NON-COLOURABLE HIERARCHICALLY HYPERBOLIC GROUPS

MARK HAGEN

Abstract. We exhibit a hierarchically hyperbolic group for which no hierarchically hyperbolic structure is *colourable*, answering an (implicit) question of Durham-Minsky-Sisto.

Introduction

An important feature of the mapping class group $MCG(S)$ of a finite-type surface S, observed by Bestvina-Bromberg-Fujiwara [BBF15], is the existence of a finite colouring of the subsurfaces of S such that subsurfaces of the same colour overlap. Together with similar statements about compact special groups (see [HW08] and [BHS17, Section 11]), this suggests a useful property of some *hierarchically hyperbolic groups (HHGs)*, which include the aforementioned examples.

This property is *colourability*, formalised in [DMS20, HP22]. One purpose of colourability is to connect the hierarchically hyperbolic geometry to the *projection systems* introduced in [BBF15]. In [HP22, Pet21], this is used to produce quasi-isometries from various hierarchically hyperbolic groups to CAT(0) cube complexes. In [DMS20], it is used to apply cubical geometry to hierarchically hyperbolic groups, for example to prove semihyperbolicity of the mapping class group. Colourability of HHGs is also used in the forthcoming paper [CRHK22] on asymptotic cones of HHG. In [DMS20, HP22], the authors (implicitly) ask for an example of a non-colourable HHG. The purpose of this note is to describe one.

We recall the definition of colourability given in [DMS20]; the definition in [HP22] is the same, up to passing to finite-index subgroups. We first recall that a hierarchically hyperbolic group $(G, \mathfrak{S})$ is a finitely generated group G and a set $\mathfrak{S}$ with three mutually exclusive relations (nesting, orthogonality, and transversality, denoted $\sqsubseteq, \perp, \pitchfork$) such that G acts cofinitely on $\mathfrak{S}$, preserving the relations. Each $U \in \mathfrak{S}$ is associated to a hyperbolic space $\mathcal{C}U$ and a coarsely lipschitz coarse map $\pi_U : G \to \mathcal{C}U$. This setup must satisfy some geometric axioms; see [BHS19, Definition 1.1]. There are equivariance conditions not needed for the definition of colourability, which we postpone. The preceding data is an *HHG structure* for G.

Definition. An HHG $(G, \mathfrak{S})$ is *colourable* if there is a finite partition $\mathfrak{S} = \bigsqcup_{i=1}^{\chi} \mathfrak{S}_i$ such that:

- for all $i \leqslant \chi$ and all $U, V \in \mathfrak{S}_i$, we have $U \pitchfork V$;
- the action of G on $\mathfrak{S}$ induces an action by permutations on the set $\{\mathfrak{S}_i\}_{i=1}^{\chi}$ of *colours*.

A group G may admit distinct HHG structures, some colourable and some not.

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1. The example

We will construct a group G as an amalgam

$$G = \Lambda \underset{\langle\lambda\rangle=\langle t\rangle}{*} T,$$

of a non-residually finite cubical group Λ and a 3–dimensional crystallographic group T. We will choose t so that, in any hierarchically hyperbolic structure on G, t takes some domain to an orthogonal domain. We will choose λ so that any finite-index subgroup of G contains $t = \lambda$. There is also some care required in choosing λ so that G is actually an HHG. This section involves CAT(0) cube complexes — we refer to [BHS17, HS20] for background.

1.1. **The group Λ and the element λ.** We start with a basic fact phrased in cubical terms:

Lemma 1.1. *Let C be a proper CAT(0) cube complex on which the group Γ acts geometrically. Let $\gamma \in \Gamma$ have a combinatorial axis α_0 and suppose that the cubical convex hull α of α_0 is contained in a neighbourhood of α_0. Let $\pi_\alpha : C \to \alpha$ be the gate map (see e.g.* [Hag22, Section 2.2]*). Then for all $h \in \Gamma$, either $\pi_\alpha : h\cdot\alpha \to \alpha$ is a cubical isomorphism, or* $\operatorname{diam}(\pi_\alpha(h\cdot\alpha)) < \infty$.

Proof. Since $\langle\gamma\rangle$ acts cocompactly on α_0 and α is $\langle\gamma\rangle$–invariant and contained in a neighbourhood of α_0, the action of $\langle\gamma\rangle$ on α is cocompact. Since the Γ–action is geometric, each translate $g\alpha$ has bounded Hausdorff distance from the image of $g\langle\gamma\rangle$ under a fixed orbit map, and hence any ball in C intersects finitely many translates of α.

Let $R = \mathsf{d}_C(\alpha, h\alpha)$, so that $\mathsf{d}_C(\alpha, gh\alpha) = R$ for all $g \in \langle\gamma\rangle$. For any $x \in \alpha$ and $s \geqslant 0$, if $g\pi_\alpha(h\alpha) \cap B^\alpha_s(x) \neq \varnothing$, then $gh\alpha \cap B^C_{R+s}(x) \neq \varnothing$. The number of such $gh\alpha$ is finite. Hence the $\langle\gamma\rangle$–translates of $\pi_\alpha(h\alpha)$ form a locally finite family in α. So by e.g. [HS20, Lemma 2.3], the action of $\operatorname{Stab}_{\langle\gamma\rangle}(\pi_\alpha(h\alpha))$ (which is virtually $\langle\gamma\rangle \cap \langle\gamma\rangle^h$) on $\pi_\alpha(h\alpha)$ is cocompact. So, if $\langle\gamma\rangle \cap h\langle\gamma\rangle h^{-1} = \{1\}$, then $\pi_\alpha(h\cdot\alpha)$ is compact. Otherwise, there exists $n \neq 0$ such that $h\gamma^n h^{-1} = \gamma^{\pm n}$, whence $\pi_\alpha(h\cdot\alpha) = \alpha$ since γ^n skewers the same hyperplanes as γ. □

Corollary 1.2. *Let T_1, T_2 be locally finite trees and let $\Gamma \leqslant \operatorname{Aut}(T_1)\times\operatorname{Aut}(T_2)$ act geometrically on $T_1 \times T_2$. Let $\gamma \in \Gamma$ fix $v \in T_2$ and act hyperbolically on $T_1 \times \{v\}$, with axis α. Let $\bar\alpha$ be the image of α under the natural projection to T_1. Then for all $h \in \Gamma$, either $h\bar\alpha = \bar\alpha$, or $h\bar\alpha \cap \bar\alpha$ has finite diameter (including the possibility that it is empty).*

Proof. Since α is a geodesic in $T_1 \times \{v\}$, we have that α is convex in $T_1 \times T_2$. Fix $h \in \Gamma$. By Lemma 1.1, either $h\alpha$ and α cross exactly the same hyperplanes, or they cross finitely many common hyperplanes, since the hyperplanes crossing $\pi_\alpha(h\alpha)$ are exactly those crossing both α and $h\alpha$ by e.g. [Hag22, Lemma 2.5]. Since α and $h\alpha$ have trivial projections to T_2, the hyperplanes crossing α (resp. $h\alpha$) are the preimages under natural projection to T_1 of midpoints of edges in $\bar\alpha$ (resp. $h\bar\alpha$), so the edges of $\bar\alpha \cap h\bar\alpha$ are in bijection with the hyperplanes crossing both α and $h\alpha$, and we are done. □

Now we discuss irreducible lattices in products of trees, following the discussion from [Cap19, Section 4]. We begin with a (finite) *BMW presentation* of a torsion-free *BMW group*:

$$\Gamma = \langle A \cup \mathcal{S} \mid R\rangle,$$

so that the associated Cayley graph is the 1–skeleton of a product $T_A \times T_{\mathcal{S}}$ of trees, where the edges in each $T_A \times \{x\}$ are labelled with the elements of A, and likewise for $T_{\mathcal{S}}$ and $\mathcal{S}$. So, $C = T_A \times T_{\mathcal{S}}$ is a proper CAT(0) square complex, and the action of Γ by left multiplication gives an inclusion $\Gamma \to \operatorname{Aut}(T_A) \times \operatorname{Aut}(T_{\mathcal{S}})$, which we assume is *irreducible.*

The element $\gamma \in \Gamma - \{1\}$ is *A–convex* if there is a vertex $v \in T_{\mathcal{S}}$ fixed by the image of γ in $\operatorname{Aut}(T_{\mathcal{S}})$. Hence γ has a combinatorial axis $\alpha \subset T_A \times \{v\}$ where the natural projection $C \to T_A$ sends α isometrically to the axis $\bar\alpha$ for γ in T_A. (We conflate α and $\bar\alpha$ with their images.)

Corollary 1.2 says that for all $h \in \Gamma$, either $h\bar{\alpha} = \bar{\alpha}$ or $\operatorname{diam}(h\bar{\alpha} \cap \bar{\alpha}) \leqslant N(h)$ for some $N(h) \in \mathbb{N}$ (with $N(h) = 0$ if $\bar{\alpha} \cap h\bar{\alpha} = \varnothing$). If there exists $N < \infty$ such that $N(h) \leqslant N$ whenever $h \in \Gamma$ satisfies $h\bar{\alpha} \neq \bar{\alpha}$, then the A–convex element γ is N*–controlled*. (There can be convex elements γ that are not N–controlled for any N; see [Wis07, Section 5] or [She22] or Remark 3.5.)

Following [Wis07], we construct a new irreducible lattice. Let $\bar{A}$ be a copy of A and let $a \mapsto \bar{a}$ be a bijection $A \to \bar{A}$. Let $\bar{R}$ be obtained from R by, in each relation, replacing each $a \in A$ by $\bar{a}$ wherever it occurs (and keeping the elements of $\mathcal{S}$). Let

$$\Lambda = \langle A \sqcup \bar{A} \sqcup \mathcal{S} \mid R \sqcup \bar{R} \rangle \cong \Gamma \underset{\langle \mathcal{S} \rangle}{*} \Gamma,$$

which is an irreducible lattice in $\operatorname{Aut}(T_{A \cup \bar{A}}) \times \operatorname{Aut}(T_{\mathcal{S}})$. Let $\widetilde{X}_\Lambda = T_{A \cup \bar{A}} \times T_{\mathcal{S}}$ (whose 1–skeleton we identify with the Cayley graph of the above BMW presentation of Λ).

Lemma 1.3. *There exists $\lambda \in \Lambda - \{1\}$ such that all of the following hold:*

- *λ is $A \cup \bar{A}$–convex,*
- *there is a finite-index subgroup $P \leqslant \langle \mathcal{S} \rangle$ centralising λ,*
- *λ is N–controlled for some $N \in \mathbb{N}$, and*
- *every finite-index subgroup of Λ contains λ.*

Proof. We imitate the discussion after Proposition 4.17 in [Cap19]. Define a homomorphism $f : \Lambda \to \Gamma$ by $f(x) = x$ for $x \in \mathcal{S}$ and $f(a) = a$ for $a \in A$ and $f(\bar{a}) = a$ for $\bar{a} \in \bar{A}$.

Let $g \in \langle A \cup \bar{A} \rangle - \{1\}$ be an $A \cup \bar{A}$–convex element fixing the image $1_{\mathcal{S}} \in T_{\mathcal{S}}$ of $1 \in \widetilde{X}_\Lambda$ under the natural projection $\widetilde{X}_\Lambda \to T_{\mathcal{S}}$. Let $\alpha \subset T_{A \cup \bar{A}} \times \{1_{\mathcal{S}}\}$ be the axis of g and let $\bar{\alpha}$ be the axis of g in $T_{A \cup \bar{A}}$. Let τ be the translation length of g (which is the same on $T_{A \cup \bar{A}} \times T_{\mathcal{S}}$ as on $T_{A \cup \bar{A}}$, by $A \cup \bar{A}$–convexity of g). By conjugating, we can assume $1_{A \cup \bar{A}} \in \bar{\alpha}$.

Consider the finitely many vertices $v_1, \ldots, v_k \in T_{A \cup \bar{A}}$ at distance at most 2τ from $1_{A \cup \bar{A}}$. For each i, let $y_i \in \Lambda$ be such that $y_i \cdot 1_{A \cup \bar{A}} = v_i$. We take $v_1 = 1_{A \cup \bar{A}}$ and $y_1 = 1$. From the definition of τ, up to relabelling, we can take $v_k = g \cdot 1_{A \cup \bar{A}}$ and $y_k = g$. These choices are not unique, but there are finitely many y_i, and they depend on g but not on the elements h discussed below.

Let

$$P = \bigcap_{i=1}^{k} y_i \langle \mathcal{S} \rangle y_i^{-1}.$$

Note that $[\langle \mathcal{S} \rangle : P] < \infty$: the subgroup $\langle \mathcal{S} \rangle$ is commensurated in Λ since the trees $\{v_i\} \times T_{\mathcal{S}}$ are all parallel in $\widetilde{X}_\Lambda$. Let $x_1, \ldots, x_r \in \langle \mathcal{S} \rangle$ be left coset representatives for P in $\langle \mathcal{S} \rangle$.

Suppose additionally that $g \in \ker f$. Then for any $p \in P$, we have $p \in \langle \mathcal{S} \rangle^g \cap \langle \mathcal{S} \rangle$, so $g^{-1}pg = p' \in \langle \mathcal{S} \rangle$. Since $f(g) = 1$ and f is the identity on $\langle \mathcal{S} \rangle$, we get

$$p' = f(p') = f(g^{-1}pg) = f(p) = p,$$

i.e. g commutes with every element of P.

Let $h \in \Lambda$ be such that $h\bar{\alpha} \cap \bar{\alpha} \neq \varnothing$ but $h\bar{\alpha} \neq \bar{\alpha}$. For some $n \in \mathbb{Z}$, we have $\mathsf{d}_{T_{A \cup \bar{A}}}(\bar{\alpha}, hg^n h^{-1} \cdot (h1_{A \cup \bar{A}})) \leqslant \tau$. Since $\bar{\alpha} \cap h\bar{\alpha} = \bar{\alpha} \cap (hg^n h^{-1})h\bar{\alpha}$, and our goal is to bound $\operatorname{diam}(h\bar{\alpha} \cap \bar{\alpha})$, we assume $n = 0$. (More precisely, let $h_1 = hg^n$. Then $h_1\bar{\alpha} = h\bar{\alpha}$, and $\mathsf{d}_{T_{A \cup \bar{A}}}(\bar{\alpha}, h_1 \cdot 1_{A \cup \bar{A}}) \leqslant \tau$, and we redefine h to be h_1.) Translating by an appropriate power of g, using $N(g^m h) = N(h)$ for all m, we can thus assume $\mathsf{d}_{T_{A \cup \bar{A}}}(1_{A \cup \bar{A}}, h \cdot 1_{A \cup \bar{A}}) \leqslant 2\tau$.

Hence there exists $i \leqslant k$ such that $h \cdot 1_{A \cup \bar{A}} = v_i$, whence $h \in y_i \langle \mathcal{S} \rangle$. So, $h = y_i x_j p$ for some $j \leqslant r$ and $p \in P$. But $h\bar{\alpha}$ is the axis of $hgh^{-1} = y_i x_j p g p^{-1} x_j^{-1} y_i^{-1} = y_i x_j g x_j^{-1} y_i^{-1}$, so $h\bar{\alpha} = y_i x_j \bar{\alpha}$. Let $N = \max_{i',j'} N(y_{i'} x_{j'})$, where $y_{i'}, x_{j'}$ range over the values for which $y_{i'} x_{j'} \bar{\alpha} \neq \bar{\alpha}$. Then $N < \infty$, and N depends on g, τ, and our choice of $y_{i'}$ and coset representatives $x_{j'}$, but not on h. So for any h for which $h\bar{\alpha} \neq \bar{\alpha}$, we have shown $N(h) \leqslant N$, i.e. g is N–controlled. Finally, Wise found (see [Cap19, Proposition 4.15]) distinct $a, b \in A$ such that

$\lambda = ab^{-1}\bar{b}\bar{a}^{-1}$ lies in every finite-index subgroup of Λ. Noting that this λ is $(A \cup \bar{A})$–convex and $f(\lambda) = 1$ completes the proof. □

Let $X_\Lambda = \Lambda\backslash(T_{A\cup\bar{A}} \times T_{\mathcal{S}})$, which is a compact nonpositively-curved square complex with one vertex. The element $\lambda \in \Lambda \cong \pi_1 X_\Lambda$ from Lemma 1.3, which lies in every finite-index subgroup of Λ, is represented by a locally convex closed immersed edge-path $\breve{\alpha}$ in X_Λ. Letting $\alpha \to T_{A\cup\bar{A}} \times T_{\mathcal{S}}$ be any elevation of $\breve{\alpha}$, we have that α is a convex (based) line (lying in a parallel copy of $T_{A\cup\bar{A}} \times \{1_{\mathcal{S}}\}$), and for any two such lifts $\alpha, h\alpha$, either $h\bar{\alpha} = \bar{\alpha}$ (and $h\alpha, \alpha$ are parallel) or $\operatorname{diam}(\pi_\alpha(h\alpha)) \leqslant N$.

1.2. **The crystallographic group T.** Let

$$T = \langle x, y, t \mid xyx^{-1}y^{-1}, txt^{-1} = y, tyt^{-1} = x^{-1}\rangle.$$

The group T is isomorphic to a uniform lattice in $\operatorname{Isom}(\mathbb{R}^3)$, where x, y act as unit translations along two orthogonal axes, and t is a screw-motion rotating the xy–plane by $\pi/2$ and translating distance 1 along its screw-axis. Tile $\mathbb{R}^3$ by 3–cubes so that the translation axes for x, y, and the screw-axis for t, are combinatorial lines. So, T acts freely and cocompactly on the resulting CAT(0) cube complex, and the quotient X_T is a nonpositively-curved cube complex. The 2–skeleton of X_T is the presentation complex of the above presentation, and the 3–cube is attached in the usual way. Let $X_T(n)$ be the cubical subdivision of X_T in which each hyperplane is replaced by n parallel copies (see e.g. [Hag07, Definition 2.3]), so that $t \in T \cong \pi_1 X_T(n)$ is represented by an embedded (oriented) combinatorial loop β of length n which is locally convex.

1.3. **The group G and its cubical structure.** Form a cube complex X by attaching X_Λ to $X_T(|\breve{\alpha}|)$ by identifying $\breve{\alpha}$ and β (preserving orientations). Then X is nonpositively-curved since it was formed by gluing nonpositively-curved cube complexes along a common locally convex immersed subcomplex. Let $G = \pi_1 X$, so that G admits a presentation

$$G \cong \langle \Lambda, x, y, t \mid xyx^{-1}y^{-1}, txt^{-1}y^{-1}, tyt^{-1}x, \lambda t^{-1}\rangle.$$

The universal cover $\widetilde{X}$ of X is a CAT(0) cube complex on which G acts freely and cocompactly. Let $\widetilde{X}_T(|\breve{\alpha}|)$ be the image of a fixed lift of the map $\widetilde{X}_T(|\breve{\alpha}|) \to X_T(|\breve{\alpha}|) \to X$, so that $\widetilde{X}_T(|\breve{\alpha}|)$ is a convex subcomplex isomorphic to the standard tiling of $\mathbb{E}^3$ by 3–cubes, with stabilizer T. Likewise, we have a convex subcomplex $\widetilde{X}_\Lambda$, isomorphic to a product of two trees, stabilised by Λ. Abusing language slightly, we call translates of α *edge spaces* and translates of $\widetilde{X}_T(|\breve{\alpha}|), \widetilde{X}_\Lambda$ *vertex spaces.*

Lemma 1.4. *Let $G' \leqslant G$ be a finite-index subgroup. Then $t \in G'$.*

Proof. We have $[\Lambda : \Lambda \cap G'] \leqslant [G : G'] < \infty$, so $\lambda \in \Lambda \cap G'$. Since $\lambda = t$ in G, we have $t \in G'$. □

Lemma 1.5. *Let $k \geqslant 0$ and let $q : \mathbb{Z}^k \to \widetilde{X}^{(0)}$ be a quasi-isometric embedding. Then $k \leqslant 3$.*

Proof. There are various ways to see this. For instance, since $\widetilde{X}$ is a 3–dimensional CAT(0) cube complex, its 0–skeleton (with the graph metric) is a coarse median space of rank 3 (see e.g.[Bow19, Section 4]), so this follows from [Bow18, Lemma 6.10]. □

1.4. **G is an HHG.** Here is the full list of equivariance properties from the definition of a hierarchically hyperbolic group $(\Gamma, \mathfrak{S})$. We saw that Γ must act on $\mathfrak{S}$ cofinitely, preserving $\sqsubseteq, \perp, \pitchfork$. For each $g, h \in \Gamma$ and $U, V \in \mathfrak{S}$, we also require:

- There is an isometry $g : \mathcal{C}U \to \mathcal{C}gU$, and the composition $\mathcal{C}U \xrightarrow{h} \mathcal{C}hU \xrightarrow{g} \mathcal{C}ghU$ is gh.
- $\pi_{gU}(gh) = g\pi_U(h)$.
- If $U \pitchfork V$ or $U \sqsubsetneq V$ and $\rho^U_V \subset \mathcal{C}V$ is as in [BHS19, Definition 1.1], then $g\rho^U_V = \rho^{gU}_{gV}$.

This definition follows [PS20] and is simpler than the coarse version in [BHS19]. The coarse version implies the above one [DHS20, Section 2]. The goal of this section is to prove:

Proposition 1.6. *The group G is a hierarchically hyperbolic group.*

We give two proofs, one using the combination theorem from [BR20] and one using cubes.

Proof using the Berlai-Robbio combination theorem. We first describe an HHS structure on $T_{A\cup\bar{A}}$. Let $\mathfrak{S}_1 = \{g\bar{\alpha} : g \in \Lambda\} \sqcup \{S_1\}$, where S_1 is just a symbol; we declare distinct translates of $\bar{\alpha}$ to be transverse, and declare them to be nested in S_1. Since λ is N–controlled, [BHS19, Theorem 9.3] implies that $(T_{A\cup\bar{A}}, \mathfrak{S}_1)$ is an HHS, where $\mathcal{C}g\bar{\alpha} = g\bar{\alpha} \cong \mathbb{R}$, and $\mathcal{C}S_1$ is the quasi-tree obtained from $T_{A\cup\bar{A}}$ by coning off the subspaces $g\bar{\alpha}$. (The same conclusion also follows from [Spr17, Theorem 4.6], where the N–controlled property is used to verify [Spr17, Definition 4.2.2].)

We equip $T_{\mathcal{S}}$ with the trivial HHS structure $(T_{\mathcal{S}}, \mathfrak{S}_2)$ where $\mathfrak{S}_2 = \{S_2\}$ and $\mathcal{C}S_2 = T_{\mathcal{S}}$. Finally, let $\mathfrak{S}_3 = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \{S_3\}$. We declare everything in $\mathfrak{S}_1$ orthogonal to everything in $\mathfrak{S}_2$, and everything is nested in S_3. The space $\mathcal{C}S_3$ is a point. By [BHS19, Proposition 8.27], $(\widetilde{X}_\Lambda, \mathfrak{S}_3)$ is an HHS, and it is readily checked that the actions of Λ on $T_{A\cup\bar{A}}$ and $T_{\mathcal{S}}$ induce an action on $\mathfrak{S}_3$ (with four orbits) and the isometries between hyperbolic spaces required to make it an HHG structure $(\Lambda, \mathfrak{S}_3)$. It is readily verified that this has the *intersection property* from [BR20] and the *clean containers* property from [ABD21].

The action of T on $\widetilde{X}_T(|\breve{\alpha}|)$ yields a hierarchically hyperbolic structure via [BHS17, Remark 13.2], which has *clean containers* by [HS20, Theorem C], while the intersection property can be deduced directly from Definition 8.1 and Remark 13.2 of [BHS17]. The t–axis is a codimension–2 combinatorial hyperplane, so its parallelism class is in the index set of this HHG structure. Moreover, by discarding some elements of the factor system that are single points, we can assume that the t–axis is $\sqsubseteq$–minimal. The associated hyperbolic space is a copy of $\mathbb{R}$, on which t acts as a length–$|\breve{\alpha}|$ translation.

Equip the edge-groups (conjugates in G of $\langle t\rangle = \langle\lambda\rangle$) with the trivial HHG structure: the underlying hyperbolic space is a line mapping isometrically to $\bar{\alpha}$ on the Λ side, and to the t–axis on the T side. This verifies the *full hieromorphism* hypothesis from [BR20, Corollary B] and the coarse lipschitz requirement from that hypothesis. The *quasiconvex* hypothesis of [BR20, Corollary B] holds since the t–axis is convex in $\widetilde{X}_T(|\breve{\alpha}|)$ while α is hierarchically quasiconvex in $(\Lambda, \mathfrak{S}_3)$ (see [BHS19, Definition 5.1]) by construction (using that λ fixes a point in $T_{\mathcal{S}}$). The *comparison maps* hypothesis holds since comparison maps are isomorphisms of combinatorial lines. So [BR20, Corollary B] — the combination theorem — yields an HHG structure. □

Remark 1.7. The proof fails as follows if λ is not N–controlled. The combination theorem [BR20, Corollary B] requires that the vertex group HHG structures contain the edge group HHG structures, which forced us to include $\bar{\alpha}$ and its translates in $\mathfrak{S}_3$. But if $\bar{\alpha}$ has arbitrarily large intersections with its translates, it cannot participate in an HHS structure on $T_{A\cup\bar{A}}$; either the consistency or complexity axiom from [BHS19, Definition 1.1] would be violated.

Proof using the cubical structure. We check that the CAT(0) cube complex $\widetilde{X}$ has a *factor system*, in the sense of [BHS17, Definition 8.1], so that the existence of an HHG structure follows from [BHS17, Remark 13.2] or [HS20, Theorem A]. Let $\mathfrak{F}$ denote the smallest collection of convex subcomplexes of $\widetilde{X}$ containing $\widetilde{X}$, containing every combinatorial hyperplane and subcomplex parallel to a combinatorial hyperplane, and having the property that if $F, F' \in \mathfrak{F}$, then the image of F' under the gate map to F is also in $\mathfrak{F}$. To check that $\mathfrak{F}$ is a factor system, it suffices to produce $\chi < \infty$ such that any vertex of $\widetilde{X}$ lies in at most χ elements of $\mathfrak{F}$.

Claim 1. For each $h \subset \widetilde{X}$ a combinatorial hyperplane, either $\pi_\alpha(h) = \alpha$, or $\operatorname{diam}(\pi_\alpha(h)) \leqslant N$.

Proof of Claim 1. By definition, h is a component of the boundary of the carrier of some hyperplane $\hat{h}$. If $\hat{h}$ crosses α, then $\pi_\alpha(h)$ is a single point, so suppose that $\hat{h}$ does not cross α.

If $\hat{h}$ crosses a vertex space containing α, then, because of the product structures of the vertex spaces, α is parallel to a subcomplex of h, so $\pi_\alpha(h) = \alpha$, or $\pi_\alpha(h)$ is a point. Otherwise, there is a sequence $g_1\alpha, g_2\alpha, \cdots, g_k\alpha$ of edge spaces separating α from h, and $\pi_\alpha(h) = \pi_\alpha(\pi_{g_1\alpha}(h))$. By induction on k, $\pi_{g_1\alpha}(h)$ is either the whole of $g_1\alpha$, or has diameter at most N. Projecting to α doesn't increase distances, and the projection of $g_1\alpha$ to α is either α or has diameter at most N since λ is N–controlled, so the claim follows. $\square$

Claim 2. There is a finite $\mathfrak{F}_0 \subset \mathfrak{F}$ such that the following holds. Let $h_1, \ldots, h_n \subset \widetilde{X}$ be combinatorial hyperplanes, let π_{h_i} be the gate map to h_i, and let $F = \pi_{h_1}(\pi_{h_2}(\cdots \pi_{h_{n-1}}(h_n)\cdots))$. Then $F \in G \cdot \mathfrak{F}_0$.

Proof of Claim 2. We consider three cases.

Case 0: Suppose $\mathsf{d}_{\widetilde{X}}(h_i, h_j) \leqslant 10N$ for all i, j. Since the convex hull of the $10N$–neighbourhood of h_i is contained in the $10N \dim \widetilde{X}$–neighbourhood of h_i (by e.g. [HP22, Lemma 4.2]), the Helly property for convex subcomplexes ([Rol16, Theorem 2.2]) says the h_i all intersect some $10N \dim \widetilde{X}$–ball B about a vertex. There are finitely many G–orbits of such balls, each intersecting finitely many hyperplanes. So up to the G–action, there are finitely many possibilities for the collection $\{h_1, \ldots, h_n\}$ and thus a finite collection $\mathfrak{F}_0'$ of subcomplexes with $F \in G \cdot \mathfrak{F}_0'$.

So, for the rest of the proof, we assume that there exist $I, J \leqslant n$ such that $\mathsf{d}_{\widetilde{X}}(h_I, h_J) > 10N$.

Case I: Suppose that any two of the h_i intersect a common vertex space. By applying the Helly property to the Bass-Serre tree of the splitting of G and the subtrees determined by the various h_i, we get a vertex space $\widetilde{Y}$ such that each $h_i \cap \widetilde{Y} \neq \varnothing$.

Subcase: no common translates of α: Suppose some h_i, h_j do not cross a common translate of α in $\widetilde{Y}$. Then any hyperplane crossing h_i and h_j crosses $h_i \cap \widetilde{Y}$ and $h_j \cap \widetilde{Y}$. So F arises by projecting the hyperplanes $h_s \cap \widetilde{Y}, 1 \leqslant s \leqslant n$ of $\widetilde{Y}$ onto one another, i.e. it lies in the factor system on $\widetilde{Y}$ (which exists since $\widetilde{Y}$ is the product of two locally finite trees or a copy of the standard cubulation of $\mathbb{R}^3$). There are finitely many orbits of such F.

So we now assume the h_i are disjoint (otherwise two cross in $\widetilde{Y}$, and so at least one cannot cross any translate of α), and any two cross some common translate of α in $\widetilde{Y}$.

A translate of α crossing all h_i: Suppose h_I, h_J both cross $g\alpha \subset \widetilde{Y}$. Suppose that some h_i does not cross $g\alpha$. Inside $\widetilde{Y}$, choose $h\alpha$ crossing h_i, h_I and $h'\alpha$ crossing h_i, h_J. Let p_i, p_I, p_J be the projections of h_i, h_I, h_J to $T_{A \cup \bar{A}}$ (which are necessarily points). The part of $g\bar{\alpha}$ between p_I, p_J has length at least $10N$. For each pair of distinct axes in $h\bar{\alpha}, h'\bar{\alpha}, g\bar{\alpha}$, the (nonempty) intersection has length at most N. This is impossible unless $h\bar{\alpha} = g\bar{\alpha}$ or $h'\bar{\alpha} = g\bar{\alpha}$, whence $p_i \in g\bar{\alpha}$, i.e. h_i crosses $g\alpha$.

Common translate case: So, by translating, $\alpha \subset \widetilde{Y}$ and $h_i \cap \alpha \neq \varnothing$ for all $i \leqslant n$. We claim

$$\mathcal{W}(F) = \bigcap_{w \in \mathcal{W}(\alpha)} \mathcal{W}(w),$$

where, for a subcomplex or hyperplane z, $\mathcal{W}(z)$ means the set of hyperplanes in $\widetilde{X}$ crossing z.

Each h_i is a component of the boundary of the carrier of a hyperplane $\hat{h}_i \in \mathcal{W}(\alpha)$, so by the definition of F, we have $\mathcal{W}(F) = \bigcap_{i=1}^n \mathcal{W}(\hat{h}_i) \supseteq \bigcap_{w \in \mathcal{W}(\alpha)} \mathcal{W}(w)$. Conversely, if $v \in \bigcap_i \mathcal{W}(\hat{h}_i)$, then $\pi_\alpha(v)$ has diameter more than N, since v crosses h_I and h_J. By Claim 1, the images of α under projections to the combinatorial hyperplanes bounding the carrier of w are thus parallel to $\alpha = \pi_\alpha(v)$, so v crosses every hyperplane crossing α, i.e. $\mathcal{W}(F) \subseteq \bigcap_{w \in \mathcal{W}(\alpha)} \mathcal{W}(w)$.

Now let U be the cubical convex hull of the union of all convex subcomplexes of $\widetilde{X}$ parallel to α. Then $U \cong \alpha \times \alpha^\perp$, where $\alpha^\perp$ is the *orthogonal complement* of α (see [HS20, Definition 1.10]). The above discussion and [HS20, Lemma 1.11] show that F is parallel to $\{x\} \times \alpha^\perp$ for some $x \in \alpha$. Since the $h_i \cap \widetilde{Y}$ are all parallel, $h_1 \cap \widetilde{Y} \subset F$, and $h_1 \cap \alpha \in h_1 \cap \widetilde{Y}$, so $F \cap U \neq \varnothing$, whence in fact $F = \{x\} \times \alpha^\perp$ for some $x \in \alpha$. Since $\lambda(\{x\} \times \alpha^\perp) = \{\lambda x\} \times \alpha^\perp$, there are finitely many orbits of such subcomplexes (at most the translation length of λ).

Case II: Suppose that some h_i, h_j are separated by the edge space α (up to translation this is the negation of the definition of Case I). Since the gate map from a hyperplane on one side of α to a hyperplane on the other factors through π_α, Claim 1 implies F is parallel to α or has diameter at most N. There are finitely many orbits of subsets of diameter at most N, so it remains to show that there are finitely many orbits of combinatorial lines parallel to α.

Consider $U = \alpha \times \alpha^\perp$, the convex hull of the union of all lines parallel to α (whose 0–skeleton is the union of the 0–skeleta of the lines). For $g \in G$, the subcomplex $gU = (g\alpha) \times (g\alpha)^\perp$ is the convex hull of the parallel copies of $g\alpha$. We claim $\mathrm{Stab}_G(U)$ acts on U cocompactly. Suppose infinitely many translates g_iU intersect some ball in $\widetilde{X}$. Since α is unbounded and $\widetilde{X}$ is locally finite, there exist $a, b \in \widetilde{X}$ such that $\mathsf{d}(a,b) > N$ and the following holds: there exist distinct g_iU, g_jU with $a, b \in g_iU \cap g_jU$. Hence there exist combinatorial lines α_i, α_j, respectively parallel to $g_i\alpha, g_j\alpha$, such that $a, b \in \alpha_i \cap \alpha_j$. This implies that $\mathrm{diam}(\pi_{g_i\alpha}(g_j\alpha)) > N$, so since λ is N–controlled, $g_i\alpha, g_j\alpha$ are parallel, so $g_iU = g_jU$. Hence the family of translates of U is locally finite, so by [HS20, Lemma 2.3], $\mathrm{Stab}_G(U)$ acts on U cocompactly. Since $\alpha \cong \mathbb{R}$ has discrete cubical isometry group, [NS13, Corollary 2.7] implies that $\mathrm{Stab}_G(U)$ virtually splits as $\mathrm{Stab}_G(\alpha) \times H$, where H acts cocompactly on each parallel copy of $\alpha^\perp$. Thus U, hence $\widetilde{X}$, contains finitely many orbits of lines parallel to α.

Conclusion: Each case added finitely many G–orbits to $\mathfrak{F}_0'$, so the claimed $\mathfrak{F}_0$ exists. □

By [HS20, Corollary 2.2] and Claim 2, $\mathfrak{F} = G \cdot \mathfrak{F}_0$. So $\mathfrak{F}$ contains finitely many G–orbits of subcomplexes, each of whose stabiliser acts on it cocompactly by [HS20, Proposition 2.7], so since the G–action on $\widetilde{X}$ is geometric, any ball intersects finitely many elements of $\mathfrak{F}$. □

2. No colourable HHG structure

On the other hand:

Proposition 2.1. *The group G does not admit a colourable HHG structure.*

We use a statement about virtually abelian subgroups of HHGs:

Proposition 2.2 (Invariant quasiflats for virtually abelian subgroups)**.** *Let $(G, \mathfrak{S})$ be an HHG. Let $T \subset G$ be a virtually $\mathbb{Z}^k$ subgroup for some $k \geqslant 1$. Then there exists $\ell \geqslant k$ and $U_1, \dots, U_\ell \in \mathfrak{S}$ such that the following hold:*

(1) $\{U_1, \dots, U_\ell\}$ is T–invariant.
(2) $U_i \perp U_j$ for $1 \leqslant i < j \leqslant \ell$.
(3) There exists $L < \infty$ such that $\mathrm{diam}(\pi_V(T)) \leqslant L$ for $V \in \mathfrak{S} - \{U_1, \dots, U_\ell\}$.
(4) For each $i \leqslant \ell$, the image $\pi_{U_i}(T)$ of T in $\mathcal{C}U_i$ is a quasi-line.

Hence the (T–invariant) hierarchically quasiconvex hull F_T of T is quasi-isometric to $\mathbb{Z}^\ell$.

Hierarchically quasiconvex hulls are discussed in [BHS19, Section 6]. The proposition is stated as [HRSS22, Proposition 2.17], where a proof can also be found, using [PS20].[1] Now return to the group G (and the subgroups Λ, T) from Section 1, with $(G, \mathfrak{S})$ an arbitrary HHG structure. We recall that Λ is an irreducible lattice in the product of two trees, and T is virtually $\mathbb{Z}^3$, being a 3–dimensional crystallographic group.

Lemma 2.3. *Recall that $T = \langle x, y, t \rangle$, and $txt^{-1} = y, tyt^{-1} = x^{-1}$. Let F_T and $U_1, \dots, U_\ell \in \mathfrak{S}$ be as in Proposition 2.2. Then $\ell = 3$, and there exists $i \in \{1, 2, 3\}$ such that $tU_i \perp U_i$.*

[1] This proposition was also proved by Paul Plummer in unpublished work.

Proof. By Proposition 2.2, $\ell \geqslant 3$, and F_T provides a quasi-isometric embedding $\mathbb{Z}^\ell \to G$. By composing with an orbit map $G \to \widetilde{X}$, we thus get a quasi-isometric embedding $\mathbb{Z}^\ell \to \widetilde{X}^{(0)}$, since the action of G on $\widetilde{X}$ is geometric by construction. Lemma 1.5 gives $\ell = 3$.

To show that $tU_i \perp U_i$ for some i, we show that t acts nontrivially on $\{U_1, U_2, U_3\}$, which consists of pairwise orthogonal elements. Assume to the contrary that $tU_i = U_i$ for all i.

Since T acts properly on F_T, and x has infinite order, up to relabelling, x^m fixes some U_j and acts loxodromically on $\mathcal{C}U_j$, for some $m > 0$. Hence $tx^m t^{-1} = y^m$ fixes $tU_j = U_j$ and acts loxdromically on $\mathcal{C}U_j$.

There are two cases. In the first case, x^m and y^{-m} translate in the "same direction" along the quasiline $\mathcal{C}U_j$, i.e. there exists L_0 such that $\mathsf{d}_{U_j}(x^{mn}, y^{-mn}) \leqslant L_0$ for all integers $n > 0$. So, applying the relations, we get

$$\begin{aligned} \mathsf{d}_{U_j}(x^{mn}, y^{mn}) &= \mathsf{d}_{U_j}(ty^{-mn}t^{-1}, tx^{mn}t^{-1}) \\ &\leqslant \mathsf{d}_{U_j}(ty^{-mn}, tx^{mn}) + \mathsf{d}_{U_j}(ty^{-mn}t^{-1}, ty^{-mn}) + \mathsf{d}_{U_j}(tx^{mn}t^{-1}, tx^{mn}) \\ &\leqslant L_0 + 2\mathsf{d}_{U_j}(1, t^{-1}), \end{aligned}$$

which is bounded independently of n. Here we have used that t, x^m, y^m fix U_j and act on $\mathcal{C}U_j$ by isometries. An application of the triangle inequality now shows that $\mathsf{d}_{U_j}(y^{-mn}, y^{mn})$ is bounded independently of n, contradicting that y^m is loxodromic.

In the second case, x^m and y^m translate in the same direction, i.e. there exists L_0 such that $\mathsf{d}_{U_j}(x^{mn}, y^{mn}) \leqslant L_0$ for all integers $n > 0$. A similar computation as above bounds $\mathsf{d}_{U_j}(x^{-mn}, y^{mn})$ independently of n, and we apply the triangle inequality to contradict that x^m is loxodromic. Either case being impossible, we conclude that x cannot act loxodromically on any U_j. Since this contradicts properness of the T–action, we must have that the $\langle t \rangle$–action on $\{U_1, U_2, U_3\}$ is nontrivial, as required. ☐

Now we are ready to prove the main proposition:

Proof of Proposition 2.1. Suppose that $(G, \mathfrak{S})$ is a colourable HHG structure on G. Then G acts on the finite set of colours; let $G' \leqslant_{f.i.} G$ be the kernel of this action. By Lemma 1.4, $t \in G'$. So, for all $U \in \mathfrak{S}$, the elements U and tU have the same colour and are thus transverse. On the other hand, by Lemma 2.3, there exists $U \in \mathfrak{S}$ such that $tU \perp U$. This is a contradiction. ☐

3. Remarks

We conclude with some remarks:

Remark 3.1. This example shows that colourability is not preserved by the Berlai-Robbio combination theorem, answering a question from [HP22].

Remark 3.2 (Refinement of Proposition 2.2)**.** In our application of Proposition 2.2, the conclusion $\ell = k$ came from considerations about the rank of G as a coarse median space, a global property of this particular group. It would be nice to know whether the cubical flat torus theorem of Woodhouse-Wise [WW17, Theore 3.6] and related results of Genevois [Gen21] have HHG analogues. Specifically, if $T \leqslant G$ is virtually $\mathbb{Z}^k$, and has no finite-index subgroup contained in a $\mathbb{Z}^{k'}$ subgroup of G with $k' > k$, does the conclusion of Proposition 2.2 hold with $\ell = k$, i.e. is every *highest* abelian subgroup hierarchically quasiconvex? If T is not highest, do orbits of T have the property that any two orbit points are joined by *at least one* hierarchy path lying in a bounded neighbourhood of the orbit? (One can't expect better, since once can choose, say, $\mathbb{Z}^2 \leqslant \mathbb{Z}^3$ that is not coarsely a median subalgebra.)

Remark 3.3 (Simpler examples)**.** Is there a rank–2 non-colourable HHG? The *rank* of $(G, \mathfrak{S})$ is the largest n such that there exists pairwise orthogonal $U_1, \ldots, U_n \in \mathfrak{S}$ with each $\mathcal{C}U_i$ unbounded. Rank-1 HHGs are hyperbolic [BHS21], so admit colourable HHG structures (the trivial one).

Remark 3.4 (Connection with separability)**.** Colourability of an HHG $(G, \mathfrak{S})$ is related to separability of certain *hierarchically quasiconvex* subgroups of G. Specifically, under the mild hypothesis that the *standard product regions* in G are cosets of a finite collection of subgroups, it is shown in [HP22, Proposition 3.2] that $(G, \mathfrak{S})$ is colourable if these *product region subgroups* are separable in G. In the forthcoming [CRHK22], this is explored a bit further; for instance, colourability of the usual HHG structure on the mapping class group is related to separability of multicurve stabilisers, established by Leininger-McReynolds in [LM07]. Examples make it hard to imagine a partial converse (colourability implying separability of the product region subgroups under reasonable additional assumptions), though: any hyperbolic group G with an infinite quasiconvex subgroup H admits an HHG structure — readily seen to be colourable — where the product regions are cosets of H [Spr17, Theorem 1]. But whether H is in general separable in G reduces to the question of residual finiteness of hyperbolic groups [AGM09].

Remark 3.5 (Shepherd's factor system–free examples)**.** We saw that Λ and λ can be chosen so that λ is N–controlled for some N. This yields other HHGs along the lines of the construction of G. For example, one could form an HNN extension conjugating λ to itself, as in [BHS17, Figure 1], or double Λ along λ. The universal covers have factor systems by arguing as in the cubical proof of Proposition 1.6. However, Sam Shepherd has observed[2] that there will not be a factor system when λ is not N–controlled for any N. Shepherd constructed examples where Λ is associated to a finite-state automaton and λ is one of the generators of the BMW presentation; Wise's anti-torus [Wis07] provides additional examples, as noted in [She22]. Before Shepherd's observation, examples of proper cocompact CAT(0) cube complexes without factor systems were not known. In [HS20], conditions are given implying existence of a factor system; examples involving uncontrolled elements of BMW groups fail the *essential index condition* from [HS20].

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School of Mathematics, University of Bristol, Bristol, United Kingdom
Email address: markfhagen@posteo.net