how the sum in Lemma 4.5 is distributed over $\mathcal{V}_k^n$. In each case, we will find a uniform lower bound on $\tau_U(g)$, which will complete the proof of the theorem.

Case 1: No relevant proper nesting.

If, for our chosen U, all of the properly nested domains are E'–irrelevant for all $n \in \mathbb{N}_\epsilon$, then the proof of the theorem concludes by applying Lemma 4.5.

Corollary 4.6. *If $\mathsf{d}_V(x, g^n x) < 5E$ for all $V \subsetneq U$ and every $n \in \mathbb{N}_\epsilon$, then $\tau_U(g) \geqslant \epsilon$.*

Proof. For each $n \in \mathbb{N}_\epsilon$, we must have $\mathcal{V}_k^n = \{U\}$, and Lemma 4.5 then gives $\mathsf{d}_U(x, g^n x) > \epsilon n$. Since $\tau_U(g) = \lim_{n \in \mathbb{N}_\epsilon} \mathsf{d}_U(x, g^n x)/n$, we conclude that $\tau_U(g) \geqslant \epsilon$. □

Since $\epsilon = \epsilon(\mathfrak{S}, E')$, this completes the proof in this case.

Case 2: Relevant proper nesting.

Suppose the assumption of Corollary 4.6 does not hold. That is, assume there is some $n \in \mathbb{N}_\epsilon$ and some $V_n \subsetneq U$ such that $\mathsf{d}_{V_n}(x, g^n x) > 5E$. If there is more than one such V_n, fix a $\sqsubseteq$–maximal choice.

If $\mathsf{d}_U(x, gx) > 14E$, then, since $\pi_U(x) \in \alpha$, Corollary 4.3 implies that $\tau_U(g) > E$, and the theorem is proved for the given g. Hence we can assume that $\mathsf{d}_U(x, gx) \leqslant 14E$.

The intuition behind the strategy of this part of the proof is as follows. First, we find a domain V that (intuitively, though not precisely) is relevant for x and any point further along the axis of g in $\mathcal{C}U$ than $g^k x$ for some k: see Figure 1. The specific way we find V also shows that for any i, the domain $g^i V$ is relevant for x and any point further along the axis than $g^{k+i}x$. If i is large enough, then there are lots of domains $g^j V$ that are relevant for the fixed pair of points x and $g^{k+i}x$; in fact, *most* of the domains $g^j V$ with $0 < j < i$ are relevant. The passing up lemma gives a uniform upper bound on the number of possible relevant domains that can appear before $\mathsf{d}_U(x, g^{k+i}x)$ must be uniformly large. From this, we deduce a uniform lower bound on translation length. Making this argument precise takes some care.

Lemma 4.7. *Under the above assumptions, there exists $V \subsetneq U$ and a natural number k such that the following hold.*

(i) $\mathsf{d}_U(x, g^k x) \leqslant 50E$.
(ii) $\mathsf{d}_U(\{x, g^k x\}, \rho_U^V) > 5E$.
(iii) *If $j < 0$, then $\mathsf{d}_V(g^j x, x) \leqslant E$.*
(iv) *If $j > k$, then $\mathsf{d}_V(g^k x, g^j x) \leqslant E$.*
(v) $\mathsf{d}_V(x, g^k x) > 5E$.
(vi) *If $V \subsetneq W \subsetneq U$, then $\mathsf{d}_W(x, g^k x) \leqslant 5E$.*

Proof. By Lemma 3.2, every geodesic from $\pi_U(x)$ to $\pi_U(g^n x)$ must come E–close to $\rho_U^{V_n}$. Since α is $2E$–quasiconvex, there is some point $y \in \alpha$ such that $\rho_U^{V_n}$ is contained in the $3E$–neighbourhood of y.

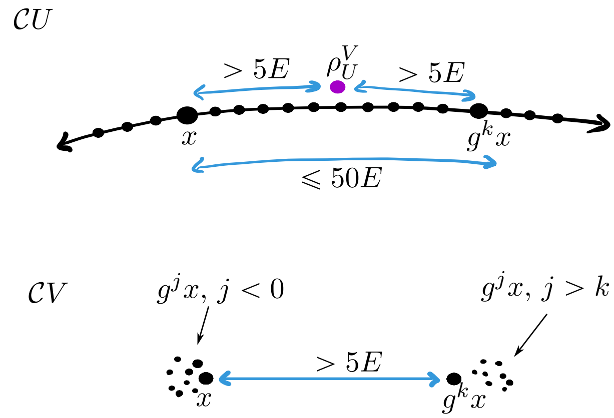

FIGURE 1. The properties of the domain $V \subsetneq U$ constructed in Lemma 4.7.

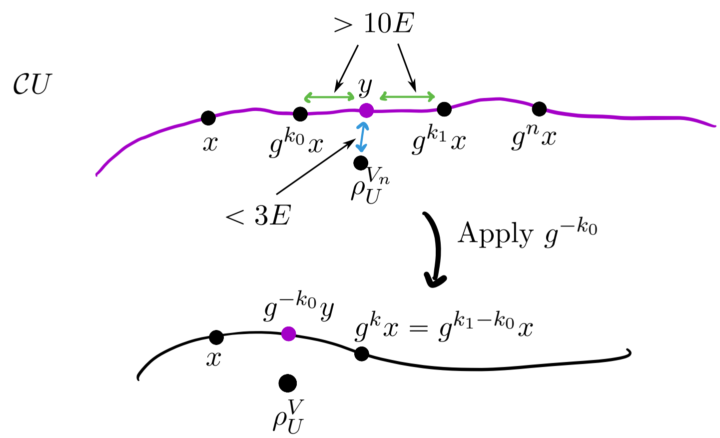

FIGURE 2. Finding the domain $V \subsetneq U$.

Because x lies on the quasiaxis α of g, there exist $k_0 < k_1$ with $k_1 - k_0$ minimal such that: $\mathsf{d}_U(g^{k_i}x, y) > 10E$ and there is some $2E$–quasigeodesic from $g^{k_0}x$ to $g^{k_1}x$ that contains y. Let $V = g^{-k_0}V_n$, and let $k = k_1 - k_0$. See Figure 2.

Item (i) holds because $k_1 - k_0$ was minimal and we are assuming that $\mathsf{d}_U(x, gx) \leqslant 14E$. Item (ii) holds by construction. Moreover, if $j < 0$, then no geodesic from $\pi_U(g^j x)$ to $\pi_U(x)$ can come $5E$–close to y, and hence cannot come E–close to ρ_U^V. Lemma 3.2 thus implies that $\mathsf{d}_V(g^j x, x) \leqslant E$, and so (iii) holds. Item (iv) holds for a similar reason. Together with the assumption on V_n, these imply (v). The final item holds since V_n is $\sqsubseteq$–maximal. □

Now fix k and $V \subsetneq U$ as in the above lemma. Let J denote the minimal natural number such that $\mathsf{d}_U(x, g^J x) > 400E$. Note that, although J is independent of V, in principle it may depend on g. The next two lemmas show that, in fact, J is bounded independently of g. We can therefore assume that $J \geqslant 12$.

We shall consider the set of all i such that $g^i \rho_U^V$ is approximately half-way between $\pi_U(x)$ and $\pi_U(g^J x)$; see Figure 3 for a schematic of the situation. Precisely, let

$$\mathcal{I} = \left\{ i \in \mathbb{Z} \,:\, \frac{J}{3} < i \leqslant k + i < \frac{2J}{3} \right\}.$$

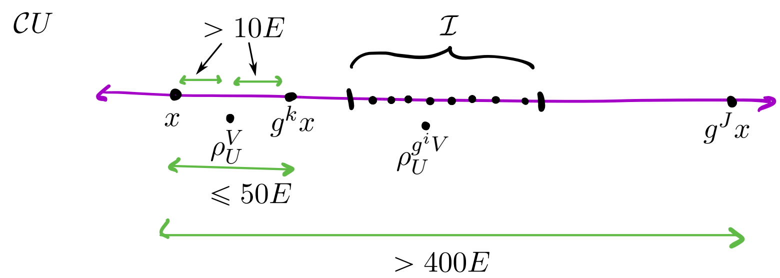

FIGURE 3. A schematic of the sets $g^i\rho_U^V = \rho_U^{g^iV}$ when $i \in \mathcal{I}$ in $\mathcal{C}U$.

Lemma 4.8. *$6k < J$, and $|\mathcal{I}| \geqslant \frac{J}{12}$.*

Proof. By definition, $|\mathcal{I}| \geqslant \frac{J}{3} - k - 1$. Moreover, the choice of k gives $\mathsf{d}_U(x, g^{6k}x) \leqslant 300E$. Because J is minimal with $\mathsf{d}_U(x, g^Jx) > 400E$, and because $\mathsf{d}_U(x, gx) \leqslant 14E$, we have $6k < J$. This shows that $|\mathcal{I}| \geqslant \frac{J}{6} - 1$, and we are done because we are assuming that $J \geqslant 12$. □

Given a number C, let $P(C)$ be the quantity given by the passing-up lemma, Lemma 3.4.

Lemma 4.9. *$|\mathcal{I}| < P(7E)$.*

Proof. If $i \in \mathcal{I}$, then $i > 0$, so by Lemma 4.7(iv) we have $\mathsf{d}_{g^iV}(x, g^ix) = \mathsf{d}_V(g^{-i}x, x) \leqslant E$. Similarly, $J - i > k$, and so by Lemma 4.7(iv), $\mathsf{d}_{g^iV}(g^{k+i}x, g^Jx) = \mathsf{d}_V(g^kx, g^{J-i}x) \leqslant E$. By the triangle inequality and Lemma 4.7(v), we therefore have

$$\mathsf{d}_{g^iV}(x, g^Jx) > \mathsf{d}_{g^iV}(g^ix, g^{k+i}x) - 2E > 5E - 2E > E.$$

Because g acts loxodromically on $\mathcal{C}U$, no power can stabilise any bounded set. In particular, ρ_U^V is not stabilised by any g^n, and hence the g^iV are pairwise distinct. Thus, if $|\mathcal{I}| \geqslant P(7E)$, then Lemma 3.4 produces a domain $W \subsetneq U$ such that $\mathsf{d}_W(x, g^Jx) > 7E$ and some g^iV is properly nested in W. Consistency then implies that $\rho_U^{g^{-i}W}$ is E–close to ρ_U^V.

Consider the domain $g^{-i}W$, into which V is properly nested. Lemma 4.7 implies that no geodesic from $\pi_U(g^{-i}x)$ to $\pi_U(x)$ can come E–close to $\rho_U^{g^{-i}W}$, and hence $\mathsf{d}_W(g^{-i}x, x) \leqslant E$ by Lemma 3.2. Also $i < \frac{2J}{3} - k$, so by Lemma 4.8 we have $J - i > 3k$. Hence Lemmas 3.2 and 4.7 similarly imply that $\mathsf{d}_{g^{-i}W}(g^{J-i}x, g^Jx) \leqslant E$. It follows from the triangle inequality that $\mathsf{d}_{g^{-i}W}(x, g^Jx) > 5E$, which contradicts Lemma 4.7. □

Corollary 4.10. *If the supposition of Corollary 4.6 fails, then $\tau_U(g) \geqslant \frac{E}{12P(7E)}$.*

Proof. By Lemmas 4.8 and 4.9, we see that there is a number $J \leqslant 12P(7E)$ such that $\mathsf{d}_U(x, g^Jx) > 14E$. The result follows from Corollary 4.3. □

Since $\frac{E}{12P(7E)}$ depends only on $(G, \mathfrak{S})$, this completes the proof of Theorem 1.5. □

5. $\mathfrak{S}$–TRANSLATION DISCRETENESS: EXAMPLES AND COUNTEREXAMPLES

We now discuss sharpness of Theorem 1.5. Recall from Lemma 3.7 that $g^{c!}U = U$ for $g \in G$ and $U \in \mathrm{Big}(g)$.

Definition 5.1. A hierarchically hyperbolic group $(G, \mathfrak{S})$ is $\mathfrak{S}$*–translation discrete* if there exists $\tau_0 > 0$ such that for all infinite-order $g \in G$, we have $\tau_U(g^{c!}) \geqslant \tau_0$ for all $U \in \mathrm{Big}(g)$.

There are two ways in which $\mathfrak{S}$–translation discreteness is stronger than the conclusion of Theorem 1.5: Theorem 1.5 only requires $\tau_U(g^{c!})$ to be uniformly bounded away from 0 for *some* $U \in \mathrm{Big}(g)$, and it does not rule out the possibility that the same U supports other elements $h \in G$ with $U \in \mathrm{Big}(h)$ but $\tau_U(h^{c!})$ arbitrarily small.

We now show that Theorem 1.5 is sharp by exhibiting HHG structures that are not $\mathfrak{S}$–translation discrete. These examples also show that a group G can admit HHG structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ such that G is $\mathfrak{S}_1$–translation discrete but not $\mathfrak{S}_2$–translation discrete.

We start with a strikingly simple example of some subtleties that can arise.

Example 5.2. Let $\mathbb{Z}^2 = \langle a, t \mid [a,t] \rangle$. For each $\varepsilon \in (0,1)$, there is an HHG structure $(\mathbb{Z}^2, \mathfrak{S}_\varepsilon)$ defined as follows. First, $\mathfrak{S}_\varepsilon = \{S, U, V\}$, where $\mathcal{C}S$ is a point and $\mathcal{C}U, \mathcal{C}V$ are copies of $\mathbb{R}$, and $U \perp V$, while $U, V \sqsubseteq S$. Define $\pi_U, \pi_V \colon \mathbb{Z}^2 \to \mathcal{C}U, \mathcal{C}V$ by $\pi_U(a^p t^q) = (p+q)\varepsilon - p$ and $\pi_V(a^p t^q) = p$ for $p, q \in \mathbb{Z}$. The maps $\pi_S \colon \mathbb{Z}^2 \to \mathcal{C}S$ and ρ^U_S, ρ^V_S are defined in the only possible way. We make two observations about $(\mathbb{Z}^2, \mathfrak{S}_\varepsilon)$:

- By construction, $\tau_U(a^p t^q) = \pi_U(a^p t^q) = (p+q)\varepsilon - p$. In particular, if ε is irrational, then τ_U takes arbitrarily small positive values, so $\mathbb{Z}^2$ is not $\mathfrak{S}_\varepsilon$–translation discrete.
- Whilst π_U and π_V vary with ε, most of the hierarchical hyperbolicity parameters need not. The exception is the *uniqueness function*, which necessarily depends on ε. One can then check that the constant A from the distance formula (Theorem 3.5) is of the order $O(\frac{1}{\varepsilon})$. Thus, although the translation length $\tau_{\mathbb{Z}^2}$ is independent of ε, the constant $\frac{K}{2A}$ from Lemma 4.5 diverges as $\varepsilon \to 0$. Hence Corollary 4.6 shows that the constant τ_0 in Theorem 1.5 is crucially dependent on the HHG structure $(G, \mathfrak{S}_\varepsilon)$.

One can vary the above examples by fixing $\delta \in (0,1)$ and redefining $\pi_V(a^p b^q) = (p+q)\delta - q$, while keeping π_U as above. There is then an uncountable subcollection of pairs (δ, ε), each yielding an HHG structure $(\mathbb{Z}^2, \mathfrak{S}_{\delta,\varepsilon})$, such that the HHS constants, including uniqueness, can be chosen uniformly over this family of HHG structures. Theorem 1.5 then *forces* the existence of a single τ_0 that works for all of these uniform structures, but for uncountably many of these (δ, ε), the HHG $(\mathbb{Z}^2, \mathfrak{S}_{\delta,\varepsilon})$ is not $\mathfrak{S}_{\delta,\varepsilon}$–translation discrete.

Example 5.2 is somewhat unsatisfying, since there is a more obvious HHG structure $\mathfrak{S}$ such that $\mathbb{Z}^2$ is $\mathfrak{S}$–translation discrete, namely the case $\varepsilon = 1$. This leaves open the possibility that every HHG has a structure $(G, \mathfrak{S})$ that is $\mathfrak{S}$–translation discrete. Below, we will produce examples of HHGs $(G, \mathfrak{S})$ that are more interesting than the above $\mathbb{Z}^2$ example for the reason that, whilst we show they are not $\mathfrak{S}$–translation discrete, we do not know whether there is some other structure $(G, \mathfrak{S}')$ such that G is $\mathfrak{S}'$–translation discrete. We achieve this by exhibiting many central extensions of HHGs that are again HHGs, and showing that *any* such central extension admits an HHG structure that is not $\mathfrak{S}$–translation discrete. First, though, we discuss $\mathfrak{S}$–translation discreteness for the most well-known HHG structures.

5.1. Positive examples

Here we motivate the upcoming construction of HHG structures that are not $\mathfrak{S}$–translation discrete by discussing some well-known HHGs.

As observed by Bowditch in [Bow08, Lem. 2.2] (or by Proposition 2.8), acylindrical actions on a hyperbolic spaces are translation discrete (positive translation lengths are uniformly bounded away from zero). Together with [BHS17, Thm 14.3], this shows that if $(G, \mathfrak{S})$ is an HHG and $S \in \mathfrak{S}$ is the unique $\sqsubseteq$–maximal element, then $\tau_S(g)$ is uniformly bounded below for $g \in G$ satisfying $\mathrm{Big}(g) = \{S\}$. This falls short of $\mathfrak{S}$–translation discreteness, but motivates the following terminology from [DHS17]. We say that $(G, \mathfrak{S})$ is *hierarchically acylindrical* if, for all $U \in \mathfrak{S}$, the action of $\mathrm{Stab}_G(U) = \{g \in G : gU = U\}$ on $\mathcal{C}U$ factors through an acylindrical action, i.e., the image of $\mathrm{Stab}_G(U) \to \mathrm{Isom}(\mathcal{C}U)$ acts acylindrically on $\mathcal{C}U$.

Proposition 5.3. *If $(G, \mathfrak{S})$ is hierarchically acylindrical, it is $\mathfrak{S}$–translation discrete.*

Examples covered by Proposition 5.3 include the standard HHG structures on fundamental groups of compact special cube complexes. For such G with the hierarchically hyperbolic structure $\mathfrak{S}$ from [BHS17], each $\mathrm{Stab}_G(U)$ is virtually a direct product of virtually compact special groups (see, e.g., [Zal23, Lemma 3.11]), one of which inherits an HHG structure where U is the $\sqsubseteq$–maximal element. By [BHS17, Thm 14.3], $(G, \mathfrak{S})$ is hierarchically acylindrical.

Nevertheless, many examples of HHGs are not hierarchically acylindrical. Indeed, Example 5.2 shows that even $\mathbb{Z}^2$ admits uncountably many HHG structures that are not hierarchically acylindrical—the following discussion is concerned with more usual structures. Examples are provided by irreducible lattices in products of trees, as constructed in [BM00, Wis07, Hug22]; see [DHS20]. In these examples, though, every $\mathcal{C}U$ is a tree, so loxodromic elements have combinatorial geodesic axes and hence the lattices are $\mathfrak{S}$–translation discrete. Mapping class groups also provide examples, as we now clarify.

Let S be a connected, orientable, hyperbolic surface S of finite type. The mapping class group $\mathrm{MCG}(S)$ admits a hierarchically hyperbolic structure $(\mathrm{MCG}(S), \mathfrak{S})$, described in [BHS19, §11] using results in [MM99, MM00, BKMM12, Beh06], where $\mathfrak{S}$ is the set of isotopy classes of essential (not necessarily connnected) non-pants subsurfaces. For each $U \in \mathfrak{S}$, the associated hyperbolic space is the curve graph $\mathcal{C}U$. When U is non-annular, the action of $\mathrm{Stab}(U)$ on $\mathcal{C}$ factors through the action of $\mathrm{MCG}(U)$ on $\mathcal{C}U$, which is acylindrical [Bow08, Thm 1.3], so translation lengths of $\mathcal{C}U$–loxodromic elements of $\mathrm{Stab}(U)$ are uniformly bounded below in terms of the topology of U (and hence in terms of the topology of S). However, this does not apply when U is an annulus, in view of the following fact, which is well-known but for which we have been unable to locate a reference.

Proposition 5.4. *Let S be a connected, orientable, finite-type surface with positive genus and one boundary component. Let γ be the boundary curve, and let $\mathcal{C}(\gamma)$ be the associated annular curve graph. The action of $\mathrm{MCG}(S)$ on $\mathcal{C}(\gamma)$ does not factor through an acylindrical action.*

The annular curve graph $\mathcal{C}(\gamma)$ and the action are described in [MM00, §2]. The proposition holds for stabilisers of curves in mapping class groups of more general surfaces, by a virtually identical argument, but we restrict our attention to surfaces with one boundary component for concreteness.

Proof of Proposition 5.4. For a surface U, write $G_U = \mathrm{MCG}(U)$. Let S_0 be the surface obtained from S by attaching a punctured disc, identifying γ with the boundary of the disc. The action of G_S on S_0 gives a surjective homomorphism $\phi\colon G_S \to G_{S_0}$ whose kernel is $T = \langle t \rangle \cong \mathbb{Z}$, where t denotes the Dehn twist about γ. This "capping homomorphism" yields a central extension

$$1 \to T \hookrightarrow G_S \xrightarrow{\phi} G_{S_0} \to 1;$$

see [FM12, Prop. 3.19], for instance.

Let $\psi\colon G_S \to \mathrm{Isom}\,\mathcal{C}(\gamma)$ be the action, and suppose that $\psi(G_S)$ acts acylindrically on $\mathcal{C}(\gamma)$. Since $\psi(t)$ acts loxodromically and $\mathcal{C}(\gamma)$ is quasi-isometric to $\mathbb{R}$ (see [MM00, §2]), the subgroup $\psi(T)$ has finite index in $\psi(G_S)$, by [DGO17, Lem. 6.7].

Let $G'_S = \psi^{-1}(\psi(T))$, which has finite index in G_S. Since $\psi|_T$ is injective, the map $r\colon g \mapsto \psi|_T^{-1}(\psi(g))$ is a retraction of G'_S onto T. Let $N = \ker(r)$. Since $N \cap T$ is trivial and T is central in G'_S, we have $G'_S = T \times N$. Also, $\phi|_N\colon N \to G_{S_0}$ is injective and has finite-index image. Letting $[\alpha] \in H^2(G_{S_0}, \mathbb{Z})$ be the cohomology class associated to the central extension given by ϕ, it follows from [FS20, Lem. 5.13] that $[\alpha]$ has finite order in $H^2(G_{S_0}, \mathbb{Z})$. On the other hand, $[\alpha]$ is the Euler class for G_{S_0}, which is known to have infinite order [FM12, §5.5.6]. This is a contradiction, so $\psi(G_S)$ cannot act acylindrically on $\mathcal{C}(\gamma)$. □

Despite Proposition 5.4, there is a uniform lower bound on $\tau_U(g)$ when U is an annulus and $g \in \mathrm{Stab}(U)$ acts on $\mathcal{C}U$ loxodromically. Hence, while $(\mathrm{MCG}(S), \mathfrak{S})$ is not hierarchically acylindrical, it is $\mathfrak{S}$–translation discrete.[1]

5.2. Quasimorphisms, central extensions, and bounded classes

Here we recall some facts needed for our construction, referring the reader to [Bro94, Ch. IV.3] for background on central extensions and [Cal09, Fri17] for quasimorphisms.

Let Γ be a group, and let $R \in \{\mathbb{Z}, \mathbb{R}\}$. A *quasimorphism* is a map $q\colon \Gamma \to R$ such that there exists $D < \infty$ for which

$$|q(g) + q(h) - q(gh)| \leqslant D$$

for all $g, h \in \Gamma$. The infimal D for which this holds is the *defect* of q, denoted $D(q)$. A quasimorphism q is *homogeneous* if $q(g^n) = nq(g)$ for all $g \in \Gamma$ and $n \in \mathbb{Z}$. Given any quasimorphism q, the *homogenisation* $\hat{q}\colon \Gamma \to \mathbb{R}$ is the homogeneous quasimorphism given by

$$\hat{q}(g) = \lim_{n\to\infty} \frac{q(g^n)}{n},$$

which has defect at most $2D(q)$.

For a group G, we consider central extensions

$$1 \to \mathbb{Z} \to E \xrightarrow{\phi} G \to 1.$$

We always use t to denote a generator of the kernel of ϕ. The group E is determined up to isomorphism by a cohomology class $[\alpha] \in H^2(G, \mathbb{Z})$ (viewing $\mathbb{Z}$ as a trivial $\mathbb{Z}G$–module). More precisely, letting the 2–cocycle $\alpha\colon G^2 \to \mathbb{Z}$ represent $[\alpha]$, there is an isomorphism $\psi_\alpha\colon E \to E_\alpha$, where E_α has underlying set $G \times \mathbb{Z}$ and group operation $*_\alpha$ given by

$$(g, p) *_\alpha (h, q) = (gh, p + q + \alpha(g, h));$$

see, e.g., [Bro94, p. 91–92]. We always assume α is *normalised*, i.e., $\alpha(g, 1) = \alpha(1, g) = 0$ for all $g \in G$, which is used implicitly in defining $*_\alpha$ but not needed later.

The extension E is said to *arise from a bounded class* if we can moreover take α to be bounded as a function to $\mathbb{Z}$. In this case, E is quasiisometric to $G \times \mathbb{R}$ [Ger92, Thm 3.1]. We are interested in certain quasimorphisms on such E.

First, consider the map $q_\alpha = \eta\psi_\alpha\colon E \to \mathbb{Z}$, where $\eta\colon E_\alpha \to \mathbb{Z}$ is the natural projection to the second factor. As observed in [HRSS22, Lem. 4.1] and [HMS21, Lem. 4.3], q_α is a quasimorphism with $q_\alpha(t^n) = n$ for all $n \in \mathbb{Z}$ (perhaps after inverting t), by boundedness of α.

Let $\hat{q}_\alpha\colon E \to \mathbb{R}$ be the homogenisation of q_α. For each infinite-order $g \in G$, the subgroup $P_g = \phi^{-1}(\langle g\rangle)$ is isomorphic to $\mathbb{Z}^2$ and contains t. The quasimorphism $\hat{q}_\alpha$ restricts to a homogeneous quasimorphism $\hat{q}_\alpha\colon P_g \to \mathbb{R}$, and, since P_g is abelian, $\hat{q}_\alpha|_{P_g}$ is a homomorphism by [Cal09, Prop. 2.65]. We will use this homomorphism to choose an element $\bar{g} \in P_g$ and a constant $\kappa_g \in \mathbb{Z}$, which will be useful in the next section.

The rank of $\ker(\hat{q}_\alpha|_{P_g})$ is 0 or 1. Suppose the kernel is nontrivial and choose a generator $\bar{g}$ of $\ker(\hat{q}_\alpha|_{P_g})$. Hence there is a unique pair of integers κ_g, θ_g such that $\psi_\alpha(\bar{g}) = (g^{\kappa_g}, \theta_g)$. Since

[1] See `https://mathoverflow.net/questions/439665/translation-length-on-annular-curve-graphs`, where a proof using stable train tracks is sketched by Sam Nead and Lee Mosher.

$\hat{q}_\alpha(t) = 1$, we have $\kappa_g \neq 0$. On the other hand, if $\hat{q}_\alpha \colon P_g \to \mathbb{R}$ is injective, choose $\bar{g} \in P_g - \langle t \rangle$ arbitrarily and let $\kappa_g = 0$.

For any homogeneous quasimorphism $\hat{p} \colon G \to \mathbb{R}$ on G, the map $\hat{p}\phi \colon E \to \mathbb{R}$ is a homogeneous quasimorphism with $\hat{p}\phi(t) = 0$. Hence $\hat{r} = \hat{q}_\alpha + \hat{p}\phi$ is a homogeneous quasimorphism on E, and $\hat{r}(t) = 1$ since $\hat{p}\phi(t) = \hat{p}(1) = 0$ by homogeneity of $\hat{p}$.

5.3. Quasimorphisms taking arbitrarily small values

We present two constructions: the first is simpler, and the second is similar to that in [BBF19] and yields more information. Various other constructions could be used instead.

5.3.1. *Generalising Example 5.2.*

Let G be an arbitrary group admitting a nontrivial homogeneous quasimorphism $\hat{p} \colon G \to \mathbb{R}$. Fix $g \in G$ with $\hat{p}(g) \neq 0$. By homogeneity, g must have infinite order, and by rescaling we can assume that $\hat{p}(g) = 1$. Let $\phi \colon E \to G$ be a $\mathbb{Z}$–central extension arising from a bounded cocycle α, and let $q_\alpha \colon E \to \mathbb{Z}$ be the quasimorphism $q_\alpha = \eta\psi_\alpha$ considered above. Let $\hat{q}_\alpha$ be its homogenisation, and, given $\delta, \epsilon \geqslant 0$, let $\hat{r} = \delta\hat{q}_\alpha + \epsilon\hat{p}\phi$. Recall that $\hat{r}$ is a homomorphism on $P_g \cong \mathbb{Z}^2$, and note that $\hat{r}(t) = \delta\hat{q}_\alpha(t) = \delta$. If $\hat{q}_\alpha$ is non-injective on P_g, then, letting $\bar{g}, \kappa_g$ be defined as above, we have $\hat{r}(\bar{g}) = \epsilon\kappa_g$. So if, for instance, $(\delta, \epsilon) = (1, \sqrt{2})$, the map $\hat{r}$ takes arbitrarily small positive values on P_g and hence on E. If $\hat{q}_\alpha$ is injective on P_g, then we can take $\delta = 1, \varepsilon = 0$. In this case, $\hat{r}(t) = 1$, so by injectivity, $\hat{r}(\bar{g})$ is irrational for the choice of $\bar{g}$ above, and thus $\hat{r}(P_g)$ is dense, so $\hat{r}$ takes arbitrarily small positive values on E.

5.3.2. *Combinations of Brooks quasimorphisms.*

Let G be a finitely generated group admitting a nonelementary acylindrical action on a hyperbolic geodesic metric space.

We first consider the case where the action is nonelementary. By [DGO17, Thm 6.14], there exist $a, b \in G$ such that: $\langle a, b \rangle = F$ is a free group; G has a maximal finite normal subgroup N; we have $\langle N, a, b \rangle \cong N \times F$; and $N \times F$ is *hyperbolically embedded* in G.

Given a reduced, cyclically reduced word $w \in F$, define $\#_w \colon F \to \mathbb{R}$ by letting $\#_w(x)$ be the maximum cardinality of a set of disjoint subwords of x, each of which is equal to w. The *small Brooks quasimorphism* $h_w \colon F \to \mathbb{Z}$ is given by $h_w(x) = \#_w(x) - \#_{w^{-1}}(x)$ [Bro81]. By [Cal09, Prop. 2.30], h_w is a quasimorphism with defect at most 2.

Define $g_i = (a^i b^i)^{101}$. This concrete choice is somewhat arbitrary, but satisfies certain small-cancellation conditions, as in [Bow98, TV00]. Observe that g_i is not a subword of $g_j^{\pm n}$ if $j \neq i$, nor is it a subword of g_i^{-n}. This shows that the corresponding small Brooks quasimorphisms satisfy $h_{g_i}(g_i^n) = n$ and $h_{g_i}(g_j^n) = 0$ for all $j \neq i$.

Let $(\lambda_i)_{i=1}^\infty$ be a sequence of nonzero real numbers with $\sum_{i=1}^\infty |\lambda_i| < \infty$. Define

$$p_F = \sum_{i=1}^{\infty} \lambda_i h_{g_i}.$$

Observe that the above sum is finite for all $x \in F$, because $h_{g_i}(x) = 0$ if $|g_i| > |x|$. Thus, because the h_{g_i} are quasimorphisms with defect at most 2, the map p_F is a quasimorphism with defect at most $2\sum_i |\lambda_i| < \infty$. The homogenisation $\hat{p}_F$ of p_F satisfies $\hat{p}_F(g_i) = \lambda_i$ for all i; in particular, $|\hat{p}_F|$ takes arbitrarily small positive values.

Extend $\hat{p}_F$ over $N \times F$ by declaring $\hat{p}_F$ to vanish on N. Viewed as a 1–cocycle, $\hat{p}_F$ is *antisymmetric* (by virtue of being homogeneous). Since $N \times F$ is hyperbolically embedded in

G, [HO13, Thm 1.4] provides a quasimorphism $p\colon G \to \mathbb{R}$ such that

$$L = \sup_{x \in N \times F} |\hat{p}_F(x) - p(x)| < \infty.$$

The homogenisation $\hat{p}\colon G \to \mathbb{R}$ satisfies $\hat{p}|F = \hat{p}_F$. In particular, $\hat{p}(g_i) = \lambda_i$ for all i, so $|\hat{p}|$ takes arbitrarily small positive values on G.

5.3.3. *Summary.*

We can now prove the following corollary, our first tool for constructing HHG structures that are not $\mathfrak{S}$–translation discrete.

Corollary 5.5. *Let $\phi\colon E \to G$ be a $\mathbb{Z}$–central extension, associated to a bounded cohomology class, of a group G that admits a nontrivial homogeneous quasimorphism. There exists a homogeneous quasimorphism $\hat{r}\colon E \to \mathbb{R}$ such that*

- *$\hat{r}(t) = 1$, and*
- *for all $\epsilon > 0$, there exists $e \in E$ such that $\hat{r}(e) \in (0, \epsilon)$.*

Moreover, if G has a nonelementary acylindrical action on a hyperbolic space, then $\hat{r}$ can be chosen with $\lim_{i\to\infty} \hat{r}(e_i) = 0$, where $(\phi(e_i))_i$ is some sequence of loxodromic elements of G.

Proof. As explained in Section 5.3.1, there exists $\hat{r}$ satisfying the itemised properties as soon as G admits a nontrivial homogeneous quasimorphism.

If G is acylindrically hyperbolic, then we can make a more specific choice of $\hat{r}$ as follows. First, let $(g_i)_i$ be loxodromic elements of G chosen as in Section 5.3.2. For each i, let κ_{g_i} be the integer chosen above by considering the restriction of $\hat{q}_\alpha$ to P_{g_i}, and let $\bar{g}_i \in P_{g_i}$ be the associated element. For each i, if $\kappa_{g_i} = 0$, let $\lambda_i = 0$, and otherwise let $\lambda_i = \frac{1}{2^i \kappa_{g_i}}$. Let $\hat{p}\colon G \to \mathbb{R}$ be the resulting homogeneous quasimorphism from Section 5.3.2.

Now let $\hat{r} = \hat{q}_\alpha + \hat{p}\phi$. As before, $\hat{r}(t) = 1$. Now, for each i such that $\kappa_{g_i} \neq 0$, we chose $\bar{g}_i$ such that $\hat{q}_\alpha(\bar{g}_i) = 0$ and we chose κ_{g_i} so that $\hat{p}\phi(\bar{g}_i) = \kappa_{g_i}\hat{p}(g_i)$. Hence $\hat{r}(\bar{g}_i) = \frac{1}{2^i}$.

If $\kappa_{g_i} = 0$, then $\hat{p}(g_i) = \lambda_i = 0$, so $\hat{p}\phi$ vanishes on P_{g_i}, so $\hat{r} = \hat{q}_\alpha$ on P_{g_i}. Also, in this case, $\hat{q}_\alpha$ is an injective homomorphism on P_{g_i}, and $\bar{g}_i$ was chosen outside of $\langle t \rangle$, and thus $\hat{q}_\alpha(\bar{g}_i) \notin \mathbb{Q}$. Thus, by applying powers of t, we can assume $0 < \hat{r}(\bar{g}_i) \leqslant \frac{1}{2^i}$.

Observing that $\phi(\bar{g}_i)$ is a nonzero power of g_i, we are done, taking $e_i = \bar{g}_i$. □

5.4. HHG constructions

Here we construct HHG structures that are not $\mathfrak{S}$–translation discrete. The next lemma is [ABO19, Lem. 4.15], except that we have extracted an additional consequence of their proof.

Lemma 5.6 (Quasilines from quasimorphisms)**.** *Let Γ be a group and let $\hat{s}\colon \Gamma \to \mathbb{R}$ be a nonvanishing homogeneous quasimorphism. There exists a graph L, quasi-isometric to $\mathbb{R}$, and a vertex-transitive, isometric action of Γ on L that fixes both ends of L. Moreover, there exists K such that for all $g \in \Gamma$ we have*

$$\frac{1}{K}|\hat{s}(g)| \leqslant \tau_L(g) \leqslant K|\hat{s}(g)|,$$

where $\tau_L(g)$ denotes the stable translation length of g on L.

Proof. Fix any positive number C_0 such that there is some $g_0 \in \Gamma$ with $|\hat{s}(g_0)| = C_0$.

Let $C \geqslant 2D(\hat{s})$ be such that there is some $g \in \Gamma$ with $\hat{s}(g) \in (0, C/2)$. According to [ABO19, Lem. 4.15], the set $\mathcal{A}$ of $g \in \Gamma$ such that $|\hat{s}(g)| < C$ generates Γ. Let $L = \mathrm{Cay}(\Gamma, \mathcal{A})$. As explained in [ABO19], L is quasi-isometric to $\mathbb{R}$, and the action of Γ fixes the ends of L. The proof of [ABO19, Lem. 4.15] shows that

$$\frac{2C|\hat{s}(g)|}{3} \leqslant \mathsf{d}_L(1,g) \leqslant \frac{|\hat{s}(g)|}{C_0}+2,$$

for all $g \in \Gamma$, from which the statement about translation lengths follows. □

Remark 5.7. One could deduce Lemma 5.6 from [Man06, Prop. 3.1]; we thank Alice Kerr for this observation. A more general statement [KL09, Cor. 1.1] about quasi-actions also works.

The following lemma is extracted from the proof of [HRSS22, Cor. 4.3].

Lemma 5.8. *Let $\phi\colon E \to G$ be a $\mathbb{Z}$–central extension of a finitely generated group G. Suppose that E acts by isometries on a graph L that is quasi-isometric to $\mathbb{R}$. Suppose further that $\tau_L(t)>0$, where t generates* $\ker\phi$. *When $G\times L$ is equipped with the ℓ^1–metric, the diagonal action of E is metrically proper and cobounded.*

Proof. Fix a base vertex $x \in L$, and let B be such that L is covered by the $\langle t\rangle$–translates of the ball $B_L(x,B)$. For each $g \in G$, choose $e_g \in \phi^{-1}(g)$ such that $d_L(x,e_gx) \leqslant B$, which is possible because t generates $\ker\phi$.

Properness. As t is loxodromic on L, there exists K such that $d_L(x,t^nx) \geqslant K|n|-K$ for all n. Given $R \geqslant 0$, let $G_R=\{g \in G : \mathsf{d}_G(1,g) \leqslant R\}$, which is finite since G is finitely generated.

Suppose $e \in E$ moves $(1,x)$ a distance at most R in $G\times L$. Then $\phi(e) \in G_R$ is one of only finitely many elements. There exists $n \in \mathbb{Z}$ such that $e=t^ne_{\phi(e)}$. From the triangle inequality,

$$d_L(x,t^nx) \leqslant \mathsf{d}_L(x,ex)+\mathsf{d}_L(t^ne_{\phi(e)}x,t^nx) \leqslant R+B.$$

Hence $|n| \leqslant (R+B+K)/K$, and so there are only finitely many such elements $e \in E$.

Coboundedness. Given $(g,y) \in G\times L$, there exists n such that $\mathsf{d}_L(t^ne_gx,y) \leqslant B$. Because $t \in \ker\phi$, we have $\phi(t^ne_g)=g$, so t^ne_g moves $(1,x)$ within distance B of (g,y). □

This lemma gives hierarchically hyperbolic structures on $\mathbb{Z}$–central extensions.

Proposition 5.9 (HHG central extensions)**.** *Let $\phi\colon E \to G$ be a $\mathbb{Z}$–central extension of an HHG $(G,\mathfrak{S})$. Suppose E acts by isometries on a graph L that is quasi-isometric to $\mathbb{R}$. Suppose that $\tau_L(t)>0$, where t generates* $\ker\phi$. *The group E admits an HHG structure $(E,\mathfrak{S}_E)$ where*

- *$\mathfrak{S}_E$ contains $\mathfrak{S}\sqcup\{A,S_E\}$, where S_E and A are two symbols not in $\mathfrak{S}$;*
- *$\mathcal{C}A=L$ and $\mathcal{C}W$ is a point for all $W \in \mathfrak{S}_E-(\mathfrak{S}\sqcup\{A\})$;*
- *$A \perp U$ for all $U \in \mathfrak{S}$;*
- *$A \subsetneq S_E$ and $U \subsetneq S_E$ for all $U \in \mathfrak{S}$.*

Moreover, E stabilises A, the induced action on $\mathcal{C}A$ is the given action on L, and $\mathrm{Big}(t)=\{A\}$.

Proof. Equip $\mathfrak{S}$ with all the same structure as in $(G,\mathfrak{S})$, but define the projections from $E \to \mathcal{C}U$ for $U \in \mathfrak{S}$ by composing the projections $\pi_U\colon G \to \mathcal{C}U$ with $\phi\colon E \to G$. The projection $\pi_A\colon E \to \mathcal{C}A$ is an orbit map $E \to L$. The remaining projections are maps to one-point spaces.

By [BHS19, Prop. 8.27], $\mathfrak{S}_E$ can be chosen so that $(G\times L,\mathfrak{S}_E)$ with the ℓ^1–metric is an HHS and all the bullet points in the statement are satisfied. From the explicit description of this construction in [BR20a, Example 2.13], the set $\mathfrak{S}_E$ consists of $A,S_E,\mathfrak{S}$, and an element V_U for each $U \in \mathfrak{S}$ except the $\sqsubseteq$–maximal element. The group E acts on $\mathfrak{S}$ via the G–action and ϕ and we declare E to act on the set of V_U in the same way, and finally to fix A and S_E. Since the G–action on $\mathfrak{S}$ is cofinite, the E–action on $\mathfrak{S}_E$ is cofinite. It is easily verified that, with the diagonal action of E on $G\times L$, the equivariance conditions of an HHG are satisfied.

It remains to check that the action of E on $G \times L$ is proper and cobounded, but this is given by Lemma 5.8. □

Now we assemble all of the ingredients:

Theorem 5.10. *Let $(G, \mathfrak{S})$ be an HHG that is not quasi-isometric to the product of two unbounded spaces. Then every $\mathbb{Z}$–central extension $E \to G$ arising from a bounded class admits an HHG structure $(E, \mathfrak{S}_E)$ such that E is not $\mathfrak{S}_E$–translation discrete.*

Proof. By [PS20, Cor. 4.7 & Rem. 4.8], G is acylindrically hyperbolic or 2–ended. In either case, Corollary 5.5 provides a homogeneous quasimorphism $\hat{r} \colon E \to \mathbb{R}$ taking arbitrarily small positive values, with $\hat{r}(t) = 1$, where t generates $\ker(\phi)$. Lemma 5.6 gives a quasiline L and an isometric E–action on L where τ_L takes arbitrarily small positive values on E but $\tau_L(t) > 0$.

According to Proposition 5.9, E admits an HHG structure $(E, \mathfrak{S}_E)$ for which there exists $A \in \mathfrak{S}_E$ such that $\mathcal{C}A = L$, and E fixes A, and the E–action on $\mathcal{C}A$ is exactly the action on L given above. In particular, $\tau_A(t) > 0$ and $\tau_A(e)$ takes arbitrarily small positive values as e varies in E. By Definition 5.1, E is thus not $\mathfrak{S}_E$–translation discrete. □

Remark 5.11. In the special case where G is not two-ended, and is hence acylindrically hyperbolic, the stronger statement in Corollary 5.5 gives an HHG structure $(E, \mathfrak{S})$ where the arbitrarily small translation lengths on the quasiline are witnessed by a sequence of elements $(\tilde{g}_i)_i$ in E whose images in G are loxodromic on the top-level hyperbolic space for the original HHG structure on G, and in fact there is a great deal of flexibility in choosing these $\tilde{g}_i$.

5.5. Questions on $\mathfrak{S}$–translation discreteness

Theorem 1.4 provides many examples of HHGs $(G, \mathfrak{S})$ that are not $\mathfrak{S}$–translation discrete (Definition 5.1), but Example 5.2 illustrates that G may admit some other structure $(G, \mathfrak{S}')$ that is $\mathfrak{S}'$–translation discrete. Hence:

Question 5.12. *Does there exist a finitely generated group G such that G admits an HHG structure but does not admit an HHG structure that is $\mathfrak{S}$–translation discrete?*

Every hierarchically hyperbolic group $(G, \mathfrak{S})$ has a *coarse median structure* associated to $\mathfrak{S}$ [BHS19, Bow13]. Distinct hierarchical structures can result in equivalent coarse median structures. In Example 5.2, however, the coarse median structures associated to each $\mathfrak{S}_\varepsilon$ are all distinct. The example shows that changing the choice of ε can change whether the group is $\mathfrak{S}_\varepsilon$–translation discrete. Thus we ask:

Question 5.13. *If $(G, \mathfrak{S})$ and $(G, \mathfrak{S}')$ are hierarchically hyperbolic group structures with the same associated coarse median structure, is it the case that G is $\mathfrak{S}$–translation discrete if and only if G is $\mathfrak{S}'$–translation discrete?*

In the direction of finding $\mathfrak{S}$–translation discrete structures, one can ask the following.

Question 5.14. *Let G be an acylindrically hyperbolic HHG, and let $\phi \colon E \to G$ be a $\mathbb{Z}$–central extension associated to a bounded cohomology class. When does E admit a homogeneous quasimorphism $\hat{r} \colon E \to \mathbb{R}$ such that $\hat{r}$ is unbounded on $\ker(\phi)$, and $|\hat{r}(e)|$ does not take arbitrarily small positive values as e varies in E (we allow $\hat{r}(e) = 0$)?*

Given such an $\hat{r}$, Proposition 5.9 produces an HHG structure $\mathfrak{S}$ on E that is $\mathfrak{S}$–translation discrete provided G admits one. In light of [ANS+19], where special consideration is given to hierarchically hyperbolic groups coarsely having a $\mathbb{Z}$ factor, one can also ask:

Question 5.15. *Let E be a hierarchically hyperbolic group quasi-isometric to $\mathbb{Z} \times A$, with A an unbounded space, one of whose asymptotic cones has a cut-point. Must E contain a finite-index subgroup E' and a infinite-order element $t \in E'$ such that t is central in E' and $E'/\langle t\rangle$ is a hierarchically hyperbolic group?*

A positive answer would show that, to strengthen the results in [ANS$^+$19], $\mathfrak{S}$–translation discreteness is most interesting for central extensions.

It may be that central extensions alone aren't enough to answer Question 5.12. Perhaps there are more elaborate examples involving complexes of groups whose vertex groups are central extensions of HHGs, assembled so that a combination theorem as in [BHMS20] provides an HHG structure, but where the induced HHG structures on the vertex groups are forced to involve translation-indiscrete actions on quasilines; the graphs of groups in [HRSS22] might be a starting point.

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