ON HIERARCHICAL HYPERBOLICITY OF CUBICAL GROUPS

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Abstract. Let $\mathcal{X}$ be a proper CAT(0) cube complex admitting a proper cocompact action by a group G. We give three conditions on the action, any one of which ensures that $\mathcal{X}$ has a *factor system* in the sense of [BHS14]. We also prove that one of these conditions is necessary. This combines with [BHS14] to show that G is a *hierarchically hyperbolic group*; this partially answers questions raised in [BHS14, BHS15]. Under any of these conditions, our results also affirm a conjecture of Behrstock-Hagen on boundaries of cube complexes, which implies that $\mathcal{X}$ cannot contain a convex *staircase*. The necessary conditions on the action are all strictly weaker than virtual cospecialness, and we are not aware of a cocompactly cubulated group that does not satisfy at least one of the conditions.

Introduction

Much work in geometric group theory revolves around generalizations of Gromov hyperbolicity: relatively hyperbolic groups, weakly hyperbolic groups, acylindrically hyperbolic groups, coarse median spaces, semihyperbolicity, lacunary hyperbolicity, etc. Much attention has been paid to groups acting properly and cocompactly on CAT(0) cube complexes, which also have features reminiscent of hyperbolicity. Such complexes give a combinatorially and geometrically rich framework to build on, and many groups have been shown to admit such actions (for a small sample, see [Sag95, Wis04, OW11, BW12, HW15]).

Many results follow from studying the geometry of CAT(0) cube complexes, often using strong properties reminiscent of negative curvature. For instance, several authors have studied the structure of quasiflats and Euclidean sectors in cube complexes, with applications to rigidity properties of right-angled Artin group [Xie05, BKS08, Hua14]. These spaces have also been shown to be median [Che00] and to have only semi-simple isometries [Hag07]. Further, under reasonable assumptions, a CAT(0) cube complex $\mathcal{X}$ either splits as a nontrivial product or $\mathrm{Isom}(\mathcal{X})$ must contain a rank-one element [CS11]. Once a given group is known to act properly and cocompactly on a CAT(0) cube complex the geometry of the cube complex controls the geometry and algebra of the group. For instance, such a group is biautomatic and cannot have Kazhdan's property (T) [NR98, NR97], and it must satisfy a Tits alternative [SW05]

Here, we examine cube complexes admitting proper, cocompact group actions from the point of view of certain convex subcomplexes. Specifically, given a CAT(0) cube complex $\mathcal{X}$, we study the following set $\mathfrak{F}$ of convex subcomplexes: $\mathfrak{F}$ is the smallest set of subcomplexes that contains $\mathcal{X}$, contains each combinatorial hyperplane, and is closed under cubical closest-point projection, i.e. if $A, B \in \mathfrak{F}$, then $\mathfrak{g}_B(A) \in \mathfrak{F}$, where $\mathfrak{g}_B : \mathcal{X} \to B$ is the cubical closest point projection.

Main results. The collection $\mathfrak{F}$ of subcomplexes is of interest for several reasons. It was first considered in [BHS14], in the context of finding *hierarchically hyperbolic structures* on $\mathcal{X}$. Specifically, in [BHS14], it is shown that if there exists $N < \infty$ so that each point of $\mathcal{X}$ is contained in at most N elements of $\mathfrak{F}$, then $\mathcal{X}$ is a *hierarchically hyperbolic space*, which has

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numerous useful consequences outlined below; the same finite multiplicity property of $\mathfrak{F}$ has other useful consequences outlined below. When this finite multiplicity condition holds, we say, following [BHS14], that $\mathfrak{F}$ is a *factor system* for $\mathcal{X}$.

We believe that if $\mathcal{X}$ is proper and some group G acts properly and cocompactly by isometries on $\mathcal{X}$, then the above finite multiplicity property holds, and thus G is a hierarchically hyperbolic group. In [BHS14], it is shown that this holds when G has a finite-index subgroup acting cospecially on $\mathcal{X}$, and it is also verified in a few non-cospecial examples.

This conjecture has proved surprisingly resistant to attack; we earlier believed we had a proof. However, a subtlety in Proposition 5.1 means that at present our techniques only give a complete proof under various conditions on the G–action, namely:

Theorem A. *Let G act properly and cocompactly on the proper CAT(0) cube complex $\mathcal{X}$. Then $\mathfrak{F}$ is a factor system for $\mathcal{X}$ provided* **any one** *of the following conditions is satisfied (up to passing to a finite-index subgroup of G):*

- *the action of G on $\mathcal{X}$ is* rotational*;*
- *the action of G on $\mathcal{X}$ satisfies the* weak finite height condition for hyperplanes*;*
- *the action of G on $\mathcal{X}$ satisfies the* essential index condition *and the* Noetherian intersection of conjugates condition (NICC) *on hyperplane-stabilisers.*

Hence, under any of the above conditions, $\mathcal{X}$ is a hierarchically hyperbolic space and G a hierarchically hyperbolic group.

Conversely, if $\mathfrak{F}$ is a factor system, then the G–action satisfies the essential index condition and the NICC.

The auxiliary conditions are as follows. The action of G is *rotational* if, whenever A, B are hyperplanes of $\mathcal{X}$, and $g \in \mathrm{Stab}_G(B)$ has the property that A and gA cross or osculate, then A lies at distance at most 1 from B. This condition is *prima facie* weaker than requiring that the action of G on $\mathcal{X}$ be cospecial, so Theorem A generalises the results in [BHS14]. (In fact, the condition above is slightly stronger than needed; compare Definition 4.1.)

A subgroup $K \leq G$ satisfies the *weak finite height* condition if the following holds. Let $\{g_i\}_{i\in I} \subset G$ be an infinite set so that $K \cap \bigcap_{i\in J} K^{g_i}$ is infinite for all finite $J \subset I$. Then there exist distinct g_i, g_j so that $K \cap K^{g_i} = K \cap K^{g_j}$. The action of G on $\mathcal{X}$ satisfies the weak finite height condition for hyperplanes if each hyperplane stabiliser satisfies the weak finite height condition.

This holds, for example, when each hyperplane stabiliser has *finite height* in the sense of [GMRS98]. Hence Theorem A implies that $\mathfrak{F}$ is a factor system when $\mathcal{X}$ is hyperbolic, without invoking virtual specialness [Ago13] because quasiconvex subgroups (in particular hyperplane stabilisers) have finite height [GMRS98]; the existence of a hierarchically hyperbolic structure relative to $\mathfrak{F}$ also follows from recent results of Spriano in the hyperbolic case [Spr17]. Also, if $\mathfrak{F}$ is a factor system and $\mathcal{X}$ does not decompose as a product of unbounded CAT(0) cube complexes, then results of [BHS14] imply that G is *acylindrically hyperbolic*. On the other hand, recent work of Genevois [Gen16] uses finite height of hyperplane-stabilisers to verify acylindrical hyperbolicity for certain groups acting on CAT(0) cube complexes. In our opinion, this provides some justification for the naturality of the weak finite height condition for hyperplanes.

The NIC condition for hyperplanes asks the following for each hyperplane-stabiliser K. Given any $\{g_i\}_{i\geq 0}$ so that $K_n = K \cap \bigcap_{i=0}^{n} K^{g_i}$ is infinite for all n, there exists ℓ so that K_n and K_ℓ are commensurable for $n \geq \ell$. Note that ℓ is allowed to depend on $\{g_i\}_{i\geq 0}$. The accompanying essential index condition asks that there exists a constant ζ so that for any $F \in \mathfrak{F}$, the stabiliser of F has index at most ζ in the stabiliser of the *essential core* of F, defined in [CS11]. These conditions are somewhat less natural than the preceding conditions, but they follow fairly easily from the finite multiplicity of $\mathfrak{F}$.

We prove Theorem A in Section 6. There is a unified argument under the weak finite height and NICC hypotheses, and a somewhat simpler argument in the presence of a rotational action.

To prove Theorem A, the main issue is to derive a contradiction from the existence of an infinite strictly ascending chain $\{F_i\}$, in $\mathfrak{F}$, using that the corresponding chain of orthogonal complements must strictly descend. The existence of such chains can be deduced from the failure of the finite multiplicity of $\mathfrak{F}$ using only the proper cocompact group action; it is in deriving a contradiction from the existence of such chains that the other conditions arise.

Any condition that allows one to conclude that the F_i have bounded-diameter fundamental domains for the actions of their stabilisers yields the desired conclusion. So, there are most likely other versions of Theorem A using different auxiliary hypotheses. We are not aware of a cocompactly cubulated group which is not covered by Theorem A.

Hierarchical hyperbolicity. *Hierarchically hyperbolic spaces/groups* (HHS/G's), introduced in [BHS14, BHS15], were proposed as a common framework for studying mapping class groups and (certain) cubical groups. Knowledge that a group is hierarchically hyperbolic has strong consequences that imply many of the nice properties of mapping class groups.

Theorem A and results of [BHS14] (see Remark 13.2 of that paper) together answer Question 8.13 of [BHS14] and part of Question A of [BHS15] — which ask whether a proper cocompact CAT(0) cube complex has a factor system — under any of the three auxiliary hypotheses in Theorem A. Hence our results expand the class of cubical groups that are known to be hierarchically hyperbolic. Some consequences of this are as follows, where $\mathcal{X}$ is a CAT(0) cube complex on which G acts geometrically, satisfying any of the hypotheses in Theorem A:

- In combination with [BHS14, Corollary 14.5], Theorem A shows that G acts acylindrically on the contact graph of $\mathcal{X}$, i.e. the intersection graph of the hyperplane carriers, which is a quasi-tree [Hag14].
- Theorem A combines with Theorem 9.1 of [BHS14] to provide a Masur-Minsky style distance estimate in G: up to quasi-isometry, the distance in $\mathcal{X}$ from x to gx, where $g \in G$, is given by summing the distances between the projections of x, gx to a collection of uniform quasi-trees associated to the elements of the factor system.
- Theorem A combines with Corollary 9.24 of [DHS16] to prove that either G stabilizes a convex subcomplex of $\mathcal{X}$ splitting as the product of unbounded subcomplexes, or G contains an element acting loxodromically on the contact graph of $\mathcal{X}$. This is a new proof of a special case of the Caprace-Sageev rank-rigidity theorem [CS11].

Proposition 11.4 of [BHS14] combines with Theorem A to prove:

Theorem B. *Let G act properly and cocompactly on the proper CAT(0) cube complex $\mathcal{X}$, with the action satisfying the hypotheses of Theorem A. Let $\mathfrak{F}$ be the factor system, and suppose that for all subcomplexes $A \in \mathfrak{F}$ and $g \in G$, the subcomplex gA is not parallel to a subcomplex in $\mathcal{F}$ which is in the orthogonal complement of A. Then $\mathcal{X}$ quasi-isometrically embeds in the product of finitely many trees.*

The set $\mathfrak{F}$ is shown in Section 2 to have a graded structure: the lowest-grade elements are combinatorial hyperplanes, then we add projections of combinatorial hyperplanes to combinatorial hyperplanes, etc. This allows for several arguments to proceed by induction on the grade. Essentially by definition, a combinatorial hyperplane H is the *orthogonal complement* of a 1–cube e, i.e. a maximal convex subcomplex H for which $\mathcal{X}$ contains the product $e \times H$ as a subcomplex. We show, in Theorem 3.3, that $\mathfrak{F}$ is precisely the set of convex subcomplexes F such that there exists a compact, convex subcomplex C so that the orthogonal complement of C is F. This observation plays an important role.

Relatedly, we give conditions in Proposition 5.1 ensuring that $\mathfrak{F}$ is closed under the operation of taking orthogonal complements. As well as being used in the proof of Theorem A, this is needed for applications of recent results about hierarchically hyperbolic spaces to cube complexes. Specifically, in [ABD17], Abbott-Behrstock-Durham introduce hierarchically hyperbolic spaces with *clean containers*, and work under that (quite natural) hypothesis. Among its applications, they produce largest, universal acylindrical actions on hyperbolic spaces for hierarchically hyperbolic groups. We will not give the definition of clean containers for general hierarchically hyperbolic structures, but for the CAT(0) cubical case, our results imply that it holds for hierarchically hyperbolic structures on $\mathcal{X}$ obtained using $\mathfrak{F}$, as follows:

Theorem C (Clean containers)**.** *Let $\mathcal{X}$ be a proper CAT(0) cube complex on which the group G acts properly and cocompactly, and suppose $\mathfrak{F}$ is a factor system. Let $F \in \mathfrak{F}$, and let $V \in \mathfrak{F}$ be properly contained in F. Then there exists $U \in \mathfrak{F}$, unique up to parallelism, such that:*

- *$U \subset F$;*
- *$V \hookrightarrow F$ extends to a convex embedding $V \times U \hookrightarrow F$;*
- *if $W \in \mathfrak{F}$, and the above two conditions hold with U replaced by W, then W is parallel to a subcomplex of U.*

Proof. Let $x \in V$ be a 0–cube and let $U' = V^{\perp}$, the orthogonal complement of V at x (see Definition 1.10). Proposition 5.1 implies that $U' \in \mathfrak{F}$, so $U = U' \cap F$ is also in $\mathfrak{F}$, since $\mathfrak{F}$ is closed under projections. By the definition of the orthogonal complement, $V \to \mathcal{X}$ extends to a convex embedding $V \times U' \to \mathcal{X}$, and $(V \times U') \cap F = V \times U$ since $V \subset F$ and $F, V \times U'$ are convex. Now, if $W \in \mathfrak{F}$ and $W \subset F$, and $V \to \mathcal{X}$ extends to a convex embedding $V \times W \to \mathcal{X}$, then $V \times W$ is necessarily contained in F, by convexity. On the other hand, by the definition of the orthogonal complement, W is parallel to a subcomplex of U'. Hence W is parallel to a subcomplex of U. This implies the third assertion and uniqueness of U up to parallelism. □

We now turn to applications of Theorem A that do not involve hierarchical hyperbolicity.

Simplicial boundary and staircases. Theorem A also gives insight into the structure of the boundary of $\mathcal{X}$. We first mention an aggravating geometric/combinatorial question about cube complexes which is partly answered by our results.

A *staircase* is a CAT(0) cube complex $\mathcal{Z}$ defined as follows. First, a *ray-strip* is a square complex of the form $S_n = [n, \infty) \times [-\frac{1}{2}, \frac{1}{2}]$, with the product cell-structure where $[n, \infty)$ has 0–skeleton $\{m \in \mathbb{Z} : m \geq n\}$ and $[-\frac{1}{2}, \frac{1}{2}]$ is a 1–cube. To build $\mathcal{Z}$, choose an increasing sequence $(a_n)_n$ of integers, collect the ray-strips $S_{a_n} \cong [a_n, \infty) \times [-\frac{1}{2}, \frac{1}{2}]$, and identify $[a_{n+1}, \infty) \times \{-\frac{1}{2}\} \subset S_{a_n+1}$ with $[a_n, \infty) \times \{\frac{1}{2}\} \subset S_{a_n}$ for each n. The model staircase is the cubical neighbourhood of a Euclidean sector in the standard tiling of $\mathbb{E}^2$ by squares, with one bounding ray in the x–axis, but for certain $(a_n)_n$, $\mathcal{Z}$ may not contain a nontrivial Euclidean sector. (One can define a d-dimensional staircase analogously for $d \geq 2$.) We will see below that the set of "horizontal" hyperplanes in $\mathcal{Z}$ – see Figure 1 for the meaning of "horizontal" – is interesting because there is no geodesic ray in $\mathcal{Z}$ crossing exactly the set of horizontal hyperplanes.

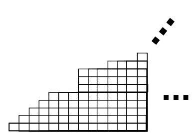

Figure 1. Part of a staircase.

Now let $\mathcal{X}$ be a proper CAT(0) cube complex with a group G acting properly and cocompactly. Can there be a convex staircase subcomplex in $\mathcal{X}$? A positive answer seems very implausible, but this question is open and has bothered numerous researchers.

In Section 7, we prove that if $\mathfrak{F}$ is a factor system, then $\mathcal{X}$ cannot contain a convex staircase. Hence, if $\mathcal{X}$ admits a geometric group action satisfying any of the hypotheses in Theorem A, then $\mathcal{X}$ cannot contain a convex staircase. In fact, we prove something more general, which is best formulated in terms of the *simplicial boundary* $\partial_\triangle \mathcal{X}$.

Specifically, the *simplicial boundary* $\partial_\triangle \mathcal{X}$ of a CAT(0) cube complex $\mathcal{X}$ was defined in [Hag13]. Simplices of $\partial_\triangle \mathcal{X}$ come from equivalence classes of infinite sets $\mathcal{H}$ of hyperplanes such that:

- if $H, H' \in \mathcal{H}$ are separated by a hyperplane V, then $V \in \mathcal{H}$;
- if $H_1, H_2, H_3 \in \mathcal{H}$ are disjoint, then one of H_1, H_2, H_3 separates the other two;
- for $H \in \mathcal{H}$, at most one halfspace associated to H contains infinitely many $V \in \mathcal{H}$.

These *boundary sets* are partially ordered by coarse inclusion (i.e., $A \preceq B$ if all but finitely many hyperplanes of A are contained in B), and two are equivalent if they have finite symmetric difference; $\partial_\triangle \mathcal{X}$ is the simplicial realization of this partial order. The motivating example of a simplex of $\partial_\triangle \mathcal{X}$ is: given a geodesic ray γ of $\mathcal{X}$, the set of hyperplanes crossing γ has the preceding properties. Not all simplices are realized by a geodesic ray in this way: a simplex in $\mathcal{X}$ is called *visible* if it is. For example, if $\mathcal{Z}$ is a staircase, then $\partial_\triangle \mathcal{Z}$ has an invisible 0–simplex, represented by the set of horizontal hyperplanes.

Conjecture 2.8 of [BH16] holds that every simplex of $\partial_\triangle \mathcal{X}$ is visible when $\mathcal{X}$ admits a proper cocompact group action; Theorem A hence proves a special case. Slightly more generally:

Corollary D. *Let $\mathcal{X}$ be a proper CAT(0) cube complex which admits a proper and cocompact group action satisfying the NICC for hyperplanes. Then every simplex is $\partial_\triangle \mathcal{X}$ is visible. Moreover, let $v \in \partial_\triangle \mathcal{X}$ be a 0–simplex. Then there exists a CAT(0) geodesic ray γ such that the set of hyperplanes crossing γ represents v.*

The above could, in principle, hold even if $\mathfrak{F}$ is not a factor system, since we have not imposed the essential index condition. The "moreover" part follows from the first part and [Hag13, Lemma 3.32]. Corollary D combines with [BH16, Theorem 5.13] to imply that $\partial_\triangle \mathcal{X}$ detects thickness of order 1 and quadratic divergence for G, under the NICC condition. Corollary D also implies the corollary about staircases at the beginning of this paper. More generally, we obtain the following from Corollary D and a simple argument in [Hag13]:

Corollary E. *Let γ be a CAT(0)-metric or combinatorial geodesic ray in $\mathcal{X}$, where $\mathcal{X}$ is as in Corollary D and the set of hyperplanes crossing γ represents a d-dimensional simplex of $\partial_\triangle \mathcal{X}$. Then there exists a combinatorially isometrically embedded $d+1$-dimensional orthant subcomplex $\mathcal{O} \subseteq Hull(\gamma)$. Moreover, γ lies in a finite neighbourhood of $\mathcal{O}$.*

(A k-dimensional orthant subcomplex is a CAT(0) cube complex isomorphic to the product of k copies of the standard tiling of $[0, \infty)$ by 1–cubes, and the convex hull $Hull(A)$ of a subspace $A \subseteq \mathcal{X}$ is the smallest convex subcomplex containing A.)

Corollary E is related to Lemma 4.9 of [Hua14] and to statements in [Xie05, BKS08] about Euclidean sectors in cocompact CAT(0) cube complexes and arcs in the Tits boundary. In particular it shows that in any CAT(0) cube complex with a proper cocompact group action satisfying NICC, nontrivial geodesic arcs on the Tits boundary extend to arcs of length $\pi/2$.

Further questions and approaches. We believe that any proper cocompact CAT(0) cube complex admits a factor system, but that some additional ingredient is needed to remove the auxiliary hypotheses in Theorem A; we hope that the applications we have outlined stimulate interest in finding this additional idea. Since the property of admitting a factor system is inherited by convex subcomplexes [BHS14], we suggest trying to use G–cocompactness of $\mathcal{X}$ to arrange for a convex (non-G–equivariant) embedding of $\mathcal{X}$ into a CAT(0) cube complex $\mathcal{Y}$ where a factor system can be more easily shown to exist. One slightly outrageous possibility is:

Question 1. Let $\mathcal{X}$ be a CAT(0) cube complex which admits a proper and cocompact group action. Does $\mathcal{X}$ embed as a convex subcomplex of the universal cover of the Salvetti complex of some right-angled Artin group?

However, there are other possibilities, for example trying to embed $\mathcal{X}$ convexly in a CAT(0) cube complex whose automorphism group is sufficiently tame to enable one to use the proof of Theorem A, or some variant of it.

Plan of the paper. In Section 1, we discuss the necessary background. Section 2 contains basic facts about $\mathfrak{F}$, and Section 3 relates $\mathfrak{F}$ to orthogonal complements. Section 4 introduces the auxiliary hypotheses for Theorem A, which we prove in Section 6. The applications to the simplicial boundary are discussed in Section 7.

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1. Background

1.1. **Basics on CAT(0) cube complexes.** Recall that a *CAT(0) cube complex* $\mathcal{X}$ is a simply-connected cube complex in which the link of every vertex is a simplicial flag complex (see e.g. [BH99, Chapter II.5], [Sag14, Wis, Che00] for precise definitions and background). In this paper, $\mathcal{X}$ always denotes a CAT(0) cube complex. Our choices of language and notation for describing convexity, hyperplanes, gates, etc. follow the account given in [BHS14, Section 2].

Definition 1.1 (Hyperplane, carrier, combinatorial hyperplane)**.** A *midcube* in the unit cube $c = [-\frac{1}{2}, \frac{1}{2}]^n$ is a subspace obtained by restricting exactly one coordinate to 0. A *hyperplane* in $\mathcal{X}$ is a connected subspace H with the property that, for all cubes c of $\mathcal{X}$, either $H \cap c = \emptyset$ or $H \cap c$ consists of a single midcube of c. The *carrier* $\mathcal{N}(H)$ of the hyperplane H is the union of all closed cubes c of $\mathcal{X}$ with $H \cap c \neq \emptyset$. The inclusion $H \to \mathcal{X}$ extends to a combinatorial embedding $H \times [-\frac{1}{2}, \frac{1}{2}] \xrightarrow{\cong} \mathcal{N}(H) \hookrightarrow \mathcal{X}$ identifying $H \times \{0\}$ with H. Now, H is isomorphic to a CAT(0) cube complex whose cubes are the midcubes of the cubes in $\mathcal{N}(H)$. The subcomplexes $H^{\pm}$ of $\mathcal{N}(H)$ which are the images of $H \times \{\pm\frac{1}{2}\}$ under the above map are isomorphic as cube complexes to H, and are *combinatorial hyperplanes* in $\mathcal{X}$. Thus each hyperplane of $\mathcal{X}$ is associated to two combinatorial hyperplanes lying in $\mathcal{N}(H)$.

Remark. The distinction between hyperplanes (which are not subcomplexes) and combinatorial hyperplanes (which are) is important. Given $A \subset \mathcal{X}$, either a convex subcomplex or a hyperplane, and a hyperplane H, we sometimes say H *crosses* A to mean that $H \cap A \neq \emptyset$. Observe that the set of hyperplanes crossing a hyperplane H is precisely the set of hyperplanes crossing the associated combinatorial hyperplanes.

Definition 1.2 (Convex subcomplex)**.** A subcomplex $\mathcal{Y} \subseteq \mathcal{X}$ is *convex* if $\mathcal{Y}$ is *full* — i.e. every cube c of $\mathcal{X}$ whose 0–skeleton lies in $\mathcal{Y}$ satisfies $c \subseteq \mathcal{Y}$ — and $\mathcal{Y}^{(1)}$, endowed with the obvious path-metric, is metrically convex in $\mathcal{X}^{(1)}$.

There are various characterizations of cubical convexity. Cubical convexity coincides with CAT(0)–metric convexity *for subcomplexes* [Hag07], but not for arbitrary subspaces.

Definition 1.3 (Convex Hull)**.** Given a subset $A \subset \mathcal{X}$, we denote by $Hull(A)$ its *convex hull*, i.e. the intersection of all convex subcomplexes containing A.

If $\mathcal{Y} \subseteq \mathcal{X}$ is a convex subcomplex, then $\mathcal{Y}$ is a CAT(0) cube complex whose hyperplanes have the form $H \cap \mathcal{Y}$, where H is a hyperplane of $\mathcal{X}$, and two hyperplanes $H \cap \mathcal{Y}, H' \cap \mathcal{Y}$ intersect if and only if H, H' intersect.

Recall from [Che00] that the graph $\mathcal{X}^{(1)}$, endowed with the obvious path metric $\mathsf{d}_{\mathcal{X}}$ in which edges have length 1, is a *median graph* (and in fact being a median graph characterizes 1–skeleta of CAT(0) cube complexes among graphs): given 0–cubes x, y, z, there exists a unique 0–cube $m = m(x, y, z)$, called the *median* of x, y, z, so that $Hull(x, y) \cap Hull(y, z) \cap Hull(x, z) = \{m\}$.

Let $\mathcal{Y} \subseteq \mathcal{X}$ be a convex subcomplex. Given a 0–cube $x \in \mathcal{X}$, there is a unique 0–cube $y \in \mathcal{Y}$ so that $\mathsf{d}_{\mathcal{X}}(x, y)$ is minimal among all 0–cubes in $\mathcal{Y}$. Indeed, if $y' \in \mathcal{Y}$, then the median m of x, y, y' lies in $\mathcal{Y}$, by convexity of $\mathcal{Y}$, but $\mathsf{d}_{\mathcal{X}}(x, y') = \mathsf{d}_{\mathcal{X}}(x, m) + \mathsf{d}_{\mathcal{X}}(m, y')$, and the same is true for y. Thus, if $\mathsf{d}_{\mathcal{X}}(x, y')$ and $\mathsf{d}_{\mathcal{X}}(x, y)$ realize the distance from x to $\mathcal{Y}^{(0)}$, we have $m = y = y'$.

Definition 1.4 (Gate map on 0–skeleton)**.** For a convex subcomplex $\mathcal{Y} \subseteq \mathcal{X}$, the *gate map* to $\mathcal{Y}$ is the map $\mathfrak{g}_{\mathcal{Y}} : \mathcal{X}^{(0)} \to \mathcal{Y}^{(0)}$ so that or all $v \in \mathcal{X}^{(0)}$, $\mathfrak{g}_{\mathcal{Y}}(v)$ is the unique 0–cube of $\mathcal{Y}$ lying closest to v.

Lemma 1.5. *Let $\mathcal{Y} \subseteq \mathcal{X}$ be a convex subcomplex. Then the map $\mathfrak{g}_{\mathcal{Y}}$ from Definition 1.4 extends to a unique cubical map $\mathfrak{g}_{\mathcal{Y}} : \mathcal{X} \to \mathcal{Y}$ so that the following holds: for any d–cube c, of $\mathcal{X}$ with vertices $x_0, \ldots, x_{2^d} \in \mathcal{X}^{(0)}$, the map $\mathfrak{g}_{\mathcal{Y}}$ collapses c to the unique k–cube c' in $\mathcal{Y}$ with 0–cells $\mathfrak{g}_{\mathcal{Y}}(x_0) \ldots, \mathfrak{g}_{\mathcal{Y}}(x_{2^d})$ in the natural way, respecting the cubical structure.*

Furthermore, for any convex subcomplex $\mathcal{Y}, \mathcal{Y}' \subseteq \mathcal{X}$, the hyperplanes crossing $\mathfrak{g}_{\mathcal{Y}}(\mathcal{Y}')$ are precisely the hyperplanes which cross both $\mathcal{Y}$ and $\mathcal{Y}'$.

Proof. The first part is proved in [BHS14, p. 1743]: observe that the integer k is the number of hyperplanes that intersect both c and $\mathcal{Y}$. The hyperplanes that intersect c' are precisely the hyperplanes which intersect both c and $\mathcal{Y}$. Indeed, the Helly property ensures that there are cubes crossing exactly this set of hyperplanes, while convexity of $\mathcal{Y}$ shows that at least one such cube lies in $\mathcal{Y}$; the requirement that it contain $\mathfrak{g}_{\mathcal{Y}}(x_i)$ then uniquely determines c'.

To prove the second statement, let H be a hyperplane crossing $\mathcal{Y}$ and $\mathcal{Y}'$. Then H separates 0–cubes $y_1, y_2 \in \mathcal{Y}'$, and thus separates their gates in $\mathcal{Y}$, since, because it crosses $\mathcal{Y}$, it cannot separate y_1 or y_2 from $\mathcal{Y}$. On the other hand, if H crosses $\mathfrak{g}_{\mathcal{Y}}(\mathcal{Y}')$, then it separates $\mathfrak{g}_{\mathcal{Y}}(y_1), \mathfrak{g}_{\mathcal{Y}}(y_2)$ for some $y_1, y_2 \in \mathcal{Y}'$. Since it cannot separate y_1 or y_2 from $\mathcal{Y}$, the hyperplane H must separate y_1 from y_2 and thus cross $\mathcal{Y}$. (Here we have used the standard fact that H separates y_i from $\mathfrak{g}_{\mathcal{Y}}(y_i)$ if and only if H separates y_i from $\mathcal{Y}$; see e.g. [BHS14, p. 1743].) Hence H crosses $\mathfrak{g}_{\mathcal{Y}}(\mathcal{Y}')$ if and only if H crosses $\mathcal{Y}, \mathcal{Y}'$. □

The next definition formalizes the relationship between $\mathfrak{g}_{\mathcal{Y}}(\mathcal{Y}'), \mathfrak{g}_{\mathcal{Y}'}(\mathcal{Y})$ in the above lemma.

Definition 1.6 (Parallel)**.** The convex subcomplexes F and F' are *parallel*, written $F \parallel F'$, if for each hyperplane H of $\mathcal{X}$, we have $H \cap F \neq \emptyset$ if and only if $H \cap F' \neq \emptyset$. The subcomplex F is *parallel into* F' if F is parallel to a subcomplex of F', i.e. every hyperplane intersecting F intersects F'. We denote this by $F \hookrightarrow_{\parallel} F'$. Any two 0–cubes are parallel subcomplexes.

The following is proved in [BHS14, Section 2] and illustrated in Figure 2:

Lemma 1.7. *Let F, F' be parallel subcomplexes of the CAT(0) cube complex $\mathcal{X}$. Then $Hull(F \cup F') \cong F \times A$, where A is the convex hull of a shortest combinatorial geodesic with endpoints on F and F'. The hyperplanes intersecting A are those separating F, F'. Moreover, if $D, E \subset \mathcal{X}$ are convex subcomplexes, then $\mathfrak{g}_E(D) \subset E$ is parallel to $\mathfrak{g}_D(E) \subset D$.*

The next Lemma will be useful in Section 2.

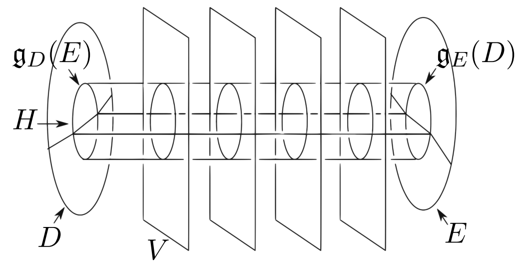

FIGURE 2. Here, D, E are convex subcomplexes. The gates $\mathfrak{g}_D(E), \mathfrak{g}_E(D)$ are parallel, and are joined by a product region, shown as a cylinder. Each hyperplane crossing $Hull\big(\mathfrak{g}_D(E) \cup \mathfrak{g}_E(D)\big)$ either separates $\mathfrak{g}_D(E), \mathfrak{g}_E(D)$ (e.g. the hyperplane V) or crosses both of $\mathfrak{g}_D(E), \mathfrak{g}_E(D)$ (e.g. the hyperplane H).

Lemma 1.8. *For convex subcomplexes C, D, E, we have $\mathfrak{g}_{\mathfrak{g}_C(D)}(E) \parallel \mathfrak{g}_C(\mathfrak{g}_D(E)) \parallel \mathfrak{g}_C(\mathfrak{g}_E(D))$.*

Proof. Let $F = \mathfrak{g}_C(D)$. Let H be a hyperplane so that $H \cap \mathfrak{g}_F(E) \neq \emptyset$. Then $H \cap E, H \cap F \neq \emptyset$ and thus $H \cap C, H \cap D \neq \emptyset$, by Lemma 1.5. Thus $\mathfrak{g}_F(E)$ is parallel into $\mathfrak{g}_C(\mathfrak{g}_D(E))$ and $\mathfrak{g}_C(\mathfrak{g}_E(D))$. However, the hyperplanes crossing either of these are precisely the hyperplanes crossing all of C, D, E. Thus, they cross F and D, and thus cross $\mathfrak{g}_F(E)$ by Lemma 1.5. □

The next lemma will be used heavily in Section 6, and gives a group theoretic description of the stabilizer of a projection.

Lemma 1.9. *Let $\mathcal{X}$ be a proper CAT(0) cube complex on which G acts properly and cocompactly. Let H, H' be two convex subcomplexes in $\mathcal{X}$ such that $\mathrm{Stab}_G(H)$ acts cocompactly on H and $\mathrm{Stab}_G(H')$ acts cocompactly on H'. Then $\mathrm{Stab}_G(\mathfrak{g}_H(H'))$ is commensurable with $\mathrm{Stab}_G(H) \cap \mathrm{Stab}_G(H')$. Further, for any finite collection $H_1, \ldots, H_n$ of convex subcomplexes whose stabilisers act cocompactly, $\mathrm{Stab}_G(\mathfrak{g}_{H_1}(\mathfrak{g}_{H_2}(\cdots \mathfrak{g}_{H_{n-1}}(H_n)\cdots)))$ is commensurable with $\bigcap_{i=1}^n \mathrm{Stab}_G(H_i)$.*

Proof. Let H and H' be two convex subcomplexes and suppose that $g \in \mathrm{Stab}_G(H) \cap \mathrm{Stab}_G(H')$. Then $g \in \mathrm{Stab}_G(\mathfrak{g}_H(H'))$, and thus $\mathrm{Stab}_G(H) \cap \mathrm{Stab}_G(H') \leq \mathrm{Stab}_G(\mathfrak{g}_H(H'))$.

Let $d = d(H, H')$. In particular, for any 0–cube in $\mathfrak{g}_H(H')$, its distance to H' is exactly d. However, there are only finitely many such translates of H' in $\mathcal{X}$, and any element of $\mathrm{Stab}_G(\mathfrak{g}_H(H'))$ must permute these. Further, there are only finitely many translates of H in $\mathcal{X}$ that contain $\mathfrak{g}_H(H')$, and any element of the stabilizer must also permute those. Thus, there is a finite index subgroup (obtained as the kernel of the permutation action on the finite sets of hyperplanes) that stabilizes both H and H'. A similar argument covers the case of finitely many complexes. □

Definition 1.10 (Orthogonal complement)**.** Let $A \subseteq \mathcal{X}$ be a convex subcomplex. Let P_A be the convex hull of the union of all parallel copies of A, so that $P_A \cong A \times A^{\perp}$, where $A^{\perp}$ is a CAT(0) cube complex that we call the *abstract orthogonal complement of A in $\mathcal{X}$*. Let $\phi_A : A \times A^{\perp} \to \mathcal{X}$ be the cubical isometric embedding with image P_A.

For any $a \in A^{(0)}$, the convex subcomplex $\phi_A(\{a\} \times A^{\perp})$ is the *orthogonal complement of A at a*. See Figures 3 and 4.

Lemma 1.11. *Let $A \subseteq \mathcal{X}$ be a convex subcomplex. For any $a \in A$, a hyperplane H intersects $\phi_A(\{a\} \times A^{\perp})$ if and only if H is disjoint from every parallel copy of A but intersects each hyperplane V with $V \cap A \neq \emptyset$. Hence $\phi_A(\{a\} \times A^{\perp}), \phi_A(\{b\} \times A^{\perp})$ are parallel for all $a, b \in A^{(0)}$.*

Proof. This follows from the definition of P_A: the hyperplanes crossing P_A are partitioned into two classes, those intersecting A (and its parallel copies) and those disjoint from A (and any of

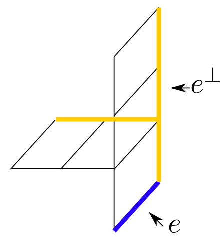

FIGURE 3. Combinatorial hyperplanes are orthogonal complements of 1–cubes.

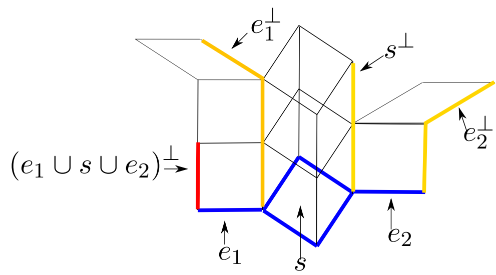

FIGURE 4. Orthogonal complements of 1–cubes e_1, e_2 and 2–cube s are shown. Note that $(e_1 \cup e_2 \cup s)^{\perp} \parallel \mathfrak{g}_{e_2^{\perp}}(\mathfrak{g}_{e_1^{\perp}}(s^{\perp}))$.

its parallel copies). By definition, $\phi_A(\{a\} \times A^{\perp})$ is the convex hull of the set of 0–cubes of P_A that are separated from a only by hyperplanes of the latter type. The product structure ensures that any hyperplane of the first type crosses every hyperplane of the second type. □

Finally, in [CS11], Caprace and Sageev defined the notion of an essential hyperplane and an essential action. We record the necessary facts here.

Definition 1.12. Let $\mathcal{X}$ be a CAT(0) cube complex, and let $F \subseteq \mathcal{X}$ be a convex subcomplex. Let $G \leq \mathrm{Aut}(\mathcal{X})$ preserve F.

(1) We say that a hyperplane H is *essential* in F if H crosses F, and each halfspace associated to H contains 0–cubes of F which are arbitrarily far from H.
(2) We say that H is *G–essential* in F if for any 0–cube $x \in F$, each halfspace associated to H contains elements of Gx arbitrarily far from H.
(3) We say that G acts *essentially* on F if every hyperplane crossing F is G–essential in F.

If G acts cocompactly on F, then a hyperplane is G–essential if and only if it is essential in F.

Proposition 1.13. *Let $\mathcal{X}$ be a proper CAT(0) cube complex admitting a proper cocompact action by a group Γ, let $F \subseteq \mathcal{X}$ be a convex subcomplex, and let $G \leq \Gamma$ act on F cocompactly. Then:*

(i) there exists a G–invariant convex subcomplex $\widehat{F}_G$, called the G–essential core of F, crossed by every essential hyperplane in F, on which G acts essentially and cocompactly;
(ii) $\widehat{F}_G$ is unbounded if and only if F is unbounded;
(iii) if $G' \leq \mathrm{Aut}(\mathcal{X})$ also acts on F cocompactly, then $\widehat{F}_{G'}$ is parallel to $\widehat{F}_G$;
(iv) if $G' \leq G$ is a finite-index subgroup, we can take $\widehat{F}_{G'} = \widehat{F}_G$.
(v) the subcomplex $\widehat{F}_G$ is finite Hausdorff distance from F.

Proof. By [CS11, Proposition 3.5], F contains a G–invariant convex subcomplex $\widehat{F}_G$ on which G acts essentially and cocompactly (in particular, $\widehat{F}_G$ is unbounded if and only if F is, and $\mathsf{d}_{Haus}(F, \widehat{F}_G) < \infty$). The hyperplanes of F crossing $\widehat{F}_G$ are precisely the G–essential hyperplanes. Observe that if H is a hyperplane crossing F essentially, then cocompactness of the G–action on F implies that H is G–essential and thus crosses $\widehat{F}_G$. It follows that if G, G' both act on F cocompactly, then a hyperplane crossing F is G–essential if and only if it is G'–essential, so $\widehat{F}_G, \widehat{F}_{G'}$ cross the same hyperplanes, i.e. they are parallel. If $G' \leq G$, then $\widehat{F}_G$ is G'–invariant, and if $[G : G'] < \infty$, then G' also acts cocompactly on F, so we can take $\widehat{F}_G = \widehat{F}_{G'}$. □

1.2. **Hyperclosure and factor systems.**

Definition 1.14 (Factor system, hyperclosure)**.** The *hyperclosure* of $\mathcal{X}$ is the intersection $\mathfrak{F}$ of all sets $\mathfrak{F}'$ of convex subcomplexes of $\mathcal{X}$ that satisfy the following three properties:

(1) $\mathcal{X} \in \mathfrak{F}'$, and for all combinatorial hyperplanes H of $\mathcal{X}$, we have $H \in \mathfrak{F}'$;
(2) if $F, F' \in \mathfrak{F}'$, then $\mathfrak{g}_F(F') \in \mathfrak{F}'$;
(3) if $F \in \mathfrak{F}'$ and F' is parallel to F, then $F' \in \mathfrak{F}'$.

Note that $\mathfrak{F}$ is $\mathrm{Aut}(\mathcal{X})$–invariant. If there exists ξ such that for all $x \in \mathcal{X}$, there are at most ξ elements $F \in \mathfrak{F}$ with $x \in F$, then, following [BHS14], we call $\mathfrak{F}$ a *factor system* for $\mathcal{X}$.

Remark 1.15. The definition of a factor system in [BHS14] is more general than the definition given above. The assertion that $\mathcal{X}$ has a factor system in the sense of [BHS14] is equivalent to the assertion that the hyperclosure of $\mathcal{X}$ has finite multiplicity, because any factor system (in the sense of [BHS14]) contains all elements of $\mathfrak{F}$ whose diameters exceed a given fixed threshold. Each of the five conditions in Definition 8.1 of [BHS14] is satisfied by $\mathfrak{F}$, except possibly the finite multiplicity condition, Definition 8.1.(3). Indeed, parts (1),(2),(4) of that definition are included in Definition 1.14 above. Part (5) asserts that there is a constant p so that $\mathfrak{g}_F(F')$ is in the factor system provided F, F' are and $\mathrm{diam}(\mathfrak{g}_F(F')) \geq p$. Hence Definition 1.14.(2) implies that this condition is satisfied by $\mathfrak{F}$, with $p = 0$.

2. Analysis of the hyperclosure

Fix a proper $\mathcal{X}$ with a group G acting properly and cocompactly. Let $\mathfrak{F}$ be the hyperclosure.

2.1. **Decomposition.** Let $\mathfrak{F}_0 = \{\mathcal{X}\}$ and, for each $n \geq 1$, let $\mathfrak{F}_n$ be the subset of $\mathfrak{F}$ consisting of those subcomplexes that can be written in the form $\mathfrak{g}_H(F)$, where $F \in \mathfrak{F}_{n-1}$ and H is a combinatorial hyperplane. Hence $\mathfrak{F}_1$ is the set of combinatorial hyperplanes in $\mathcal{X}$.

Lemma 2.1 (Decomposing $\mathfrak{F}$)**.** *Each $F \in \mathfrak{F} - \{\mathcal{X}\}$ is parallel to a subcomplex of the form*

$$\mathfrak{g}_{H_1}(\mathfrak{g}_{H_2}(\cdots \mathfrak{g}_{H_{n-1}}(H_n)\cdots))$$

for some $n \geq 1$, where each H_i is a combinatorial hyperplane, i.e. $\mathfrak{F}/_{\parallel} = (\cup_{n\geq 1}\mathfrak{F}_n)/_{\parallel}$.

Proof. This follows by induction, Lemma 1.8, and the definition of $\mathfrak{F}$. □

Corollary 2.2. $\mathfrak{F} = \cup_{n\geq 0}\mathfrak{F}_n$.

Proof. It suffices to show $\mathfrak{F} \subseteq \cup_{n\geq 0}\mathfrak{F}_n$. Let $F \in \mathfrak{F}$. If $F = \mathcal{X}$, then $F \in \mathfrak{F}_0$. Otherwise, by Lemma 2.1, there exists $n \geq 1$, a combinatorial hyperplane H, and a convex subcomplex $F' \in \cup_{k\leq n}\mathfrak{F}_k$ with $F \parallel \mathfrak{g}_H(F')$. Consider $\phi_F(P_F \cong F \times F^{\perp})$ and choose $f \in F^{\perp}$ so that $\phi_F(F \times \{f\})$ coincides with F. Then $\phi_F(F \times \{f\})$ lies in some combinatorial hyperplane H' – either $H' = H$ and $F = \mathfrak{g}_H(F)$, or F is non-unique in its parallelism class, so lies in a combinatorial hyperplane in the carrier of a hyperplane crossing $F^{\perp}$. Consider $\mathfrak{g}_{H'}(\mathfrak{g}_H(F'))$. On one hand, $\mathfrak{g}_{H'}(\mathfrak{g}_H(F')) \in \cup_{k\leq n+1}\mathfrak{F}_k$. But $\mathfrak{g}_{H'}(\mathfrak{g}_H(F')) = F$. Hence $F \in \cup_{n\geq 1}\mathfrak{F}_n$. □

2.2. **Stabilizers act cocompactly.** The goal of this subsection is to prove that $\mathrm{Stab}_G(F)$ acts cocompactly on F for each $F \in \mathfrak{F}$. The following lemma is standard but we include a proof in the interest of a self-contained exposition.

Lemma 2.3 (Coboundedness from finite multiplicity)**.** *Let X be a metric space and let $G \to$ $\mathrm{Isom}(X)$ act cocompactly, and let $\mathcal{Y}$ be a G–invariant collection of subspaces such that every ball intersects finitely many elements of $\mathcal{Y}$. Then $\mathrm{Stab}_G(P)$ acts coboundedly on P for every $P \in \mathcal{Y}$.*

Proof. Let $P \in \mathcal{Y}$, choose a basepoint $r \in X$, and use cocompactness to choose $t < \infty$ so that $\mathsf{d}(x, G \cdot r) \leq t$ for all $x \in X$. Choose $g_1, \ldots, g_s \in G$ so that the G–translates of P intersecting $\mathcal{N}_{10t}(r)$ are exactly $g_i P$ for $i \leq s$. Since $\mathcal{Y}$ is G–invariant and locally finite, $s < \infty$. (In other words, the assumptions guarantee that there are finitely many cosets of $\mathrm{Stab}_G(P)$ whose corresponding translates of P intersect $\mathcal{N}_{10t}(r)$, and we have fixed a representative of each of these cosets.) Let $K_r = \max_{i \leq s} \mathsf{d}(r, g_i r)$. For each $g \in G$, the translates of P that lie within distance $10t$ of $g \cdot r$ are precisely $gg_1P, \ldots, gg_sP$. Letting $K_{gr} = \max_{i \leq s} \mathsf{d}(gr, gg_i r)$, we have $K_{gr} = K_r$ just because $\mathsf{d}(r, g_i \cdot r) = \mathsf{d}(g \cdot r, gg_i \cdot r)$.

Fix a basepoint $p \in P$ and let $q \in P$ be an arbitrary point; choose $h_p, h_q \in G$ so that $\mathsf{d}(h_p \cdot r, p) \leq t, \mathsf{d}(h_q \cdot r, q) \leq t$. Without loss of generality, we may assume that $h_q = 1$. Then $\{h_p g_i P\}_{i=1}^s$ is the set of P–translates intersecting $\mathcal{N}_{10t}(h_p \cdot r)$. Now, $p \in P$ and $\mathsf{d}(h_p \cdot r, p) < 10t$, so there exists i so that $h_p g_i P = P$, i.e. $h_p g_i \in \mathrm{Stab}_G(P)$. Finally,

$$\mathsf{d}(h_p g_i \cdot q, p) \leq \mathsf{d}(h_p g_i \cdot r, h_p g_i \cdot q) + \mathsf{d}(h_p \cdot r, p) + \mathsf{d}(h_p g_i \cdot r, h_p \cdot r) \leq 2t + K_{h_p r} = 2t + K_r,$$

which is uniformly bounded. Hence the action of $\mathrm{Stab}_G(P)$ on P is cobounded. □

Remark 2.4. We use Lemma 2.3 when X and P are proper, to get a cocompact action.

Lemma 2.5. *Let $\mathcal{X}$ be a proper CAT(0) cube complex with a group G acting cocompactly. Let $Y, Y' \subset \mathcal{X}$ be parallel convex subcomplexes, then $\mathrm{Stab}_G(Y)$ and $\mathrm{Stab}_G(Y')$ are commensurable. Thus, if $\mathrm{Stab}_G(Y)$ acts cocompactly on Y, then $\mathrm{Stab}_G(Y) \cap \mathrm{Stab}_G(Y')$ acts cocompactly on Y'.*

Proof. Let T be the set of $\mathrm{Stab}_G(Y)$–translates of Y'. Then each $gY' \in T$ is parallel to Y, and $\mathsf{d}_{\mathcal{X}}(gY', Y) = \mathsf{d}_{\mathcal{X}}(Y', Y)$. Since $Y^{\perp}$ is locally finite, $|T| < \infty$. Hence $K = \ker(\mathrm{Stab}_G(Y) \to \mathrm{Sym}(T))$ has finite index in $\mathrm{Stab}_G(Y)$ but lies in $\mathrm{Stab}_G(Y) \cap \mathrm{Stab}_G(Y')$. By Lemma 1.7, K acts cocompactly on $Hull(Y \cup Y')$, stabilizing Y', and thus acts cocompactly on Y'. □

Definition 2.6. Let $H \in \mathfrak{F}_1$. For $n \geq 1, k \geq 0$, let $\mathfrak{F}_{n,H,k}$ be the set of $F \in \mathfrak{F}_n$ so that $F = \mathfrak{g}_H(F')$ for some $F' \in \mathfrak{F}_{n-1}$ with $\mathsf{d}(H, F') \leq k$. Let $\mathfrak{F}_{n,H} = \cup_{k \geq 0} \mathfrak{F}_{n,H,k}$ and $\mathfrak{F}_{n,k} = \cup_{H \in \mathfrak{F}_1} \mathfrak{F}_{n,H,k}$.

Proposition 2.7 (Cocompactness)**.** *Let $n \geq 1$. Then, for any $F \in \mathfrak{F}_n$, $\mathrm{Stab}_G(F)$ acts cocompactly on F. Hence $\mathrm{Stab}_G(F)$ acts cocompactly on F for each $F \in \mathfrak{F}$.*

Proof. The second assertion follows from the first and Corollary 2.2. We argue by double induction on n, k to prove the first assertion, with k as in Definition 2.6. First, observe that $\mathfrak{F}_n$, $\mathfrak{F}_{n,k}$ are G–invariant for all n, k. Similarly, $\mathfrak{F}_{n,H,k}$ is $\mathrm{Stab}_G(H)$–invariant for all $H \in \mathfrak{F}_1$.

Base Case: $n = 1$**.** From local finiteness of $\mathcal{X}$, cocompactness of the action of G and Lemma 2.3, we see that $\mathrm{Stab}_G(H)$ acts cocompactly on H for each $H \in \mathfrak{F}_1$.

Inductive Step 1: (n, k) **for all** k **implies** $(n+1, 0)$**.** Let $F \in \mathfrak{F}_{n+1,0}$. Then $F = H \cap F'$, where $H \in \mathfrak{F}_1$ and $F' \in \mathfrak{F}_n$. By definition, $F' = \mathfrak{g}_{H'}(F'')$ for some $F'' \in \mathfrak{F}_{n-1}$ and $H' \in \mathfrak{F}_1$. Thus $K = \mathrm{Stab}_G(F')$ acts cocompactly on F' by induction.

Let $\mathcal{S} = \{k(H \cap F') : k \in K\}$, which is a K–invariant set of convex subcomplexes of F'. Moreover, since the set of all K–translates of H is a locally finite collection, because $\mathcal{X}$ is locally finite and H is a combinatorial hyperplane, $\mathcal{S}$ has the property that every ball in F' intersects finitely many elements of $\mathcal{S}$. Lemma 2.3, applied to the cocompact action of K on F', shows that $\mathrm{Stab}_K(H \cap F')$ (which equals $\mathrm{Stab}_K(F)$), and hence $\mathrm{Stab}_G(F)$, acts cocompactly on F.

Inductive Step 2: (n, k) **implies** $(n, k+1)$**.** Let $F \in \mathfrak{F}_{n,k+1}$ so that $F = \mathfrak{g}_H(F')$ with $H \in \mathfrak{F}_1$, $F' \in \mathfrak{F}_{n-1}$ and $d = \mathsf{d}(H, F') \leq k+1$. If $d \leq k$, induction applies. Thus, we can assume that $d = k+1$. Then there is a product region $F \times [0, d] \subset \mathcal{X}$ with $F \times \{0\} = F$, and $F \times \{d\} \subset F'$. Now, $F_1 := F \times \{1\}$ is a parallel copy of F contained in the carrier of the hyperplane H'' dual to the edge $[0,1]$ of $[0,d]$. Letting H' be the combinatorial hyperplane parallel to H'' in $\mathcal{N}(H'')$ and separated from F by H'', we have $F_1 \subset \mathfrak{g}_{H'}(F')$. Moreover, $d(H', F') \leq d - 1 = k$. By induction $L = \mathrm{Stab}_G(\mathfrak{g}_{H'}(F'))$ acts cocompactly on $\mathfrak{g}_{H'}(F')$.

We claim that $F_1 = \mathfrak{g}_{H'}(F') \cap \mathfrak{g}_{H'}(H)$. To see this, note that the hyperplanes that cross F_1 are exactly the hyperplanes that cross F' and H. However, those are the hyperplanes which cross H' and F' which also cross H. It easily follows that the two subcomplexes are equal.

Now let $\mathcal{T}$ be the set of L–translates of $F_1 = \mathfrak{g}_{H'}(F') \cap \mathfrak{g}_{H'}(H)$ in $\mathfrak{g}_{H'}(F')$. This is an L–invariant collection of convex subcomplexes of $\mathfrak{g}_{H'}(F')$. Moreover, each ball in $\mathfrak{g}_{H'}(F')$ intersects finitely many elements of $\mathcal{T}$. Indeed, $\mathcal{T}$ is a collection of subcomplexes of the form $T_\ell = \mathfrak{g}_{\ell H'}(\ell H) \cap \mathfrak{g}_{H'}(F')$, where $\ell \in L$. Recall that $\mathsf{d}_{\mathcal{X}}(H, H') = 1$. Hence, fixing $y \in \mathfrak{g}_{H'}(F')$ and $t \geq 0$, if $\{T_{\ell_i}\}_{i\in I} \subseteq \mathcal{T}$ is a collection of elements of $\mathcal{T}$, all of which intersect $\mathcal{N}_t(y)$, then $\{\ell_i H, \ell_i H'\}_{i\in I}$ all intersect $\mathcal{N}_{t+1}(y)$. However, by local finiteness of $\mathcal{X}$ there are only finitely many distinct $\ell_i H, \ell_i H'$. Further, if $\ell_i H = \ell_j H$ and $\ell_i H' = \ell_j H'$, then $T_{\ell_i} = T_{\ell_j}$. Thus, the index set I must be finite. Hence, by Lemma 2.3 and cocompactness of the action of L on $\mathfrak{g}_{H'}(F')$, we see (as in Inductive Step 1) that $\mathrm{Stab}_G(F_1)$ acts cocompactly on F_1. Now, since F_1 is parallel to F, we see by Lemma 2.5 that $\mathrm{Stab}_G(F)$ acts cocompactly on F. □

The next Lemma explains how to turn the algebraic conditions on the G–action described in Section 4 into geometric properties of the convex subcomplexes in $\mathfrak{F}$. This is of independent interest, giving a complete algebraic characterization of when two cocompact subcomplexes have parallel essential cores.

Lemma 2.8 (Characterization of commensurable stabilizers)**.** *Let Y_1 and Y_2 be two convex subcomplexes of $\mathcal{X}$ and let $G_i = \mathrm{Stab}_G(Y_i)$. Suppose further that G_i acts on Y_i cocompactly. Then G_1 and G_2 are commensurable if and only of the G_1–essential core $\widehat{Y_1}$ and the G_2–essential core $\widehat{Y_2}$ are parallel.*

Proof. First, if $\widehat{Y_1}, \widehat{Y_2}$ are parallel, then Lemma 2.5 shows that $\mathrm{Stab}_G(\widehat{Y_1}), \mathrm{Stab}_G(\widehat{Y_2})$ contain $\mathrm{Stab}_G(\widehat{Y_1}) \cap \mathrm{Stab}_G(\widehat{Y_2})$ as a finite-index subgroup. Since $\mathrm{Stab}_G(\widehat{Y_i})$ contains G_i as a finite-index subgroup, it follows that $G_1 \cap G_2$ has finite index in G_1 and in G_2.

Conversely, suppose that G_1, G_2 have a common finite-index subgroup. Thus, $G_1 \cap G_2$ acts cocompactly on both Y_1 and Y_2. This implies that Y_1, Y_2 lie at finite Hausdorff distance, since choosing $r > 0$ and $y_i \in Y_i$ so that $(G_1 \cap G_2)B_r(y_i) = Y_i$, we see that Y_1 is in the $d(y_1, y_2) + r$ neighbourhood of Y_2, and vice-versa. Further, this implies that $\widehat{Y_1}, \widehat{Y_2}$ lie at finite Hausdorff distance, since $\widehat{Y_i}$ is finite Hausdorff distance from Y_i.

Suppose that $\widehat{Y_1}, \widehat{Y_2}$ are not parallel. Then, without loss of generality, some hyperplane H of $\mathcal{X}$ crosses $\widehat{Y_1}$ but not $\widehat{Y_2}$. Since G_1 acts on $\widehat{Y_1}$ essentially and cocompactly, [CS11] provides a hyperbolic isometry $g \in G_1$ of $\widehat{Y_1}$ so that $g\overleftarrow{H} \subsetneq \overleftarrow{H}$, where $\overleftarrow{H}$ is the halfspace of $\mathcal{X}$ associated to H and disjoint from Y_2. Choosing $n > 0$ so that the translation length of g^n exceeds the distance from Y_2 to the point in which some g–axis intersects H, we see that H cannot separate $g^n\widehat{Y}_2$ from the axis of g. Thus, $g^n\widehat{F}' \cap \overleftarrow{H} \neq \emptyset$, whence $\langle g\rangle \cap G_2 = \{1\}$, contradicting that G_1 and G_2 are commensurable (since g has infinite order). Thus $\widehat{Y}_1, \widehat{Y}_2$ are parallel. □

2.3. **Ascending or descending chains.**

We reduce Theorem A to a claim about chains in $\mathfrak{F}$.

Lemma 2.9 (Finding chains)**.** *Let $\mathfrak{U} \subseteq \mathfrak{F}$ be an infinite subset satisfying $\bigcap_{U\in\mathfrak{U}} U \ni x$ for some $x \in \mathcal{X}$. Then one of the following holds:*

- *there exists a sequence $\{F_i\}_{i\geq 1}$ in $\mathfrak{F}$ so that $x \in F_i \subsetneq F_{i+1}$ for all i;*
- *there exists a sequence $\{F_i\}_{i\geq 1}$ in $\mathfrak{F}$ so that $x \in F_i$ and $F_i \supsetneq F_{i+1}$ for all i.*

Proof. Let $\mathfrak{F}_x \supseteq \mathfrak{U}$ be the set of $F \in \mathfrak{F}$ with $x \in F$. Let Ω be the directed graph with vertex set $\mathfrak{F}_x$, with (F, F') a directed edge if $F \subsetneq F'$ and there does not exist $F'' \in \mathfrak{F}_x$ with $F \subsetneq F'' \subsetneq F'$.

Let $F_0 = \{x\}$. Since x is the intersection of the finitely many hyperplane carriers containing it, $F_0 \in \mathfrak{F}$ and in particular $F_0 \in \mathfrak{F}_x$. Moreover, note that F_0 has no incoming Ω–edges, since

F_0 cannot properly contain any other subcomplex. For any $F \in \mathfrak{F}_x$, either Ω contains an edge from F_0 to F, or there exists $F' \in \mathfrak{F}_x$ such that $F_0 \subset F' \subset F$.

Hence either $\mathfrak{F}_x$ contains an infinite ascending or descending $\subseteq$–chain, or Ω is a connected directed graph in which every non-minimal vertex as an immediate predecessor, and every non-maximal vertex has an immediate successor. In the first two cases, we are done, so assume that the third holds. In the third case, there is a unique vertex namely F_0, with no incoming edges and there is a finite length directed path from F_0 to any vertex.

Let $F \in \mathfrak{F}_x$ and suppose that $\{F_i\}_i$ is the set of vertices of Ω so that (F, F_i) is an edge. For $i \neq j$, we have $F \subseteq F_i \cap F_j \subsetneq F_i$, so since $F_i \cap F_j = \mathfrak{g}_{F_i}(F_j) \in \mathfrak{F}$, we have $F_i \cap F_j = F$.

The set $\{F_i\}_i$ is invariant under the action of $\mathrm{Stab}_G(F)$. Also, by Proposition 2.7, $\mathrm{Stab}_G(F)$ acts cocompactly on F. A 0–cube $y \in F$ is *diplomatic* if there exists i so that y is joined to a vertex of $F_i - F$ by a 1–cube in F_i. Only uniformly finitely many F_i can witness the diplomacy of y since $\mathcal{X}$ is uniformly locally finite and $F_i \cap F_j = F$ whenever $i \neq j$. Also, y is diplomatic, witnessed by $F_{i_1}, \ldots, F_{i_k}$, if and only if gy is diplomatic, witnessed by $gF_{i_1}, \ldots, gF_{i_k}$, for each $g \in \mathrm{Stab}_G(F)$. Since $\mathrm{Stab}_G(F) \curvearrowright F$ cocompactly, we thus get $|\{F_i\}_i / \mathrm{Stab}_G(F)| < \infty$.

Let $\widehat{\Omega}$ be the graph with a vertex for each $F \in \mathfrak{F}$ containing a point of $G \cdot x$ and a directed edge for minimal containment as above. Then $\widehat{\Omega}$ is a graded directed graph as above. For each $n \geq 0$, let $\mathcal{S}_n$ be the set of vertices in $\widehat{\Omega}$ at distance n from a minimal element. The above argument shows that G acts cofinitely on each $\mathcal{S}_n$, and thus $\widehat{\Omega}/G$ is locally finite. Hence, by König's infinity lemma, either $\widehat{\Omega}/G$ contains a directed ray or $\widehat{\Omega}^{(0)}/G$ is finite. In the former case, $\widehat{\Omega}$ must contain a directed ray, in which case there exists $\{F_i\} \subseteq \mathfrak{F}$ with $F_i \subsetneq F_{i+1}$ for all i. Up to translating by an appropriate element of G, we can assume that $x \in F_1$. The latter case means that the set of $F \in \mathfrak{F}$ such that $F \cap G \cdot x \neq \emptyset$ is G–finite. But since G acts properly and cocompactly on $\mathcal{X}$, any G–invariant G–finite collection of subcomplexes whose stabilizers act cocompactly has finite multiplicity, a contradiction. □

3. Orthogonal complements of compact sets and the hyperclosure

We now characterise $\mathfrak{F}$ in a CAT(0) cube complex $\mathcal{X}$, without making use of a group action.

Lemma 3.1. *Let $A \subseteq B \subseteq \mathcal{X}$ be convex subcomplexes and let $a \in A$. Then $\phi_B(\{a\} \times B^{\perp}) \subseteq \phi_A(\{a\} \times A^{\perp})$.*

Proof. Let $x \in \phi_B(\{a\} \times B^{\perp})$. Then every hyperplane H separating x from a separates two parallel copies of B and thus separates two parallel copies of A, since $A \subseteq B$. It follows from Lemma 1.11 that every hyperplane separating a from x crosses $\phi_A(\{a\} \times A^{\perp})$, whence $x \in \phi_A(\{a\} \times A^{\perp})$. □

Given a convex subcomplex $F \subseteq \mathcal{X}$, fix a base 0–cube $f \in F$ and for brevity, let $F^{\perp} = \phi_F(\{f\} \times F^{\perp}) \subseteq \mathcal{X}$. Note that $f \in F^{\perp}$, and so we let $F^{\perp^{\perp}} = \phi_{F^{\perp}}\left(\{f\} \times (F^{\perp})^{\perp}\right)$ (here, the $(F^{\perp})^{\perp}$ is the abstract orthogonal complement of $F^{\perp}$), which again contains f, and so we can similarly define $\left((F^{\perp})^{\perp}\right)^{\perp}$ etc.

Lemma 3.2. *Let F be a convex subcomplex of $\mathcal{X}$. Then $\left((F^{\perp})^{\perp}\right)^{\perp} = F^{\perp}$.*

Proof. If F is a convex subcomplex, there is a parallel copy of $F^{\perp}$ based at each 0–cube of F, since $F \times F^{\perp}$ is a convex subcomplex of $\mathcal{X}$. Thus $F \hookrightarrow_{\|} (F^{\perp})^{\perp}$, and by Lemma 3.1 we have $F^{\perp} \supseteq \left((F^{\perp})^{\perp}\right)^{\perp}$. To obtain the other inclusion, we show that every parallel copy of F is contained in a parallel copy of $(F^{\perp})^{\perp}$. This is clear since, letting $A = F^{\perp}$, we have that

$\phi_A(A \times A^\perp)$ is a convex subcomplex of $\mathcal{X}$, but $F \subset A^\perp$ by the above, and thus $\phi_F(F \times F^\perp) \subseteq \phi_{F^\perp}(F^\perp \times (F^\perp)^\perp)$, both of which are convex subcomplexes of $\mathcal{X}$. Hence $F^\perp \subseteq \left((F^\perp)^\perp\right)^\perp$, completing the proof. $\square$

3.1. **Characterisation of $\mathfrak{F}$ using orthogonal complements of compact sets.** In this section, we assume that $\mathcal{X}$ is locally finite, but do not need a group action.

Theorem 3.3. *Let $F \subset \mathcal{X}$ be a convex subcomplex. Then $F \in \mathfrak{F}$ if and only if there exists a compact convex subcomplex C so that $C^\perp = F$.*

Proof. Let C be a compact convex subcomplex of $\mathcal{X}$. Let $H_1, \ldots, H_k$ be the hyperplanes crossing C. Fix a basepoint $x \in C$, and suppose the H_i are labeled so that $x \in \mathcal{N}(H_i)$ for $1 \leq i \leq m$, and $x \notin \mathcal{N}(H_i)$ for $i > m$, for some $m \leq k$. Let $F = \bigcap_{i=1}^k \mathfrak{g}_{H_1}(H_i)$, which contains x. Any hyperplane H crosses $\phi_C(\{x\} \times C^\perp)$ if and only if H crosses each H_i, which occurs if and only if H crosses F. Hence $F = \phi_C(\{x\} \times C^\perp)$, as required.

We now prove the converse. Let $F \in \mathfrak{F}_n$ for $n \geq 1$. If $n = 1$ and F is a combinatorial hyperplane, $F = e^\perp$ for some 1–cube e of $\mathcal{X}$. Next, assume that $n \geq 2$ and write $F = \mathfrak{g}_H(F')$ where $F' \in \mathfrak{F}_{n-1}$ and H is a combinatorial hyperplane. Induction on n gives $F' = (C')^\perp$ for some compact convex subcomplex C'.

Let e be a 1–cube with orthogonal complement $H \in \mathfrak{F}_1$, chosen as close as possible to C', so that $\mathsf{d}(e, C') = \mathsf{d}(H, C')$. In particular, any hyperplane separating e from C' separates H from C'. Moreover, we can and shall assume that C' was chosen in its parallelism class so that $\mathsf{d}(e, C')$ is minimal when e, C' are allowed to vary in their parallelism classes.

Let C be the convex hull of (the possibly disconnected set) $e \cup C'$.

We claim that $\mathfrak{g}_H(F') = \{x\} \times C^\perp$. First, suppose that V is a hyperplane crossing $\{x\} \times C^\perp$. Then V separates two parallel copies of C, each of which contains a parallel copy of e and one of C'. Hence V crosses H and F', so V crosses $\mathfrak{g}_H(F')$. Thus $\{x\} \times C^\perp \subseteq \mathfrak{g}_H(F')$.

Conversely, suppose V is a hyperplane crossing $\mathfrak{g}_H(F')$, i.e. crossing H and F'. To show that V crosses $\{x\} \times C^\perp$, it suffices to show that V crosses every hyperplane crossing C. If W crosses C, then either W separates e, C' or crosses $e \cup C'$. In the latter case, V crosses W since V crosses H and $(C')^\perp = F'$. In the former case, since e, C' are as close as possible in their parallelism classes, W separates e, C' only if it separates H from $C' \times (C')^\perp$, so W must cross V. Hence $\mathfrak{g}_H(F') \subseteq \{x\} \times C^\perp$. Since only finitely many hyperplanes V either cross e, cross C', or separate e from C', the subcomplex C is compact. $\square$

Corollary 3.4. *If $F \in \mathfrak{F}$, then $(F^\perp)^\perp = F$.*

Proof. If $F \in \mathfrak{F}$, then $F = C^\perp$ for some compact C, by Theorem 3.3, and hence $((C^\perp)^\perp)^\perp = (F^\perp)^\perp = C^\perp = F$, by Lemma 3.2. $\square$

4. Auxiliary conditions

In this section, the group G acts geometrically on the proper CAT(0) cube complex $\mathcal{X}$.

4.1. **Rotation.**

Definition 4.1 (Rotational)**.** The action of G on $\mathcal{X}$ is *rotational* if the following holds. For each hyperplane B, there is a finite-index subgroup $K_B \leq \mathrm{Stab}_G(B)$ so that for all hyperplanes A with $\mathsf{d}(A, B) > 0$, and all $k \in K_B$, the carriers $\mathcal{N}(A)$ and $\mathcal{N}(kA)$ are either equal or disjoint.

Remark 4.2. For example, if $G\backslash\mathcal{X}$ is (virtually) special, then G acts rotationally on $\mathcal{X}$, but one can easily make examples of non-cospecial rotational actions on CAT(0) cube complexes.

To illustrate how to apply rotation, we first prove a lemma about $\mathfrak{F}_2$.

Lemma 4.3 (Uniform cocompactness in $\mathfrak{F}_2$ under rotational actions)**.** *Let G act properly, cocompactly, and rotationally on $\mathcal{X}$. Then for any ball Q in $\mathcal{X}$, there exists $s \geq 0$, depending only on $\mathcal{X}$, and the radius of Q, so that for all $A, B \in \mathfrak{F}_1$, at most s distinct translates of $\mathfrak{g}_B(A)$ can intersect Q.*

Proof. Note that if B, gB are in the same G–orbit, and $K_B \leq \mathrm{Stab}_G(B)$ witnesses the rotation of the action at B, then K_B^g does the same for gB, so we can assume that the index of $K_B \in \mathrm{Stab}_G(B)$ is uniformly bounded by some constant ι as B varies over the (finitely many orbits of) combinatorial hyperplanes.

Next, note that it suffices to prove the claim for Q of radius 0, since the general statement will then follow from uniform properness of $\mathcal{X}$.

Finally, it suffices to fix combinatorial hyperplanes B and A and bound the number of $\mathrm{Stab}_G(B)$–translates of A whose projections on B contain some fixed 0–cube $x \in B$, since only boundedly many translates of B can contain x.

We can assume that A is disjoint from B, for otherwise $\mathfrak{g}_B(A) = A \cap B$, and the number of translates of A containing x is bounded in terms of G and $\mathcal{X}$ only.

Now suppose $\mathsf{d}(A, B) > 0$. First, let $\{g_1, \ldots, g_k\} \subset K_B$ be such that the translates $g_i\mathfrak{g}_B(A)$ are all distinct and $x \in \bigcap_{i=1}^k \mathfrak{g}_B(g_iA) = \bigcap_{i=1}^k g_i\mathfrak{g}_B(A)$. For simplicity, we can and shall assume that $g_1 = 1$.

We can also assume, by multiplying our eventual bound by 2, that the g_iA all lie on the same side of B, i.e. the hyperplane B' whose carrier is bounded by B and a parallel copy of B does not separate any pair of the g_iA. By rotation, A, g_2A are disjoint, and hence separated by some hyperplane V.

Since V cannot separate $\mathfrak{g}_B(A), \mathfrak{g}_B(g_2A)$, we have that V separates either A or g_2A from B. (The other possibility is that $V = B$ but we have ruled this out above.) Up to relabelling, we can assume the former. Then, for $i \geq 2$, we have that g_iV separates g_iA from $g_iB = B$. Moreover, by choosing V as close as possible to B among hyperplanes that separate A from B and g_2A, we see that the hyperplanes $\{g_iV\}_{i=1}^k$ have pairwise-intersecting carriers, and at least two of them are distinct. This contradicts the rotation hypothesis unless $k = 1$.

More generally, the above argument shows that if $\{g_1, \ldots, g_k\} \subset \mathrm{Stab}_G(B)$ are such that the translates $g_i\mathfrak{g}_B(A)$ are all distinct and $x \in \bigcap_{i=1}^k \mathfrak{g}_B(g_iA) = \bigcap_{i=1}^k g_i\mathfrak{g}_B(A)$, then the number of g_i belonging to any given left coset of K_B in $\mathrm{Stab}_G(B)$ is uniformly bounded. Since $[\mathrm{Stab}_G(B) : K_B] \leq \iota$, the lemma follows. $\square$

More generally:

Lemma 4.4. *Let G act properly, cocompactly, and rotationally on $\mathcal{X}$. Then for all $\rho \geq 0$, there exists a constant s' so that the following holds. Let $F \in \mathfrak{F}$. Then at most s' distinct G–translates of F can intersect any ρ–ball in $\mathcal{X}$.*

Proof. As in the proof of Lemma 4.3, it suffices to bound the number of G–translates of F containing a given 0–cube x. As in the same proof, it suffices to bound the number of $\mathrm{Stab}_G(B)$–translates of F containing x, where B is a combinatorial hyperplane for which $x \in F \subset B$.

By the first paragraph of the proof of Theorem 3.3, there exists n and combinatorial hyperplanes $A_1, \ldots, A_n$ such that $F = \bigcap_{i=1}^n \mathfrak{g}_B(A_i)$. If A_n is parallel to B, then $F = \bigcap_{i=1}^{n-1} \mathfrak{g}_B(A_i)$, so by choosing a smallest such collection, we have that no A_i is parallel to B.

Let $g_1, \ldots, g_k \in K_B$ have the property that the g_iF are all distinct and contain x. Note that $g_iF = \bigcap_{j=1}^n \mathfrak{g}_B(g_iA_j)$.

Now, if A_j is disjoint from B, then rotation implies that for all i, either $g_iA_j = A_j$ or $g_iA_j \cap A_j = \emptyset$. If $g_iA_j \neq A_j$, there is a hyperplane V separating them, and V cannot cross or coincide with B (as in the proof of Lemma 4.3). Hence V separates A_j, say, from B. So g_iV separates g_iA_j from B, and $g_iV \neq g_jV$. By choosing V as close as possible to B, we have (again as in Lemma 4.3) that V and g_iV cross or osculate, which contradicts rotation. Hence $g_iA_j = A_j$ for all such i, j.

Let J be the set of $j \leq n$ so that A_j is disjoint from B, so that $\bigcap_{j\in J} \mathfrak{g}_B(A_j)$ is fixed by each g_i. Let $J' = \{1, 2, \ldots, n\} - J$. Note that for all $j \in J'$, the combinatorial hyperplane A_j is one of at most χ combinatorial hyperplanes that contain x and χ is the maximal degree of a vertex in $\mathcal{X}$. Moreover, $\mathfrak{g}_B(A_j) = A_j \cap B$.

Since each g_i fixes $\bigcap_{j\in J} \mathfrak{g}_B(A_j)$, k must be bounded in terms of the number of translates of $\bigcap_{j\in J'} \mathfrak{g}_B(A_j)$ containing x; since we can assume that the A_j contain x, this follows.

As in the proof of Lemma 4.3, if $g_1, \ldots, g_k \in \mathrm{Stab}_G(B)$ have the property that the g_iF are all distinct and contain x, then the number of g_i belonging to any particular $hK_B, h \in \mathrm{Stab}_G(B)$ is uniformly bounded, and the number of such cosets is bounded by ι, so the number of such $\mathrm{Stab}_G(B)$–translates of F containing x is uniformly bounded. □

We now prove Theorem A in the special case where G acts on $\mathcal{X}$ rotationally.

Corollary 4.5. *Let G act properly, cocompactly, and rotationally on the proper CAT(0) cube complex $\mathcal{X}$. Then $\mathfrak{F}$ is a factor system.*

Proof. By Lemma 4.4, there exists $s' < \infty$ so that for all $F \in \mathfrak{F}$, at most s' distinct G–translates of F can contain a given point. By uniform properness of F and the proof of Lemma 2.3, there exists $R < \infty$ so that each $F \in \mathfrak{F}$ has the following property: fix a basepoint $x \in F$. Then for any $y \in F$, there exists $g \in \mathrm{Stab}_G(F)$ so that $\mathsf{d}(gx, y) \leq R$. Hence there exists k so that for all F, the complex F contains at most k $\mathrm{Stab}_G(F)$–orbits of cubes.

Conclusion in the virtually torsion-free case: If G is virtually torsion-free, then (passing to a finite-index torsion-free subgroup) $G\backslash\mathcal{X}$ is a compact nonpositively-curved cube complex admitting a local isometry $\mathrm{Stab}_G(F)\backslash F \to G\backslash\mathcal{X}$, where $\mathrm{Stab}_G(F)\backslash F$ is a nonpositively-curved cube complex with at most k cubes. Since there are only finitely many such complexes, and finitely many such local isometries, the quotient $G\backslash\mathfrak{F}$ is finite. Since each $x \in \mathfrak{S}$ is contained in boundedly many translates of each $F \in \mathfrak{F}$, and there are only finitely many orbits in $\mathfrak{F}$, it follows that x is contained in boundedly many elements of $\mathfrak{F}$, as required.

General case: Even if G is not virtually torsion-free, we can argue essentially as above, except we have to work with nonpositively-curved orbi-complexes instead of nonpositively-curved cube complexes.

First, let $\mathcal{Y}$ be the first barycentric subdivision of $\mathcal{X}$, so that G acts properly and cocompactly on $\mathcal{Y}$ and, for each cell y of $\mathcal{Y}$, we have that $\mathrm{Stab}_G(y)$ fixes y pointwise (see [BH99, Chapter III.$\mathcal{C}$.2].) Letting F' be the first barycentric subdivision of F, we see that F' is a subcomplex of $\mathcal{Y}$ with the same properties with respect to the $\mathrm{Stab}_G(F)$–action. Moreover, F' has at most k' $\mathrm{Stab}_G(F)$–orbits of cells, where k' depends on $\dim \mathcal{X}$ and k, but not on F.

The quotient $G\backslash\mathcal{Y}$ is a complex of groups whose cells are labelled by finitely many different finite subgroups, and the same is true for $\mathrm{Stab}_G(F)\backslash F$. Moreover, we have a morphism of complexes of groups $\mathrm{Stab}_G(F)\backslash F \to G\backslash\mathcal{Y}$ which is injective on local groups. Since G acts on $\mathcal{X}$ properly, the local groups in $G\backslash\mathcal{Y}$ are finite. Hence there are boundedly many cells in $\mathrm{Stab}_G(F)\backslash F$, each of which has boundedly many possible local groups (namely, the various subgroups of the local groups for the cells of $G\backslash\mathcal{Y}$). Hence there are finitely many choices of $\mathrm{Stab}_G(F)\backslash F$, and thus finitely many G–orbits in $\mathfrak{F}$, and we can conclude as above. □

4.2. **Weak finite height and essential index conditions.**

Definition 4.6 (Weak finite height condition)**.** Let G be a group and $H \leq G$ a subgroup. The subgroup H satisfies the *weak finite height condition* if the following holds. Let $\{g_i\}_{i\in I}$ be an infinite subset of G so that $H \cap \bigcap_{i\in J} H^{g_i}$ is infinite whenever $J \subset I$ is a finite subset. Then there exists i, j so that $H \cap H^{g_i} = H \cap H^{g_j}$.

Definition 4.7 (Noetherian Intersection of Conjugates Condition (NICC))**.** Let G be a group and $H \leq G$ a subgroup. The subgroup H satisfies the *Noetherian intersection of conjugates condition* (NICC) if the following holds. Let $\{g_i\}_{i=1}^{\infty}$ be an infinite subset of distinct elements of G so that $H_n = H \cap \bigcap_{i=1}^{n} H^{g_i}$ is infinite for all n, then there exists $\ell > 0$ so that for all $j, k \geq \ell$, H_j and H_k are commensurable.

Definition 4.8 (Conditions for hyperplanes)**.** Let G act on the CAT(0) cube complex $\mathcal{X}$. Then the action satisfies the *weak finite height condition for hyperplanes* or respectively *NICC for hyperplanes* if, for each hyperplane B of $\mathcal{X}$, the subgroup $\mathrm{Stab}_G(B) \leq G$ satisfies the weak finite height condition, or NICC, respectively.

Remark 4.9. Recall that $H \leq G$ has *finite height* if there exists n so that any collection of at least $n+1$ distinct left cosets of H has the property that the intersection of the corresponding conjugates of H is finite. Observe that if H has finite height, then it satisfies both the weak finite height condition and NICC, but that the converse does not hold.

Definition 4.10 (Essential index condition)**.** The action of G on $\mathcal{X}$ satisfies the *essential index condition* if there exists $\zeta \in \mathbb{N}$ so that for all $F \in \mathfrak{F}$ we have $[\mathrm{Stab}_G(\widehat{F}) : \mathrm{Stab}_G(F)] \leq \zeta$, where $\widehat{F}$ is the $\mathrm{Stab}_G(F)$–essential core of F.

4.3. **Some examples where the auxiliary conditions are satisfied.** We now briefly consider some examples illustrating the various hypotheses. Our goal here is just to illustrate the conditions in simple cases.

4.3.1. *Special groups.* Stabilizers of hyperplanes in a right-angled Artin groups are simply special subgroups generated by the links of vertices. Let Γ be a graph generating a right angled Artin group A_Γ and let Λ be any inducted subgraph. Then A_Λ is a special subgroup of A_Γ, and $A_\Lambda^{g_i}$ has non-trivial intersection with $A_\Lambda^{g_j}$ if and only if $g_i g_j^{-1}$ commutes with some subgraph Λ' of Λ. Further, their intersection is conjugate to the special subgroup $A_{\Lambda'}$. The weak finite height condition and NICC follow. The essential index condition also holds since each A_Λ acts essentially on the corresponding element of $\mathfrak{F}$, which is just a copy of the universal cover of the Salvetti complex of A_Λ. In fact, these considerations show that hyperplane stabilisers in RAAGs have finite height. It is easily verified that these properties are inherited by subgroups arising from compact local isometries to the Salvetti complex, reconfirming that (virtually) compact special cube complexes have factor systems in their universal covers.

4.3.2. *Non-virtually special lattices in products of trees.* The uniform lattices in products of trees from [Wis96, BM97, Rat07, JW09] do not satisfy the weak finite height condition, but they do satisfy NICC and the essential index condition.

Indeed, let G be a cocompact lattice in $\mathrm{Aut}(T_1 \times T_2)$, where T_1, T_2 are locally finite trees. If A, B are disjoint hyperplanes, then $\mathfrak{g}_B(A)$ is a parallel copy of some T_i, i.e. $\mathfrak{g}_B(A)$ is again a hyperplane; otherwise, if A, B cross, then $\mathfrak{g}_B(A)$ is a single point. The essential index condition follows immediately, as does the NICC. However, G can be chosen so that there are pairs of parallel hyperplanes A, B so that $\mathrm{Stab}_G(A) \cap \mathrm{Stab}_G(B)$ has arbitrarily large (finite) index in $\mathrm{Stab}_G(B)$, so the weak finite height condition fails.

4.3.3. *Graphs of groups.* Let Γ be a finite graph of groups, where each vertex group G_v acts properly and cocompactly on a CAT(0) cube complex $\mathcal{X}_v$ with a factor system $\mathfrak{F}_v$, and each edge group G_e acts properly and cocompactly on a CAT(0) cube complex $\mathcal{X}_e$, with a factor system $\mathfrak{F}_e$, so that the following conditions are satisfied, where v, w are the vertices of e:

- there are G–equivariant convex embeddings $\mathcal{X}_e \to \mathcal{X}_v, \mathcal{X}_w$
- these embedding induce injective maps $\mathfrak{F}_e \to \mathfrak{F}_v, \mathfrak{F}_w$.

If the action of G on the Bass-Serre tree is acylindrical, then one can argue essentially as in the proof of [BHS15, Theorem 8.6] to prove that the resulting tree of CAT(0) cube complexes has a factor system. Moreover, ongoing work on improving [BHS15, Theorem 8.6] indicates that one can probably obtain the same conclusion in this setting without this acylindricity hypothesis.

Of course, one can imagine gluing along convex cocompact subcomplexes that don't belong to the factor systems of the incident vertex groups. Also, we believe that the property of being cocompactly cubulated with a factor system is preserved by taking graph products, and that one can prove this by induction on the size of the graph by splitting along link subgroups. This is the subject of recent work in the hierarchically hyperbolic setting; see [BR18].

4.3.4. *Cubical small-cancellation quotients.* There are various ways of building more exotic examples of non-virtually special cocompactly cubulated groups using groups. In [JW17], Jankiewicz-Wise construct a group G that is cocompactly cubulated but does not virtually split. They start with a group G' of the type discussed in Remark 4.2 and consider a small-cancellation quotient of the free product of several copies of G'. This turns out to satisfy strong cubical small-cancellation conditions sufficient to produce a proper, cocompact action of G on a CAT(0) cube complex. However, it appears that the small-cancellation conditions needed to achieve this are also strong enough to ensure that the NICC and essential index properties pass from G' to G. The key points are that G is hyperbolic relative to G', and each wall in G intersects each coset of G' in at most a single wall (Lemma 4.2 and Corollary 4.5 of [JW17]).

4.3.5. *A non-rotational example.* Let Y be a compact nonpositively-curved cube complex whose fundamental group G has the following two properties:

- G has no proper finite-index subgroup;
- there exists $g \in G$ such that g is represented by a based combinatorial path $L \to Y$ that is a local isometry.

The examples mentioned in Subsection 4.3.2 show that we can take Y to have universal cover the product of two trees.

Let M' be a copy of $[0, 2] \times [0, |L|]$, endowed with the product cubical structure in which $[0, 2]$ and $[0, |L|]$ are regarded as cube complexes with 0–cubes at integer points. Let M be the quotient of M' obtained by identifying $[0, 2] \times \{0\}$ and $[0, 2] \times \{|L|\}$ by an orientation-reversing combinatorial isometry, so that M is a Möbius strip tiled by squares. Form a cube complex X from $Y \times [0, 1] \sqcup M$ by identifying L with $\{1\} \times [0, |L|]$ (here we think of L as a the path $L \to Y \times \{0\} \hookrightarrow Y \times [0, 1]$). Then X is nonpositively curved, because $L \to Y \times [0, 1]$ is a local isometry and $L \to M$ is a local isometry.

Let $\widetilde{X}$ be the universal cover of X, on which $\pi_X = G$ acts freely and cocompactly. Now, the preimage of $Y \times \{\frac{1}{2}\}$ under the covering map $\widetilde{X} \to X$ has a single component, which is a hyperplane that we call B. By construction, the stabiliser of B is G, which has no proper finite-index subgroups. Now, let A be a hyperplane of $\widetilde{X}$ projecting to an immersed hyperplane of M dual to the image of $[0, 1] \times \{0\}$. Then $\mathcal{N}(A) \cap \mathcal{N}(gA)$ intersect (along an elevation of L), while A and B do not cross. Hence the action of G on $\widetilde{X}$ is non-rotational; because $G = \mathrm{Stab}_G(B)$ has no proper finite-index subgroup, it was sufficient to find a single hyperplane A and a single $g \in G$ violating the condition in Definition 4.1.

For a less self-contained example, it appears that G can be chosen so that the action of G on Y is not rotational. In fact [BM97] and [Wis96] contain examples where G is acting geometrically on $T_1 \times T_2$, with T_1, T_2 trees, but the induced actions on the two factors are not discrete. In particular, it seems that for each edge e of T_1, and any $r \geq 0$, there is a vertex of T_1 at distance r from e so that the stabiliser of e nontrivially permutes the edges incident to that vertex, and this seems to be an obstruction to the action being rotational. (This is in stark contrast to the case where G is virtually a product of free groups, in which case the action is rotational.)

5. $\mathfrak{F}$ is closed under orthogonal complementation, given a group action

We now assume that $\mathcal{X}$ is a locally finite CAT(0) cube complex on which the group G acts properly and cocompactly. Let $\mathfrak{F}$ be the hyperclosure in $\mathcal{X}$ and let B be a constant so that each 0–cube x of $\mathcal{X}$ lies in $\leq B$ combinatorial hyperplanes.

For convex subcomplexes D, F of $\mathcal{X}$, we write $F = D^{\perp}$ to mean $F = \phi_D(\{f\} \times D^{\perp})$ for some $f \in D$, though we may abuse notation, suppress the ϕ_D, and write e.g. $\{f\} \times D^{\perp}$ to mean $\phi_D(\{f\} \times D^{\perp})$ when we care about the specific point f.

Proposition 5.1 ($\mathfrak{F}$ is closed under orthogonal complements)**.** *Let G act on $\mathcal{X}$ properly and cocompactly. Suppose that **one of the following holds**:*

- *the G–action on $\mathcal{X}$ satisfies the weak finite height property for hyperplanes;*
- *the G–action on $\mathcal{X}$ satisfies the essential index condition and the NICC for hyperplanes;*
- *$\mathfrak{F}$ is a factor system.*

Let A be a convex subcomplex of $\mathcal{X}$. Then $A^{\perp} \in \mathfrak{F}$. Hence $\mathrm{Stab}_G(A^{\perp})$ acts on $A^{\perp}$ cocompactly. In particular, for all $F \in \mathfrak{F}$, we have that $F^{\perp} \in \mathfrak{F}$.

We first need a lemma.

Lemma 5.2. *Let $A \subset \mathcal{X}$ be a convex subcomplex with $\mathrm{diam}(A) > 0$ and let $x \in A^{(0)}$. Let $H_1, \ldots, H_k$ be all of the hyperplanes intersecting A whose carriers contain x, so that for each i, there is a combinatorial hyperplanes H_i^+ associated to H_i with $x \in H_i^+$. Let $Y = \cap_{i=1}^k H_i^+$. Let $\mathcal{S}$ be the set of all combinatorial hyperplanes associated to hyperplanes crossing A. Then*

$$A^{\perp} = \bigcap_{H' \in \mathcal{S}} \mathfrak{g}_Y(H'),$$

where $A^{\perp}$ denotes the orthogonal complement of A at x. If A is unique in its parallelism class, then $A^{\perp} = \{x\}$. Finally, if $\mathrm{diam}(A) = 0$, then $A^{\perp} = \mathcal{X}$.

Proof. If A is a single 0–cube, then $A^{\perp} = \mathcal{X}$ by definition. Hence suppose that $\mathrm{diam}(A) > 0$.

Let $H' \in \mathcal{S}$. Since $\mathfrak{g}_Y(H' \cap A) \subseteq Y \cap A = \{x\}$, we see that $x \in \mathfrak{g}_Y(H')$. Suppose that $y \in A^{\perp}$. Then every hyperplane V separating y from x crosses each of the hyperplanes H' crossing A, and thus crosses Y, whence $y \in \mathfrak{g}_Y(H')$ for each $H' \in \mathcal{S}$. Thus $A^{\perp} \subseteq \bigcap_{H' \in \mathcal{S}} \mathfrak{g}_Y(H')$. On the other hand, suppose that $y \in \bigcap_{H' \in \mathcal{S}} \mathfrak{g}_Y(H')$. Then every hyperplane H' separating x from y crosses every hyperplane crossing A, so $y \in A^{\perp}$. This completes the proof that $A^{\perp} = \bigcap_{H' \in \mathcal{S}} \mathfrak{g}_Y(H')$.

Finally, A is unique in its parallelism class if and only if $A^{\perp} = \{x\}$, by definition of $A^{\perp}$. □

We can now prove the proposition:

Proof of Proposition 5.1. The proof has several stages.

Setup using Lemma 5.2: If A is a single point, then $A^{\perp} = \mathcal{X}$, which is in $\mathfrak{F}$ by definition. Hence suppose $\mathrm{diam}(A) > 0$, and let $H_1, \ldots, H_k$, $x \in A$, $Y \subset \mathcal{X}$, and $\mathcal{S}$ be as in Lemma 5.2, so

$$A^{\perp} = \bigcap_{H' \in \mathcal{S}} \mathfrak{g}_Y(H').$$

Thus, to prove the proposition, it is sufficient to produce a finite collection $\mathfrak{H}$ of hyperplanes H' crossing A so that

$$\bigcap_{H' \in \mathcal{S}} \mathfrak{g}_Y(H') = \bigcap_{H' \in \mathfrak{H}} \mathfrak{g}_Y(H').$$

Indeed, if there is such a collection, then we have shown $A^{\perp}$ to be the intersection of finitely many elements of $\mathfrak{F}_k$, whence $A^{\perp} \in \mathfrak{F}_{k+|\mathfrak{H}|}$, as required. Hence suppose for a contradiction that for any finite collection $\mathfrak{H} \subset \mathcal{S}$, we have

$$\bigcap_{H' \in \mathcal{S}} \mathfrak{g}_Y(H') \subsetneq \bigcap_{H' \in \mathfrak{H}} \mathfrak{g}_Y(H').$$

Bad hyperplanes crossing Y: For each m, let $\mathcal{H}_m$ be the (finite) set of hyperplanes H' intersecting $\mathcal{N}_m^A(x) = A \cap \mathcal{N}_m(x)$ (and hence satisfying $x \in \mathfrak{g}_Y(H')$).

Consider the collection $\mathcal{B}_m$ of all hyperplanes W such that W crosses each element of $\mathcal{H}_m$ and W crosses Y, but W fails to cross $A^\perp$. (This means that there exists $j > m$ and some $U \in \mathcal{H}_j$ so that $W \cap U = \emptyset$.)

Suppose that there exists m so that $\mathcal{B}_n = \emptyset$ for $n > m$. Then we can take $\mathcal{H}_m$ to be our desired set $\mathfrak{H}$, and we are done. Hence suppose that $\mathcal{B}_m$ is nonempty for arbitrarily large m. Note that if $U \in \mathcal{B}_m$, then $\mathcal{N}_m^A(x)$ is parallel into U, but there exists $j > m$ so that $\mathcal{N}_j^A(x)$ is not parallel into U. (Here $\mathcal{N}_j^A(x)$ denotes the j–ball in A about x.)

Elements of $\mathcal{B}_m$ osculating $A^\perp$: Suppose that $U \in \mathcal{B}_m$, so that $\mathcal{N}_j^A(x)$ is not parallel into U for some $j > m$. Suppose that U' is a hyperplane separating U from $A^\perp$.

Then U' separates U from x so, since U intersects Y and $x \in Y$, we have that U' intersects Y. Since $\mathfrak{g}_Y(A) = \{x\}$ is not crossed by any hyperplanes, U' cannot cross A. Hence U' separates U from $\mathcal{N}_m^A(x)$, so $\mathcal{N}_m^A(x)$ is parallel into U' (since it is parallel into U). On the other hand, since U' separates U from $A^\perp$, U' cannot cross $A^\perp$, and thus fails to cross some hyperplane crossing A. Hence $U' \in \mathcal{B}_m$. Thus, for each m, there exists $U_m \in \mathcal{B}_m$ whose carrier intersects $A^\perp$. Indeed, we have shown that any element of $\mathcal{B}_m$ as close as possible to $A^\perp$ has this property.

Hence we have a sequence of radii r_n and hyperplanes U_n so that:

- U_n crosses Y;
- $\mathcal{N}(U_n) \cap A^\perp \neq \emptyset$;
- $\mathcal{N}_{r_n}^A(x)$ is parallel into U_n for all n;
- $\mathcal{N}_{r_{n+1}}^A(x)$ is not parallel into U_n, for all n.

The above provides a sequence $\{V_n\}$ of hyperplanes so that for each n:

- V_n crosses A;
- V_n crosses U_m for $m \geq n$;
- V_n does not cross U_m for $m < n$.

Indeed, for each n, choose V_n to be a hyperplane crossing $\mathcal{N}_{r_{n+1}}^A(x)$ but not crossing U_n. For each n, let $\bar{V}_n$ be one of the two combinatorial hyperplanes (parallel to V_n) bounding $\mathcal{N}(V_n)$.

Claim 1. For each n, the subcomplex $A^\perp$ is parallel into $\bar{V}_n$.

Proof of Claim 1. Let H be a hyperplane crossing $A^\perp$. Then, by definition of $A^\perp$, H crosses each hyperplane crossing A. But V_n crosses A, so H must also cross V_n. Thus $A^\perp$ is parallel into V_n. □

Next, since G acts on $\mathcal{X}$ cocompactly, it acts with finitely many orbits of hyperplanes, so, by passing to a subsequence (but keeping our notation), we can assume that there exists a hyperplane V crossing A and elements $g_n \in G, n \geq 1$ so that $V_n = g_n V$ for $n \geq 1$. For simplicity, we can assume $g_1 = 1$.

Next, we can assume, after moving the basepoint $x \in A$ a single time, that $x \in \bar{V}$, i.e. $\bar{V}$ is among the k combinatorial hyperplanes whose intersection is Y. This assumption is justified by the fact that $\mathfrak{F}$ is closed under parallelism, so it suffices to prove that any given parallel copy of $A^\perp$ lies in $\mathfrak{F}$. Thus we can and shall assume $Y \subset \bar{V}$.

For each m, consider the inductively defined subcomplexes $Z_1 = \bar{V}$ and for each $m \geq 2$, $Z_m = \mathfrak{g}_{\bar{V}}(\mathfrak{g}_{g_1\bar{V}}(\cdots(\mathfrak{g}_{g_{m-1}\bar{V}}(g_m\bar{V}))\cdots))$, which is an element of $\mathfrak{F}$.

Claim 2. For all $m \geq 1$, we have $A^\perp \subseteq Z_m$.

Proof of Claim 2. Indeed, since $A^\perp \subset Y$ by definition, and $Y \subset \bar{V}$, we have $A^\perp \subset \bar{V}$. On the other hand, for each n, Claim 1 implies that $A^\perp$ is parallel into $g_n\bar{V}$ for all n, so by induction, $A^\perp \subset Z_m$ for all $m \geq 1$, as required. □

Claim 3. For all $m \geq 1$, we have $Z_m \supsetneq Z_{m+1}$.

Proof of Claim 3. For each m, the hyperplane U_m crosses Y, by construction. Since $Y \subseteq \bar{V}$, this implies that U_m crosses $\bar{V}$. On the other hand, U_m does not cross $\bar{V}_{m+1}$. This implies that $Z_m \neq Z_{m+1}$. On the other hand, $Z_{m+1} \subset Z_m$ just by definition. □

Let $K_m = \mathrm{Stab}_G(V) \cap \bigcap_{n=1}^m \mathrm{Stab}_G(V)^{g_n}$; by Lemma 1.9 K_m has finite index in $\mathrm{Stab}_G(Z_m)$. Thus, since $Z_m \in \mathfrak{F}$, we see that K_m acts on Z_m cocompactly. Claim 3 implies that no Z_m is compact, for otherwise we would be forced to have $Z_m = Z_{m+1}$ for some m. Since K_m acts on Z_m cocompactly, it follows that K_m is infinite for all m.

Thus far, we have not used any of the auxiliary hypotheses. We now explain how to derive a contradiction under the weak finite height hypothesis.

Claim 4. Suppose that the G–action on $\mathcal{X}$ satisfies weak finite height for hyperplanes. Then, after passing to a subsequence, we have $K_m = K_2$ for all $m \geq 2$.

Proof of Claim 4. Let $I \subset \mathbb{N}$ be a finite set and let $m = \max I$. Then $\bigcap_{n\in I} \mathrm{Stab}_G(V)^{g_i}$ contains K_m, and is thus infinite, since K_m was shown above to be infinite. Hence, since $\mathrm{Stab}_G(V)$ satisfies the weak finite height condition, there exist distinct m, m' so that $\mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g_m} = \mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g'_m}$.

Declare $m \sim m'$ if $\mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g_m} = \mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g'_m}$, so that $\sim$ is an equivalence relation on $\mathbb{N}$. If any $\sim$–class $[m]$ is infinite, then we can pass to the subsequence $[m]$ and assume that $\mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g_n} = \mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g'_n}$ for all n. Otherwise, if every $\sim$–class is finite, then there are infinitely many $\sim$–classes, and we can pass to a subsequence containing one element from each $\sim$–class. This amounts to assuming that $\mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g_m} \neq \mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g'_m}$ for all distinct m, m', but this contradicts weak finite height, as shown above.

Hence, passing to a subsequence, we can assume that $\mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g_m} = \mathrm{Stab}_G(V) \cap \mathrm{Stab}_G(V)^{g'_m}$ for all $m, m' \geq 2$, and hence $K_m = K_2$ for all m. □

From Claim 4 and the fact that K_m stabilises Z_m for each m, we have that K_2 stabilises each Z_m. Moreover, K_2 acts on Z_m cocompactly.

The inclusion $Z_{m+1} \hookrightarrow Z_m$ descends to an inclusion $Z_{m+1}/K_2 \hookrightarrow Z_m/K_2$, and since the latter spaces are compact, we must have some M such that $Z_m/K_2 = Z_M/K_2$ for $M \geq m$. Hence $Z_m = Z_M$ for all $m \geq M$. This contradicts Claim 3. Hence the the claimed sequence of U_m cannot exist, whence $A^\perp \in \mathfrak{F}$.

Having proved the proposition under the weak finite height assumption, we now turn to the other hypotheses. Let $\{Z_m\}$ and $\{K_m\}$ be as above.

Claim 5. Suppose that the G–action on $\mathcal{X}$ satisfies the NICC and the essential index condition. Then there exists ℓ so that $\mathrm{Stab}_G(Z_m) = \mathrm{Stab}_G(Z_\ell)$ for all $m \geq \ell$.

Proof of Claim 5. Let $I \subset \mathbb{N}$ be a finite set and let $m = \max I$. Then $\bigcap_{n\in I} \mathrm{Stab}_G(V)^{g_i}$ contains K_m, and is thus infinite, since K_m was shown above to be infinite. Hence, since $\mathrm{Stab}_G(V)$ satisfies the NICC, there exists ℓ so that K_m is commensurable with K_ℓ for all $m \geq \ell$.

Hence, by Lemma 2.8, we have that $\widehat{Z}_m$ and $\widehat{Z}_\ell$ are parallel for all $m \geq \ell$. Moreover, since $K_m \leq K_\ell$, Proposition 1.13 implies that we can choose essential cores within their parallelism classes so that $\widehat{Z}_m = \widehat{Z}_\ell$ for all $m \geq \ell$.

Let $L = \mathrm{Stab}_G(\widehat{Z}_\ell) = \mathrm{Stab}_G(\widehat{Z}_m)$ for all $m \geq \ell$. The essential index condition implies that $\mathrm{Stab}_G(Z_m)$ has uniformly bounded index in L as $m \to \infty$, so by passing to a further infinite subsequence, we can assume that $\mathrm{Stab}_G(Z_m) = \mathrm{Stab}_G(Z_\ell)$ for all $m \geq \ell$ (since L has finitely many subgroups of each finite index). □

Claim 5 implies that (up to passing to a subsequence), $\mathrm{Stab}_G(Z_\ell) = \mathrm{Stab}_G(Z_m)$ preserves Z_m (and acts cocompactly on Z_m) for all $m \geq \ell$.

Recall that $Z_m \subset Z_\ell$ for $\ell \leq m$. The inclusion $Z_m \hookrightarrow Z_\ell$ descends to an inclusion $Z_m/\mathrm{Stab}_G(Z_\ell) \hookrightarrow Z_\ell/\mathrm{Stab}_G(Z_\ell)$, and since the latter spaces are compact, there exists M such that $Z_m = Z_M$ for all $m \geq M$. This again contradicts Claim 3. As before, we therefore cannot have the sequences $(U_m), (V_m)$ with the given properties, and hence $A^\perp \in \mathfrak{F}$.

Applying the factor system assumption: If $\mathfrak{F}$ is a factor system, then since $Z_m \in \mathfrak{F}$ for all m, and $Z_m \supsetneq Z_{m+1}$ for all m, we have an immediate contradiction, so $A^\perp \in \mathfrak{F}$.

Conclusion: We have shown that under any of the additional hypotheses, $A^\perp \in \mathfrak{F}$ when $A \subset \mathcal{X}$ is a convex subcomplex. This holds in particular if $A \in \mathfrak{F}$. □

The preceding proposition combines with earlier facts to yield:

Corollary 5.3 (Ascending and descending chains)**.** *Let G act properly and cocompactly on $\mathcal{X}$, satisfying any of the hypotheses of Proposition 5.1. Suppose that for all $N \geq \infty$, there exists a 0–cube $x \in \mathcal{X}$ so that x lies in at least N elements of $\mathfrak{F}$. Then there exist sequences $(F_i)_{i\geq1}, (F_i')_{i\geq1}$ of subcomplexes in $\mathfrak{F}$ so that all of the following hold for all $i \geq 1$:*

- $F_i \subsetneq F_{i+1}$;
- $F_i' \supsetneq F_{i+1}'$;
- $F_i' = F_i^\perp$.

Moreover, there exists a 0–cube x that lies in each F_i and each F_i'.

Proof. Lemma 2.9, cocompactness, and G–invariance of $\mathfrak{F}$ provide a sequence (F_i) in $\mathfrak{F}$ and a point x so that $x \in F_i$ for all i and either $F_i \subsetneq F_{i+1}$ for all i, or $F_i \supsetneq F_{i+1}$ for all i. For each i, let $F_i' = \phi_{F_i}(\{x\} \times F_i^\perp)$. Proposition 5.1 implies that each $F_i' \in \mathfrak{F}$, and Lemma 3.1 implies that (F_i') is an ascending or descending chain according to whether (F_i) was descending or ascending. Assume first that $F_i \subsetneq F_{i+1}$ for all i. Now, if $F_i' = F_i^\perp = F_{i+1}^\perp = F_{i+1}'$, then by Corollary 3.4, we have $F_i = F_{i+1}$, a contradiction. Hence (F_i') is properly descending, i.e. $F_i' \supsetneq F_{i+1}'$ for all i. The case where (F_i) is descending is identical. This completes the proof. □

6. Proof of Theorem A

We first establish the setup. Recall that $\mathcal{X}$ is a proper CAT(0) cube complex with a proper, cocompact action by a group G. We denote the hyperclosure by $\mathfrak{F}$; our goal is to prove that there exists $N < \infty$ so that each 0–cube of $\mathcal{X}$ is contained in at most N elements of $\mathfrak{F}$, under any of the three additional hypotheses of Theorem A.

If there is no such N, then Corollary 5.3 implies that there exists a 0–cube $x \in \mathcal{X}$ and a sequence $(F_i)_{i\geq1}$ in $\mathfrak{F}$ so that $x \in F_i \subsetneq F_{i+1}$ for each $i \geq 1$. For the sake of brevity, given any subcomplex $E \ni x$, let $E^\perp$ denote the orthogonal complement of E *based at* x. Corollary 5.3 also says that $F_i^\perp \in \mathfrak{F}$ for all i and $F_i^\perp \supsetneq F_{i+1}^\perp$ for all i. Proposition 2.7 shows that $\mathrm{Stab}_G(F_i^\perp)$ acts on $F_i^\perp$ cocompactly for all i.

Let $U = \bigcup_i F_i$ and let $I = \bigcap_i F_i^\perp$, and note that $U^\perp = I$ and $I^\perp = U$. In particular, $U^{\perp\perp} = I^\perp = U$ and $I^{\perp\perp} = U^\perp = I$. From here, we can now prove our main theorem:

Proof of Theorem A. We have already proved the theorem under the rotation hypothesis, in Corollary 4.5. Hence suppose that either weak finite height holds or the NICC and essential index conditions both hold, so that Proposition 5.1 implies that $U, I \in \mathfrak{F}$.

By Corollary 3.4, we have compact convex subcomplexes D, E with $U = D^\perp$ and $I = E^\perp$. Moreover, we can take $D \subset I$ and $E \subset U$. Now, Corollary 3.4, Theorem 3.3, and Proposition 5.1

provide, for each $i \geq 1$, a compact, convex subcomplex C_i, containing x and contained in F_i, so that $C_i^{\perp} = F_i^{\perp}$.

Let $C'_1 = C_1$. For $i \geq 2$, let $C'_i = Hull(\cup_{j<i} C_j)$. Note that C'_i is a compact, convex subcomplex contained in F_i, so $C_i \subseteq C'_i \subseteq F_i$ for $i \geq 1$.

By Lemma 3.1, for each i, $F_i^{\perp} \subseteq (C'_i)^{\perp} \subseteq C_i^{\perp} = F_i^{\perp}$ since $C_i \subseteq C'_i$. Hence $(C'_i)_{i\geq 1}$ is an ascending sequence of convex, compact subcomplexes, containing x, with $(C'_i)^{\perp} = F_i^{\perp}$ for all i.

Note that $\bigcap_i (C'_i)^{\perp} = \bigcap_i F_i^{\perp} = I$. However, $I = (\bigcup C'_i)^{\perp}$. But, $\bigcup C'_i$ cannot be compact since $C'_i \subsetneq C'_{i+1}$, and by Corollary 3.4, we can choose $E \subseteq \bigcup C'_i$. Thus $E \subseteq C'_R$ for some R. But by Lemma 3.1 this means that $I = E^{\perp} \supseteq (C'_R)^{\perp} = F_R{}^{\perp}$, a contradiction. Thus, $\mathfrak{F}$ must have finite multiplicity, as desired. □

We show now that for cube complexes that admit geometric actions, having a factor system implies the NICC for hyperplanes and essential index conditions for hyperplanes. Thus, any proof that $\mathfrak{F}$ forms a factor system for all cocompact cubical groups must necessarily show that any group acting geometrically on a CAT(0) cube complex satisfies these conditions.

Theorem 6.1. *Let $\mathcal{X}$ be a CAT(0) cube complex admitting a geometric action by a group G. If $\mathfrak{F}$ is a factor system, then G satisfies NICC for hyperplanes and the essential index condition.*

Proof. Suppose that $\mathfrak{F}$ is a factor system and at most N elements of $\mathfrak{F}$ can contain any given $x \in \mathcal{X}^{(0)}$. Then for any $A, B \in \mathfrak{F}_1$, there are at most N elements of $\mathfrak{F}$ which can contain $\widehat{F}$, the $\mathrm{Stab}_G(F)$–essential core of F. In particular, there are at most N distinct $\mathrm{Stab}_G(\widehat{F})$–translates of F. Thus, $[\mathrm{Stab}_G(\widehat{F}), \mathrm{Stab}_G(F)] \leq N$, verifying the essential index condition.

To verify NICC for hyperplanes, let H be a hyperplane and let $K = \mathrm{Stab}_G(H)$. Let $\{g_i\}_{i=1}^{\infty}$ be sequence of distinct elements of G so that for $n \geq 1$, the subgroup $K \cap \bigcap_{i=1}^{n} K^{g_i}$ is infinite.

Consider the hyperplane H, notice that K^{g_i} is the stabilizer of $g_i H$. Now, consider $F_1 = \mathfrak{g}_H(g_1 H)$ and inductively define $F_k = \mathfrak{g}_{F_{k-1}}(\mathfrak{g}_H(g_k H))$. Since $\mathfrak{F}$ is a factor system, the set of G–translates of F_{k-1} and $\mathfrak{g}_H(g_k H)$ have finite multiplicity for all $k \geq 2$, and so we can apply the argument of Lemma 1.9 and induction to conclude that $\mathrm{Stab}_G(F_k)$ is commensurable with $G_k = K \cap \bigcap_{i=1}^{k} K^{g_k}$, which is infinite by assumption.

Since $\mathfrak{F}$ is a factor system, there must be some ℓ so that for all $k \geq \ell$, $F_k = F_\ell$. In this case, G_k and G_ℓ are commensurable for all $k \geq \ell$, and in particular G_k and $G_{k'}$ are commensurable for all $k, k' \geq \ell$, and thus G satisfies NICC for hyperplanes. □

7. Factor systems and the simplicial boundary

Corollary D follows from Theorem A, Proposition 7.1 and [Hag13, Lemma 3.32]. Specifically, the first two statements provide a combinatorial geodesic ray representing each boundary simplex v, and when v is a 0–simplex, [Hag13, Lemma 3.32] allows one to convert the combinatorial geodesic ray into a CAT(0) ray. Proposition 7.1 is implicit in the proof of [DHS16, Theorem 10.1]; we give a streamlined proof here.

Proposition 7.1. *Let $\mathcal{X}$ be a CAT(0) cube complex with a factor system $\mathfrak{F}$. Then each simplex σ of $\partial_{\triangle}\mathcal{X}$ is* visible*, i.e. there exists a combinatorial geodesic ray α such that the set of hyperplanes intersecting α is a boundary set representing the simplex σ.*

Remark 7.2. Proposition 7.1 does not assume anything about group actions on $\mathcal{X}$, but instead shows that the existence of an invisible boundary simplex is an obstruction to the existence of a factor system. The converse does not hold: counterexamples can be constructed by beginning with a single combinatorial ray, and gluing to the n^{th} vertex a finite staircase S_n, along a single vertex. The staircase S_n is obtained from $[0, n]^2$ by deleting all squares that are strictly above the diagonal joining $(0, 0)$ to (n, n). In this case, $\mathfrak{F}$ has unbounded multiplicity, and any factor system must contain all elements of $\mathfrak{F}$ exceeding some fixed threshold diameter, so the complex cannot have a factor system.

Proof of Proposition 7.1. Let σ be a simplex of $\partial_{\triangle}\mathcal{X}$. Let σ' be a maximal simplex containing σ, spanned by $v_0, \ldots, v_d$. The existence of σ' follows from [Hag13, Theorem 3.14], which says that maximal simplices exist since $\mathcal{X}$ is finite-dimensional (otherwise, it could not have a factor system). By Theorem 3.19 of [Hag13], which says that maximal simplices are visible, σ' is visible, i.e. there exists a combinatorial geodesic ray γ such that the set $\mathcal{H}(\gamma)$ of hyperplanes crossing γ is a boundary set representing σ'. We will prove that each 0–simplex v_i is visible. It then follows from [Hag13, Theorem 3.23] that any face of σ' (hence σ) is visible.

Let $\mathcal{Y}$ be the convex hull of γ. The set of hyperplanes crossing $\mathcal{Y}$ is exactly $\mathcal{H}(\gamma)$. Since $\mathcal{Y}$ is convex in $\mathcal{X}$, Lemma 8.4 of [BHS14], which provides an induced factor system on convex subcomplexes of cube complexes with factor systems, implies that $\mathcal{Y}$ contains a factor system.

By Theorem 3.10 of [Hag13], we can write $\mathcal{H}(\gamma) = \bigsqcup_{i=1}^{d} \mathcal{V}_i$, where each $\mathcal{V}_i$ is a minimal boundary set representing the 0–simplex v_i. Moreover, up to reordering and discarding finitely many hyperplanes (i.e. moving the basepoint of γ) if necessary, whenever $i < j$, each hyperplane $H \in \mathcal{V}_j$ crosses all but finitely many of the hyperplanes in $\mathcal{V}_i$.

For each $1 \leq i \leq d$, minimality of $\mathcal{V}_i$ provides a sequence of hyperplanes $(V_n^i)_{n\geq 0}$ in $\mathcal{V}_i$ so that V_n^i separates $V_{n\pm 1}^i$ for $n \geq 1$ and so that any other $U \in \mathcal{V}_i$ separates V_m^i, V_n^i for some m, n, by the proof of [Hag13, Lemma 3.7] or [CFI16, Lemma B.6] (one may have to discard finitely many hyperplanes from $\mathcal{V}_i$ for this to hold; this replaces γ with a sub-ray and shrinks $\mathcal{Y}$).

We will show that, after discarding finitely many hyperplanes from $\mathcal{H}(\gamma)$ if necessary, every element of $\mathcal{V}_i$ crosses every element of $\mathcal{V}_j$, whenever $i \neq j$. Since every element of $\mathcal{V}_i$ either lies in $(V_n^i)_n$ or separates two elements of that sequence, it follows that U and V cross whenever $U \in \mathcal{V}_i, V \in \mathcal{V}_j$ and $i \neq j$. Then, for any i, choose $n \geq 0$ and let $H = \bigcap_{j\neq i} V_n^j$. Projecting γ to H yields a geodesic ray in $\mathcal{Y}$, all but finitely many of whose dual hyperplanes belong to $\mathcal{V}_i$, as required. Hence it suffices to show that V_n^i and V_m^j cross for all m, n whenever $i \neq j$.

Fix $j \leq d$ and $i < j$. For each $n \geq 0$, let $m(n) \geq 0$ be minimal so that $V_{m(n)}^j$ fails to cross V_n^i. Note that we may assume that this is defined: if V_n^i crosses all V_m^j, then, since V_m^j crosses all but finitely many of the hyperplanes from $\mathcal{V}_i$, it crosses V_k^i for $k >> n$. Since it also crosses V_n^i, it must also cross V_r^i for all $n \leq r \leq k$. By discarding V_k^i for $k \leq n$ we complete the proof. Now suppose that $m(n)$ is bounded as $n \to \infty$. Then there exists N so that V_n^i, V_m^j cross whenever $m, n \geq N$, and we are done, as before.

Hence suppose that $m(n) \to \infty$ as $n \to \infty$. In other words, for all $m \geq 0$, there exists $n \geq 0$ so that V_m^j crosses V_k^i if and only if $k \geq n$. Choose $M \gg 0$ and choose n maximal with $m(n) < M$. Then all of the hyperplanes $V_{m(k)}^j$ with $k \leq n$ cross $V_k^i, \ldots, V_n^i$ but do not cross V_t^i for $t < k$. Hence the subcomplexes $\mathfrak{g}_{V_n^i}(V_k^i), k \leq n$ are all different: $\mathfrak{g}_{V_n^i}(V_k^i)$ intersects $V_{m(k)}^j$ but $\mathfrak{g}_{V_n^i}(V_{k-1}^i)$ does not. On the other hand, since V_k^i separates V_ℓ^i from V_n^i when $\ell < k < n$, every hyperplane crossing V_n^i and V_ℓ^i crosses V_k^i, so $\mathfrak{g}_{V_n^i}(V_k^i) \cap \mathfrak{g}_{V_n^i}(V_\ell^i) \neq \emptyset$. Thus the factor system on $\mathcal{Y}$ has multiplicity at least n. But since $m(n) \to \infty$, we could choose n arbitrarily large in the preceding argument, violating the definition of a factor system. □

Proof of Corollary E. If γ is a CAT(0) geodesic, then it can be approximated, up to Hausdorff distance depending on $\dim \mathcal{X}$, by a combinatorial geodesic, so assume that γ is a combinatorial geodesic ray. By Corollary D, the simplex of $\partial_{\triangle}\mathcal{X}$ represented by γ is spanned by 0–simplices $v_0, \ldots, v_d$ with each v_i represented by a combinatorial geodesic ray γ_i. Theorem 3.23 of [Hag13] says that $\mathcal{X}$ contains a cubical orthant $\prod_i \gamma_i'$, where each γ_i' represents v_i. Hence $Hull(\cup_i \gamma_i') = \prod_i Hull(\gamma_i)$. Up to truncating an initial subpath of γ, we have that γ is parallel into $Hull(\cup_i \gamma_i')$ (and thus lies in a finite neighbourhood of it). The projection of the original CAT(0) geodesic approximated by γ to each $Hull(\gamma_i')$ is a CAT(0) geodesic representing v_i. The product of these geodesics is a combinatorially isometrically embedded $(d+1)$-dimensional orthant subcomplex of $\mathcal{Y}$ containing (the truncated) CAT(0) geodesic in a regular neighbourhood. □

In the presence of a proper, cocompact group action, we can achieve full visibility under slightly weaker conditions than those that we have shown suffice to obtain a factor system:

Proposition 7.3. *Let $\mathcal{X}$ be a proper CAT(0) cube complex on which the group G acts properly and cocompactly. Suppose that the action of G on $\mathcal{X}$ satisfies NICC for hyperplanes. Then each simplex σ of $\partial_\triangle \mathcal{X}$ is* visible*, i.e. there exists a combinatorial geodesic ray α such that the set of hyperplanes intersecting α is a boundary set representing the simplex σ.*

Proof. We adopt the same notation as in the proof of Proposition 7.1. As in that proof, if $\partial_\triangle \mathcal{X}$ contains an invisible simplex, then we have two infinite sets $\{V_i\}_{i\geq 0}, \{H_j\}_{j\geq 0}$ of hyperplanes with the following properties:

- for each $i \geq 1$, the hyperplane H_i separates H_{i-1} from H_{i+1};
- for each $j \geq 1$, the hyperplane V_j separates V_{j-1} from V_{j+1};
- there is an increasing sequence (i_j) so that for all j, V_j crosses H_i if and only if $i \leq i_j$.

This implies that for all $i \geq 1$, the subcomplex $F_i = \mathfrak{g}_{H_0}(\mathfrak{g}_{H_1}(\cdots(\mathfrak{g}_{H_{i-1}}(H_i))\cdots))$ is unbounded. Since $\mathrm{Stab}_G(F_i)$ acts cocompactly, by Proposition 2.7, $\mathrm{Stab}_G(F_i)$ is infinite. By Lemma 1.9, $\mathrm{Stab}_G(F_i)$ is commensurable with $K_i = \bigcap_{j=1}^{i} \mathrm{Stab}_G(H_j)$, and so by NICC, there exists N so that K_i is commensurable with K_N for all $i \geq N$. Thus, after passing to a subsequence, we see that for all i, the K_i–essential core of F_i is a fixed nonempty (indeed, unbounded) convex subcomplex $\widehat{F}$ of H_0.

Now, for each j, the hyperplane V_j cannot cross $\widehat{F}$, because $\widehat{F}$ lies in F_i for all i, and V_j fails to cross H_i for all sufficiently large i. Moreover, this shows that $\widehat{F}$ must lie in the halfspace associated to V_j that contains V_{j+1}. But since this holds for all j, we have that $\widehat{F}$ is contained in an infinite descending chain of halfspaces, contradicting that $\widehat{F} \neq \emptyset$. □

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